REMARKS ON EXCHANGEABILITY OF INVARIANT MEASURES FOR SYMMETRIC EXCLUSION

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Abstract. We explore the use of an entropy functional to deduce exchangeability of invariant measures, which are also shift-invariant, in a class of symmetric exclusion processes on $\mathbb{Z}^d$.

1. Introduction

The purpose of this note is to explore the use of an entropy functional to deduce exchangeability of invariant measures, which are also shift-invariant, in a class of symmetric exclusion particle systems on $\mathbb{Z}^d$. 'Entropy' arguments were used in Section IV.5 in [8] to characterize the invariant measures of stochastic Ising models as Gibbs states. From this point of view, we were motivated to consider such methods in the context of exclusion processes.

Informally, the symmetric exclusion process follows a collection of continuous time symmetric motions subject to the condition that jumps to occupied vertices are suppressed. More carefully, the system follows the unlabeled Markov evolution of particle occupations on $\mathbb{Z}^d$. Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$ be the configuration space $\eta = (\eta(x))_{x\in\mathbb{Z}^d}$, where $\eta(x) = 1$ means that x is occupied and $\eta(x) = 0$ that it is empty.

Let $p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty)$ be a symmetric, $p(x,y) = p(y,x)$ for $x, y \in \mathbb{Z}^d$, irreducible transition probability. For $x, y \in \mathbb{Z}^d$, let $\tau_{x,y}\eta = \eta^{x,y} \in \Omega$ be the configuration obtained by swapping $\eta(x)$ and $\eta(y)$. Denote the process generator on local functions f, that is those which depend only on a finite number of $\{\eta(x)\}_{x\in\mathbb{Z}^d}$, by

$$\begin{aligned}(Lf)(\eta) &= \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} p(x,y)\,\eta(x)\big(1-\eta(y)\big)\,\big(f(\eta^{x,y}) - f(\eta)\big)\\ &= \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} p(x,y)\,\big(f(\eta^{x,y}) - f(\eta)\big),\end{aligned}$$

where that last line follows from observing that for any pair (x,y), via symmetry of $p(\cdot,\cdot)$,

$$\eta(x)\big(1-\eta(y)\big) + \eta(y)\big(1-\eta(x)\big) = \mathbf{1}_{\{\eta(x)\neq\eta(y)\}},$$

and $f(\eta^{x,y}) - f(\eta) = 0$ whenever $\eta(x) = \eta(y)$.

We recall that a probability measure μ on Ω is invariant for L if $\int Lf\,d\mu = 0$ for all local f. Let τ_x be the shift operator so that $(\tau_x\eta)(y) = \eta(y-x)$, and let $\tau_x\mu$ be the induced measure.

Let $\mathcal{I}$ denote the convex set of invariant measures, and $\mathcal{I}_e$ their extreme points. Let also $\mathcal{S}$ be the convex set of translation-invariant measures μ on Ω, that is $\tau_x\mu = \mu$ for all $x \in \mathbb{Z}^d$. For a convex set of measures $\mathcal{P}$, we denote their extreme points by $\mathcal{P}_e$.

For a density $\rho \in [0,1]$, let $\nu_\rho = \prod_{x\in\mathbb{Z}^d}\mathrm{Bern}(\rho)$ denote the Bernoulli product measure on Ω. It is a straightforward calculation to see that each ν_ρ is invariant with respect to L.

Let also $\mathcal{H}$ be the set of bounded harmonic functions $\alpha : \mathbb{Z}^d \to [0,1]$ of $p(\cdot,\cdot)$, that is when $\sum_{y\in\mathbb{Z}^d} p(x,y)\alpha(y) = \alpha(x)$ for all $x \in \mathbb{Z}^d$. For such an $\alpha(\cdot)$, define the possibly inhomogeneous

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Bernoulli product measure $\nu_{\alpha(\cdot)} = \prod_{x\in\mathbb{Z}^d} \mathrm{Bern}(\alpha(x))$. If $\{P_t\}_{t\geq 0}$ denotes the semigroup with respect to L, then by Theorem VIII.1.24(a) in [8],

$$\hat{\nu}_{\alpha(\cdot)} := \lim_{t\to\infty} \nu_{\alpha(\cdot)} P_t$$

exists. When $\alpha(\cdot) \equiv \rho$ is constant, then $\hat{\nu}_{\alpha(\cdot)} = \nu_{\alpha(\cdot)} = \nu_\rho$ by invariance of ν_ρ.

We now state the well-known characterization of invariant measures, seen in Theorem VIII.1.44 in [8]. The proof in [8] makes use of 'duality' of the symmetric process (cf. Theorem VIII.1.1 [8]) and coupling ideas, along with certain semigroup comparisons with independent particle systems.

Theorem 1.1. *If $p(\cdot,\cdot)$ is symmetric and irreducible, then*

$$\mathcal{I}_e = \{\hat{\nu}_{\alpha(\cdot)} : \alpha(\cdot) \in \mathcal{H}\}.$$

One can ask if all shift-invariant invariant measures $\mu \in \mathcal{I}\cap\mathcal{S}$ are exchangeable, that is if $(\mathcal{I}\cap\mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$. Such a statement seems natural. However, as the form of $\hat{\nu}_{\alpha(\cdot)}$ is in general not explicit, such an inference does not seem immediate from Theorem 1.1.

Nevertheless, when $p(\cdot,\cdot)$ is such that $\mathcal{H}$ consists of constants, all invariant measures are in fact shift-invariant and moreover exchangeable (recall de Finetti's theorem; cf. [3]), that is $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$. Sufficient conditions include when the Markov chain $\{X_n\}_{n\geq 0}$ with transition probability $p(\cdot,\cdot)$ is recurrent (by considering the bounded martingale $\{\alpha(X_n)\}_{n\geq 0}$ for $\alpha(\cdot)\in\mathcal{H}$) or when $p(\cdot,\cdot)$ is translation-invariant, that is $p(x,y) = p(y-x)$ for all $x,y\in\mathbb{Z}^d$ (cf. Choquet-Deny theorem in Corollary II.7.2 in [8]).

Consider now the following class of transition probabilities $p(\cdot,\cdot)$.

Assumption 1.2. Suppose $p(\cdot,\cdot)$ satisfies the following items.

(1) $p(\cdot,\cdot)$ is a symmetric transition probability,
(2) $p(\cdot,\cdot)$ is such that

$$\frac{1}{|S_n|}\sum_{x\in S_n}\sum_{y\notin S_n} p(x,y) \xrightarrow[n\to\infty]{} 0,$$

where $S_n = \{x\in\mathbb{Z}^d : |x| \leq n\}$ is the box of width $2n+1$, with $|x| = \max_{i=1,\ldots,d}|x_i|$.
(3) $p(\cdot,\cdot)$ is minorized: $\beta(z) := \inf_{x\in\mathbb{Z}^d} p(x,x+z)$ for $z\in\mathbb{Z}^d$ is an irreducible sub-probability on $\mathbb{Z}^d$.

Here, although $p(\cdot,\cdot)$ may be non-translation-invariant, $\beta(\cdot)$ in Item (3) may be thought of as an irreducible 'translation-invariant' proxy. One suspects that the associated $\mathcal{H}$ consists of constants, but this is not pursued here.

Note also that Item (2) includes all finite-range kernels, that is when $p(x,y) = 0$ for $|x-y| > K$ for some $K < \infty$: Indeed, let $\partial_K B = \{x \in B : \mathrm{dist}(x, B^c) \leq K\}$. Then, $\frac{1}{|S_n|}\sum_{x\in S_n}\sum_{y\notin S_n} p(x,y) \leq \frac{|\partial_K S_n|}{|S_n|}\sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} p(x,y) = \frac{|\partial_K S_n|}{|S_n|} \to 0$. In addition, if $p(\cdot,\cdot)$ is translation-invariant, Item (3) specifies that $\beta(z) \equiv p(x,x+z)$, and Item (2) would hold when $\sum_{j\geq 1} j^{d-1}\hat{\beta}(j) < \infty$ where $\hat{\beta}(j) = \sup_{z:|z|=j}\beta(z)$.

The focus of this note will be on the use of an entropy functional to give a brief, self-contained characterization of invariant measures that are shift-invariant, with respect to the class specified by Assumption 1.2.

Theorem 1.3. *Suppose $p(\cdot,\cdot)$ satisfies Assumption 1.2. Then,*

$$(\mathcal{I}\cap\mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}.$$

In Section 2, we prove Theorem 1.3. Then, in Section 3, we end with some remarks on extensions and open problems.

2. Proof of Theorem 1.3

Let $\mu \in \mathcal{I} \cap \mathcal{S}$, that is an invariant measure that is also shift-invariant. Fix a finite box $B \subset \mathbb{Z}^d$ and write $\Omega_B = \{0,1\}^B$. For $\xi \in \Omega_B$, let $C_\xi := \{\eta \in \Omega : \eta|_B = \xi\}$ and write μ_B for the B-marginal of μ.

For a configuration $\xi \in \Omega_B$, denote by $\xi^{x,y}$ the 'internal' swap on B with respect to $x, y \in B$. If $x \in B$, write $\sigma_x \xi$ for the configuration in Ω_B obtained by flipping the bit at x.

Let μ_t be the law of the process at time t with $\mu_0 = \mu$, and let $\mu_{t,B}$ be the B-marginal of μ_t. Note that with this notation, we have $\mu_t(C_\xi) = \mu_{t,B}(\xi)$ for $\xi \in \Omega_B$.

Step 1. Let $\xi \in \Omega_B$, and consider the indicator $f_\xi(\eta) = \mathbf{1}_{C_\xi}(\eta)$. By stationarity of μ and the backward equation,

$$\begin{aligned} 0 = \int L f_\xi \, d\mu &= \frac{1}{2} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} p(x,y) \big(\mu(\tau_{x,y}^{-1} C_\xi) - \mu(C_\xi) \big) \\ &= \frac{d}{dt} \mu_{t,B}(\xi) \Big|_{t=0}. \end{aligned} \tag{1}$$

In the summation above, only bonds (x,y) involving at least one site in B contribute. Internal swaps $(x,y) \subset B$ are such that $\tau_{x,y}^{-1} C_\xi = C_{\xi^{x,y}}$. Boundary-crossing swaps where $x \in B$, $y \notin B$ satisfy

$$\begin{aligned} \mu(\tau_{x,y}^{-1} C_\xi) - \mu(C_\xi) &= \mu\big((\tau_{x,y}^{-1} C_\xi) \cap \{\eta(x) \neq \eta(y)\} \big) - \mu\big(C_\xi \cap \{\eta(x) \neq \eta(y)\} \big) \\ &= \mu\big(C_{\sigma_x \xi} \cap \{\eta(y) = 1 - \eta(x)\} \big) - \mu\big(C_\xi \cap \{\eta(y) = 1 - \eta(x)\} \big). \end{aligned}$$

Hence, for each $\xi \in \Omega_B$, noting symmetry $p(x,y) = p(y,x)$, equation (1) equals

$$0 = \frac{d}{dt} \mu_{t,B}(\xi)|_{t=0} = \sum_{x \in B} \sum_{y \in B} p(x,y) \left(\mu_B(\xi^{x,y}) - \mu_B(\xi) \right) + \sum_{x \in B} \sum_{y \notin B} \big(\Gamma_{x,y}(\sigma_x \xi) - \Gamma_{x,y}(\xi) \big), \tag{2}$$

where for $x \in B$ and $y \notin B$ we define

$$\Gamma_{x,y}(\xi) := p(x,y) \, \mu\big(C_\xi \cap \{\eta(y) = 1 - \xi(x)\} \big).$$

Step 2. Consider the finite-volume entropy functional

$$H_B(\mu_t) := \sum_{\xi \in \Omega_B} \mu_{t,B}(\xi) \log \mu_{t,B}(\xi),$$

with the convention $0 \log 0 = 0$. We will also use later the convention $0/0 = 1$. We comment that the functional H_B may be seen in terms of relative entropy $\widetilde{H}$ with respect to $\nu := \nu_{1/2}$ restricted to B, namely $H_B(\mu) = \widetilde{H}(\mu_B; \nu_B) - |B| \log 2$. Consider further an $\epsilon > 0$ variation:

$$H_B^\epsilon(\mu_t) := \sum_{\xi \in \Omega_B} (\mu_{t,B}(\xi) + \epsilon) \log(\mu_{t,B}(\xi) + \epsilon).$$

Differentiating H_B^ϵ by the chain rule and using (1) gives

$$\frac{d}{dt} H_B^\epsilon(\mu_t) \Big|_{t=0} = \sum_{\xi \in \Omega_B} \big(\log(\mu_B(\xi) + \epsilon) + 1 \big) \frac{d}{dt} \mu_{t,B}(\xi) \Big|_{t=0} = 0.$$

Since $\sum_{\xi \in \Omega_B} \frac{d}{dt} \mu_{t,B}(\xi) \big|_{t=0} = \frac{d}{dt} 1 = 0$, the constant $+1$ terms do not contribute, so we have

$$\frac{d}{dt} H_B^\epsilon(\mu_t) \Big|_{t=0} = \sum_{\xi \in \Omega_B} \log(\mu_B(\xi) + \epsilon) \frac{d}{dt} \mu_{t,B}(\xi) \Big|_{t=0} = 0. \tag{3}$$

We now observe that if $\xi \in \Omega_B$ is such that $\mu_B(\xi) = 0$, and so that $\mu\big(C_\xi \cap \{\eta(y) = 1-\eta(x)\}\big) = 0$, then equation (2) reduces to the vanishing of a sum of nonnegative terms, in particular

$$\sum_{x\in B}\sum_{y\in B} p(x,y)\,\mu_B(\xi^{x,y}) + \sum_{x\in B}\sum_{y\notin B}\Gamma_{x,y}(\sigma_x\xi) = 0.$$

This implies that $p(x,y)\mu_B(\xi^{x,y}) = 0$ in the case $x, y \in B$ and $\Gamma_{x,y}(\sigma_x\xi) = 0$ in the case $x \in B$, $y \notin B$. Therefore, given our conventions $0\log 0 = 0$ and $0/0 = 1$, we may take $\epsilon \downarrow 0$ in (3), noting (2), to obtain

$$\sum_{\xi\in\Omega_B} \log\mu_B(\xi)\,\frac{d}{dt}\mu_{t,B}(\xi)\Big|_{t=0} = 0, \tag{4}$$

understanding in this sum and later calculations that when probabilities $\mu_B(\xi)$ vanish, the whole term indexed by ξ vanishes.

Step 3. Inserting the right-hand side of (2) into (4), we have

$$\begin{aligned}0 &= \sum_{\xi\in\Omega_B}\log\mu_B(\xi)\Big[\sum_{x\in B}\sum_{y\in B}p(x,y)\big(\mu_B(\xi^{x,y})-\mu_B(\xi)\big) + \sum_{x\in B}\sum_{y\notin B}\big(\Gamma_{x,y}(\sigma_x\xi)-\Gamma_{x,y}(\xi)\big)\Big]\\
&= \sum_{x\in B}\sum_{y\in B}p(x,y)\sum_{\xi\in\Omega_B}\big(\mu_B(\xi^{x,y})-\mu_B(\xi)\big)\log\mu_B(\xi) + \sum_{x\in B}\sum_{y\notin B}\sum_{\xi\in\Omega_B}\big(\Gamma_{x,y}(\sigma_x\xi)-\Gamma_{x,y}(\xi)\big)\log\mu_B(\xi).\end{aligned}$$

We write this as

$$0 = -R_{B,1} + R_{B,2}, \tag{5}$$

where, from adding the sum to itself with the change of variables $\xi \to \xi^{x,y}$,

$$\begin{aligned}R_{B,1} &:= -\sum_{x\in B}\sum_{y\in B}p(x,y)\sum_{\xi\in\Omega_B}\big(\mu_B(\xi^{x,y})-\mu_B(\xi)\big)\log\mu_B(\xi)\\
&= \frac{1}{2}\sum_{x\in B}\sum_{y\in B}p(x,y)\sum_{\xi\in\Omega_B}\big(\mu_B(\xi^{x,y})-\mu_B(\xi)\big)\log\frac{\mu_B(\xi^{x,y})}{\mu_B(\xi)},\end{aligned} \tag{6}$$

and from the change of variables $\xi \to \sigma_x\xi$,

$$\begin{aligned}R_{B,2} &:= \sum_{x\in B}\sum_{y\notin B}\sum_{\xi\in\Omega_B}\big(\Gamma_{x,y}(\sigma_x\xi)-\Gamma_{x,y}(\xi)\big)\log\mu_B(\xi)\\
&= \sum_{x\in B}\sum_{y\notin B}\sum_{\xi\in\Omega_B}\Gamma_{x,y}(\xi)\log\frac{\mu_B(\sigma_x\xi)}{\mu_B(\xi)}.\end{aligned} \tag{7}$$

Each summand in (6) for which $p(x,y) \neq 0$ is of the form $(a-b)\log(a/b)$ with $a = \mu_B(\xi^{x,y})$ and $b = \mu_B(\xi)$, and is therefore nonnegative and equal to 0 exactly when $a = b$, in other words exactly when μ_B is invariant under $\tau_{x,y}$ for $x, y \in B$.

Step 4. For $x, y \in B$ define the bond-wise entropy

$$J_B(x,y) := \sum_{\xi\in\Omega_B}\big(\mu_B(\xi^{x,y})-\mu_B(\xi)\big)\log\frac{\mu_B(\xi^{x,y})}{\mu_B(\xi)}.$$

With this notation, (6) reads

$$R_{B,1} = \frac{1}{2}\sum_{x\in B}\sum_{y\in B}p(x,y)\,J_B(x,y). \tag{8}$$

We now show that the 'boundary flux' $R_{B,2}$ is at most of order $|\partial_K B|$.

Lemma 2.1. *For every finite box B,*

$$0 \le R_{B,2} \;\le\; 2 \sum_{x \in B} \sum_{y \notin B} p(x,y).$$

Proof. The left-hand inequality follows as $R_{B,1} \ge 0$ and $R_{B,2} = R_{B,1}$.

For the right-hand inequality, fix a boundary pair (x,y) with $x \in B$, $y \notin B$, and write $\sigma = \sigma_x$. Noting the second representation in (7), the contribution of this pair to $R_{B,2}$ equals $\sum_{\xi \in \Omega_B} \Gamma_{x,y}(\xi) \log(\mu_B(\sigma\xi)/\mu_B(\xi))$. Set $u_\xi := \mu_B(\sigma\xi)/\mu_B(\xi)$. Using $\log u \le u-1$ and $\mu(C_\xi \cap \{\eta(y) = 1-\eta(x)\})/\mu(C_\xi) \le 1$, we have

$$\begin{aligned}\sum_{\xi \in \Omega_B} \Gamma_{x,y}(\xi) \log u_\xi &\le p(x,y) \sum_{\xi \in \Omega_B} \mu(C_\xi \cap \{\eta(y) = 1-\eta(x)\})\,(u_\xi - 1)\\ &\le p(x,y) \sum_{\xi \in \Omega_B} \big|\mu_B(\sigma\xi) - \mu_B(\xi)\big|.\end{aligned}$$

The last sum equals $2\|\mu_B - \mu_B \circ \sigma\|_{\mathrm{TV}} \le 2$, so the contribution of (x,y) is at most $2\,p(x,y)$. Summing over all boundary pairs yields $0 \le R_{B,2} \le 2\sum_{x\in B}\sum_{y \notin B} p(x,y)$. ☐

Step 5. Recall $S_n = \{x \in \mathbb{Z}^d : |x| \le n\}$. Combining (5), (8), and Lemma 2.1 yields

$$\sum_{x \in S_n} \sum_{y \in S_n} p(x,y)\, J_{S_n}(x,y) \;\le\; 4 \sum_{x \in S_n} \sum_{y \notin S_n} p(x,y). \tag{9}$$

Therefore, by Item (2) in Assumption 1.2, the inequality (9) gives

$$\frac{1}{|S_n|} \sum_{x \in S_n} \sum_{y \in S_n} p(x,y)\, J_{S_n}(x,y) \;\le\; \frac{4}{|S_n|} \sum_{x \in S_n} \sum_{y \notin S_n} p(x,y) \;\xrightarrow[n\to\infty]{}\; 0. \tag{10}$$

Step 6. We now try to lower bound this average to obtain a pointwise statement. If $A \subset B$ and $x,y \in A$, then projecting from B to A can only decrease the (x,y)-bond entropy (by the log-sum inequality, say). Hence,

$$J_A(x,y) \;\le\; J_B(x,y). \tag{11}$$

Recall Item (3) in Assumption 1.2. Fix a finite set $\Lambda \subset \mathbb{Z}^d$ and fix a displacement z_0 with $\beta(z_0) > 0$. Let $a \in \Lambda$ be such that $(a, a+z_0) \subset \Lambda$. Since all terms in (10) are nonnegative, we have

$$\frac{1}{|S_n|} \sum_{\substack{x \in S_n\\ x+z_0 \in S_n}} J_{S_n}(x, x+z_0) \;\le\; \frac{1}{\beta(z_0)} \cdot \frac{1}{|S_n|} \sum_{x \in S_n} \sum_{y \in S_n} p(x,y)\, J_{S_n}(x,y) \;\xrightarrow[n\to\infty]{}\; 0. \tag{12}$$

For n large, let $U_n := \{u \in \mathbb{Z}^d :\ \Lambda + u \subset S_n\}$. For every $u \in U_n$, the projection monotonicity (11) gives

$$J_{\Lambda+u}(a+u, a+u+z_0) \;\le\; J_{S_n}(a+u, a+u+z_0).$$

Averaging over $u \in U_n$ and using that $|U_n|/|S_n| \to 1$ yields

$$\frac{1}{|U_n|} \sum_{u \in U_n} J_{\Lambda+u}(a+u, a+u+z_0) \;\le\; \frac{|S_n|}{|U_n|} \cdot \frac{1}{|S_n|} \sum_{\substack{x \in S_n\\ x+z_0 \in S_n}} J_{S_n}(x, x+z_0) \;\xrightarrow[n\to\infty]{}\; 0, \tag{13}$$

where the last limit follows from (12).

Step 7. Our goal now is to show that the Λ–marginal of μ is invariant under $\tau_{a,a+z_0}$. By shift-invariance of μ, the left-hand side of (13) equals the constant $J_\Lambda(a, a+z_0)$. Hence, $J_\Lambda(a, a+z_0) = 0$, and therefore $\mu_\Lambda(\xi^{a,a+z_0}) = \mu_\Lambda(\xi)$ for all $\xi \in \Omega_\Lambda$. That is, μ_Λ is invariant under the transposition $\tau_{a,a+z_0}$.

We will say that the bond $(a, a+z) \subset \Lambda$ is admissible when $\beta(z) \neq 0$. Since Λ and the admissible bond transposition $\tau_{a,a+z_0}$ were arbitrary, μ is invariant under all admissible bond transpositions $\tau_{x,x+z}$: Indeed, for f bounded functions, then $E_\mu[f(\eta^{x,x+z})] = \sum_{\xi\in\Omega_\Lambda} \mu_\Lambda(\xi) E_\mu[f(\eta^{x,x+z})|\eta_\Lambda = \xi] = \sum_{\xi\in\Omega_\Lambda} \mu_\Lambda(\xi) E_\mu[f(\eta)|\eta_\Lambda = \xi^{x,x+z}] = \sum_{\xi\in\Omega_\Lambda} \mu_\Lambda(\xi^{x,x+z}) E_\mu[f(\eta)|\eta_\Lambda = \xi] = E_\mu[f(\eta)]$, when $(x, x+z) \subset \Lambda$. Because the admissible-bond graph on $\mathbb{Z}^d$ is connected by irreducibility of $\beta(\cdot)$, these transpositions generate the group of finitely supported permutations. Hence, μ is exchangeable.

By de Finetti's theorem, μ is a mixture of $\{\nu_\rho : 0 \leq \rho \leq 1\}$. Note that each ν_ρ is an invariant measure and also shift-invariant, and cannot be expressed as a nontrivial mixture: If so, $\nu_\rho = \int \nu_m \, \lambda(dm)$. Then, $\rho = \int_0^1 m \, \lambda(dm)$ and $\int_\Omega \eta(x)\eta(y) \, \nu_\rho(d\eta) = \rho^2 = \int_0^1 m^2 \, \lambda(dm)$ for $x \neq y$. Hence, the variance of $\lambda(dm)$ vanishes.

We conclude from these observations that $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$. □

3. Concluding remarks

To summarize the strategy of the entropy proof of Theorem 1.3, (1) stationarity of μ forces the bulk entropy production to be supported near the boundary, and (2) shift-invariance determines that μ is invariant to interior transpositions, yielding exchangeability of μ.

One may ask if the entropy technique may be pursued in other symmetric mass conservative processes such as zero-range models to characterize their translation-invariant stationary measures, but this is left to the interested reader. Such identifications for the translation-invariant model are already known in [2] when the zero-range rate $g(k)$ is increasing in k, among others; see also [10] in this context.

We now recall an open problem: Consider finite-range, irreducible, translation-invariant kernels $p(\cdot)$ with mean $\sum zp(z) = 0$, but that are not necessarily symmetric. In $d = 1$, it is known that $\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 1\}$; see Theorem VIII.3.14 in [8]. The same should be true in $d \geq 2$, as stated in Problem 5 on p. 416 in [8], however this still seems to be an open problem. One wonders if the entropy technique would be helpful for this question.

See also [1], [4], [5], [6], and [9], and references therein, for more comprehensive discussions, including open problems, on invariant measures for exclusion on $\mathbb{Z}^d$ and its relatives.

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