

# ON A NONLINEAR SPDE DERIVED FROM A HYDRODYNAMIC LIMIT IN A SINAI-TYPE RANDOM ENVIRONMENT

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ABSTRACT. With the recent developments on nonlinear SPDE's, where smoothing of rough noises is needed, one is naturally led to study interacting particle systems whose macroscopic evolution is described by these equations and which possess an in-built smoothing. In this article, our main results are to derive regularized versions of the ill-posed one dimensional SPDE

$$\partial_t \rho = \frac{1}{2} \Delta \Phi(\rho) - 2 \nabla (W' \Phi(\rho)),$$

where the spatial white noise  $W'$  is replaced by a regularization  $W'_\varepsilon$ , as quenched and annealed hydrodynamic limits of zero-range interacting particle systems in  $\varepsilon$ -regularized Sinai-type random environments. Some computations are also made about annealed mean hydrodynamic limits in unregularized Sinai-type random environments with respect to independent particles.

## 1. INTRODUCTION

Given recent developments in paracontrolled and regularity structures analysis of nonlinear SPDE's, where smoothing of noises is involved, it is natural to investigate microscopic interacting particle systems whose macroscopic hydrodynamic limit already captures the smoothing of the noise. In this article, we are interested in interacting particle systems whose hydrodynamic evolution is given by a smoothing of the ill-posed one dimensional nonlinear SPDE

$$\partial_t \rho = \frac{1}{2} \Delta \Phi(\rho) - 2 \nabla (W' \Phi(\rho)) \tag{1.1}$$

where the spatial white noise  $W'$  is replaced by a regularization  $W'_\varepsilon$ , and  $\Phi \in C^1$ . Such an equation might be considered a 'mass conserved' generalized parabolic Anderson equation, noting that the parabolic Anderson equation is when  $\Phi(v) \equiv v$  and  $\nabla$  is not present.

More specifically, with respect to a zero-range particle system in a common Sinai-type  $\varepsilon$ -regularized random environment on a one dimensional discrete lattice, we derive 'quenched' and 'annealed' hydrodynamic limits, which feature a regularized noise  $W'_\varepsilon$ , emerging from a scaling of the random environment. Some computations are also made on the annealed mean hydrodynamic limit in unregularized Sinai-type random environments with respect to independent particles. We remark, as part of a two-scale approach, these results connect to the work Funaki et. al. [13], where the limit of the hydrodynamic density as  $\varepsilon \downarrow 0$  is shown to satisfy a paracontrolled version of the SPDE (1.1).

In the next subsections, we introduce the random environments studied, summarize the main results and their proofs, review some previous literature, and discuss directions for future work.

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1991 *Mathematics Subject Classification.* 60K35; 60K37; 60L50.

*Key words and phrases.* Sinai random environment, Brox diffusion, SPDE, interacting particle system, zero-range, hydrodynamic, quasilinear, inhomogeneous, annealed, quenched, regularization.

**1.1. ‘Sinai’ random environments.** To introduce elements of the environments studied in the article, we recall that a ‘Sinai’ random environment on  $\mathbb{Z}$  is a sequence of independent and identically distributed (i.i.d.) random variables  $\{u_i\}_{i \in \mathbb{Z}}$ , indexed over vertices, with the property that  $c \leq u_0 \leq 1 - c$  for some constant  $0 < c < 1/2$  and  $E[\log(u_0/(1 - u_0))] = 0$ . Define  $\sigma^2 = E[(\log(u_0/(1 - u_0)))^2]$ . Let  $U_n$  be the position of a discrete-time random walk in this random environment (RWRE):

$$P(U_{n+1} = U_n + 1 | U_n, \{u_i\}) = 1 - P(U_{n+1} = U_n - 1 | U_n, \{u_i\}) = u_{U_n}$$

for  $n \geq 1$  and  $U_0 = 0$ . When  $\sigma^2 > 0$ , Sinai [32] showed that  $\sigma^2 U_n / (\log(n))^2$  converges in the annealed sense weakly to a non-trivial random variable  $U_\infty$ , whose law was identified in Kesten [22] and Golosov [14]. A functional limit theorem to a non-trivial process is problematic however as  $\sigma^2 U_{\lfloor nt \rfloor} / (\log(n))^2 \Rightarrow U_\infty$ , the limit process here being constant in time  $t$ .

However, a continuous analog  $X_t$  of the Sinai RWRE on  $\mathbb{R}$  was introduced in Brox [4]: Formally,

$$dX_t = dB_t - \frac{1}{2} W'(X_t) dt$$

and  $X_0 = 0$  or that the generator  $\mathcal{L}_{Brox}$  takes form

$$\mathcal{L}_{Brox} = \frac{1}{2} e^{W(x)} \frac{d}{dx} (e^{-W(x)} \frac{d}{dx}).$$

Here,  $B$  is a standard Brownian motion and  $W$  is a two-sided Brownian motion on  $\mathbb{R}$ :  $W(0) = 0$ ,  $W(x) = \sigma W_+(x)$  for  $x > 0$ , and  $W(x) = \sigma W_-(-x)$  for  $x < 0$ , where  $W_\pm$  are independent standard Brownian motions. This description is only short hand, as  $W$  is not differentiable a.s. More carefully, this ‘ $\sigma$ -Brox’ diffusion is defined in terms of speed and scale measures:

$$X_t = A^{-1}(B_{T^{-1}(t)}) \quad \text{where} \quad A(y) = \int_0^y e^{W(z)} dz \quad \text{and} \quad T(t) = \int_0^t e^{-2W(A^{-1}(B_s))} ds$$

for  $y \in \mathbb{R}$  and  $t \geq 0$ . Brox [4] showed that  $X_t / (\log t)^2 \Rightarrow U_\infty$ , the same limit as for the discrete Sinai RWRE convergence.

To connect the two models, related random environments  $\{u_i^N\}_{i \in \mathbb{Z}}$ , in terms of a scaling parameter  $N$ , were introduced in Seignourel [31]. An example is where  $u_i^N = 1/2 + r_i / \sqrt{\sigma^2 N}$  where  $\{r_i\}_{i \in \mathbb{Z}}$  is an i.i.d. sequence of bounded random variables which are mean-zero and with positive variance  $\sigma^2$ . Let  $U_n^N$  be the corresponding RWRE with respect to the scaled environment  $\{u_i^N\}$ . Seignourel [31] showed that  $\{N^{-1} U_{\lfloor N^2 t \rfloor}^N : t \geq 0\}$  converges weakly in the annealed sense to the standard 1-Brox diffusion  $\{X_t : t \geq 0\}$ . See also Andriopolous [2] and Pacheco [28] for extensions and variations of this convergence, and Hu, Le and Mytnik [16], Matzavinos, Roitershtein and Seol [26] and references therein in this context.

**1.2. Sketch of results.** To specify the ‘regularized’ Sinai-type environment that we consider and the associated hydrodynamic limits of the zero-range process in this environment, we discuss first what might hold with respect to an ‘unregularized’ Seignourel environment  $\{u_i^N = 1/2 + r_i / \sqrt{\sigma^2 N}\}$  on a discrete torus  $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ . Informally, in the zero-range process, a particle at site  $i$ , with  $k$  particles, will jump with rate  $g(k)/k$  and move to a neighbor  $j = i \pm 1$  with probability  $p(i, j)$ . We point out the case  $g(k) \equiv k$  is when the random walks are all independent. When a random environment is imposed,  $p(i, j) = u_i^N$  when  $j = i + 1$  and  $= 1 - u_i^N$  when  $j = i - 1$ . Since a random walker in such an environment experiences many traps and moves slowly, one expects the hydrodynamic limit to involve ‘drift’ terms reflecting the environment.

In principle, since infinitesimally the rate of change of the number of particles  $\eta_i(t)$  at site  $i$  at time  $t$  varies according to the generator action,

$$L\eta_i = \frac{1}{2} \{g(\eta_{i+1}) + g(\eta_{i-1}) - 2g(\eta_i)\} \\ + \{g(\eta_{i-1})(r_{i-1}/\sqrt{\sigma^2 N}) - g(\eta_{i+1})(r_{i+1}/\sqrt{\sigma^2 N})\},$$

and  $r_i/\sqrt{\sigma^2 N} = [S(i) - S(i-1)]/\sqrt{N}$  where  $S(\cdot)$  is the partial sum of  $\{r_i\}$ , one might expect, scaling space and time diffusively, that the hydrodynamic evolution of the mass satisfies the SPDE (1.1), where  $\Phi$  is a homogenized version of the jump rate  $g$ . Of course, since  $W$  is not differentiable, the spatial white noise  $W'$  in (1.1) has to be interpreted as a distribution, and the equation is ill-posed. But, in a sense, an equation like this should represent the hydrodynamic equation, the term  $W'$  reflecting the random environment, even after some averaging in the scaling limit has been taken.

As a way to obtain the singular hydrodynamic equation (1.1), we consider a two-scale approach: We show here a hydrodynamic limit with respect to more regular random environments, those which average  $\{u_i^N\}$  in small macroscopic blocks of width  $\varepsilon$ , in which the noise  $W'$  is replaced by an  $\varepsilon$ -smoothing  $W'_\varepsilon$ . Then, later we connect to work on the  $\varepsilon \downarrow 0$  limit of the associated hydrodynamic densities.

Let  $\psi : [-1, 1] \mapsto \mathbb{R}$  be a continuously differentiable function such that  $\int_{-1}^1 \psi(x) dx = 1$ . Define  $\{\bar{u}_i^N = 1/2 + r_i^N/\sqrt{N}\}$  where  $r_i^N = (\sigma N \varepsilon)^{-1} \sum_{|j-i| \leq \lfloor N\varepsilon \rfloor} r_j \psi((j-i)/(N\varepsilon))$  and  $\varepsilon > 0$  is fixed. In this formula,  $[a]$  stands for the integer part of  $a \in \mathbb{R}$ . Summing by parts,

$$r_i^N = \frac{1}{\sigma N \varepsilon} \sum_{j=-\lfloor N\varepsilon \rfloor}^{\lfloor N\varepsilon \rfloor - 1} S_{i+j} \left[ \psi\left(\frac{j}{N\varepsilon}\right) - \psi\left(\frac{j+1}{N\varepsilon}\right) \right] \\ + \frac{1}{\sigma N \varepsilon} S_{i+\lfloor N\varepsilon \rfloor} \psi\left(\frac{\lfloor N\varepsilon \rfloor}{N\varepsilon}\right) - \frac{1}{\sigma N \varepsilon} S_{i-1-\lfloor N\varepsilon \rfloor} \psi\left(-\frac{\lfloor N\varepsilon \rfloor}{N\varepsilon}\right).$$

In part (I) of Theorem 3.5, under a fixed random environment with the same distribution as  $\{\bar{u}_i^N\}$ , that is in a ‘quenched’ environment afforded by Skorokhod representation, we obtain the hydrodynamic equation

$$\partial_t \rho^{(\varepsilon)}(t, x) = \frac{1}{2} \Delta \Phi(\rho^{(\varepsilon)}(t, x)) - 2 \nabla (W'_\varepsilon \Phi(\rho^{(\varepsilon)}(t, x))) \quad (1.2)$$

where

$$W'_\varepsilon(v) := -\frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} W(v+u) \psi'\left(\frac{u}{\varepsilon}\right) du + \frac{1}{\varepsilon} \left\{ W(v+\varepsilon) \psi(1) - W(v-\varepsilon) \psi(-1) \right\}$$

is a form of  $\varepsilon$ -smoothing of  $W'$ . Observe that  $W'_\varepsilon \in \cap_{0 < \varepsilon < 1/2} C^{1/2-\varepsilon}$  when either  $\psi(1)$  or  $\psi(-1)$  does not vanish. If  $\psi(\pm 1) = 0$  and  $\psi' \in C^\gamma$  for some  $\gamma \geq 1/2$ , then  $W'_\varepsilon \in C^\gamma$ . In contrast, if  $\psi' \in C^\gamma$  for some  $\gamma < 1/2$ , then  $W'_\varepsilon \in \cap_{0 < \varepsilon < 1/2} C^{1/2-\varepsilon}$ . A natural case is  $\psi(x) = (1/2) \mathbb{1}_{[-1, 1]}(x)$  where  $W'_\varepsilon(x) = (2\varepsilon)^{-1} \{W(x+\varepsilon) - W(x-\varepsilon)\} \in \cap_{0 < \varepsilon < 1/2} C^{1/2-\varepsilon}$ .

In part (II) of Theorem 3.5, with respect to the random environments  $\{\bar{u}_i^N\}$ , we obtain that the annealed hydrodynamic limit of the space-time mass distributions is given by the distribution of  $\rho^{(\varepsilon)}$  satisfying (1.2).

At this point, we see that the quenched versions of  $\{\bar{u}_i^N\}$  fit into the general form  $\{\hat{u}_i^N = 1/2 + \alpha_i^N/N\}$  where  $\{\alpha_i^N : i \in \mathbb{T}_N\}_{N \geq 1}$  is a deterministic array such that piecewise continuous interpolations of  $\{\alpha_i^N : i \in \mathbb{T}_N\}$  converge uniformly, as  $N \uparrow \infty$ , to a (necessarily) continuous function  $\alpha(\cdot)$  on the unit torus  $\mathbb{T}$ : Indeed, we may take  $\alpha_i^N = \sqrt{N} r_i^N$  and note

that  $\alpha_i^N \rightarrow \alpha(x) = \tilde{W}'_\varepsilon(x)$  when  $i/N \rightarrow x$ . In Theorem 3.3, of its own interest, we state that a hydrodynamic limit holds for this spatially inhomogeneous weakly asymmetric system with hydrodynamic equation

$$\partial_t \rho = \frac{1}{2} \Delta \Phi(\rho) - 2 \nabla(\alpha \Phi(\rho)). \quad (1.3)$$

Via Theorem 3.3 and its proof, we deduce parts (I) and (II) of Theorem 3.5.

We comment, as the regularization vanishes in the SPDE (1.2), that one might expect some form of (1.1) should emerge. Indeed, as noted earlier, in the companion work Funaki et. al. [13], the limit of  $\rho^{(\varepsilon)}$  as  $\varepsilon \downarrow 0$  is considered and shown to solve (1.1) in the ‘paracontrolled’ sense.

In this respect, we discuss a computation, Theorem 3.6, later in the context of subsection 1.5, on the form of the annealed mean hydrodynamic limit in unregularized random environments for independent particles.

**1.3. Idea of the proofs.** The proof of Theorem 3.3 broadly employs the ‘entropy’ method of Guo-Papanicolaou-Varadhan (GPV) (cf. Kipnis and Landim [23]), by applying an Ito’s formula to the empirical measure of particle mass with respect to the zero-range evolution. However, the weak asymmetry is not homogeneous and in particular is not translation-invariant, a key feature of the ‘GPV’ technique to homogenize resulting nonlinear terms of the process in a replacement scheme. The main technical work is to introduce ‘local’ averages to piece together the ‘global’ average in the hydrodynamic limit.

To derive the homogenization, we make use of some averaging in time, afforded by spectral gap or mixing estimates of localized processes, to perform ‘local’ 1 and 2-block replacements, leading to more ‘global’ replacements. These ‘local’ replacements are not so broadly known although related notions were used in Jara, Landim, and Sethuraman [18], [19], and Fatkullin, Sethuraman, and Xue [12] to analyze tagged particle motion and Young diagram evolutions. However, differently, in our context, estimates on the random environment play a significant role in making the ‘local’ replacements work in the context of the spatial inhomogeneity. We remark that in this work we do not assume the process rate  $g$  is an increasing function, as it is in [19], [12], that is that the process is ‘attractive’, a technical condition which would allow the use of ‘basic’ particle couplings.

The limiting equation (1.2), a case of equation (1.3), is not a standard one as the factor  $W'_\varepsilon$  may not be smooth. To finish the proof of the hydrodynamic limit in Theorem 3.3, we need to show uniqueness of weak solutions of (1.3). We can derive a certain continuum energy estimate by considering the microscopic particle system, namely that  $\partial_x \Phi(\rho(t, x))$  can be defined in a weak sense. Uniqueness of weak solutions to (1.3) in an appropriate class, with respect to bounded, measurable  $\alpha(\cdot)$ , is then shown in Theorem 10.1 by a self-contained argument, perhaps of separate interest.

**1.4. Hydrodynamics in random environments.** To put our results in context, we remark that, with respect to systems of continuous-time random walks in a common random environment of different types, a ‘quenched’ deterministic hydrodynamic limit for the bulk space time mass density of the walks has been shown in some models when there is enough ‘averaging’ with respect to the random environment. For exclusion process with random conductances, see Faggionato [8], [9], Jara and Landim [17], and Nagy [27], and with site disorder, see Quastel [30]. For independent random walks in a ballistic random environment, see Peterson [29]. For zero-range process, see Faggionato [10] and Gonçalves and Jara [15].

When the random environment does not allow sufficient ‘averaging’, a ‘quenched’ hydrodynamic may involve random terms. In Jara, Landim, and Teixeira [20], the hydrodynamic limit is shown for a system of symmetric independent random walks in a common scaled ‘trap’ environment on a torus, that one of the particles at a site  $i$  jumps at rate  $\xi_i^N$  to a nearest neighbor, where  $\xi_i^N$  is random and heavy-tailed; here the limit equation involves a heavy-tailed subordinator arising from the random environment. In Faggionato, Jara, and Landim [11], the hydrodynamic limit is shown for symmetric simple exclusion processes on a torus, with heavy-tailed random conductances on the bonds, which also involves a heavy-tailed subordinator coming from the random environment. In Jara and Peterson [21], the hydrodynamic limit is shown for independent random walks in a random environment on  $\mathbb{Z}$ , where a single particle is transient but not ballistic, which incorporates a random term arising from the environment.

**1.5. Discussion and open problems.** Finally, given the development with respect to the regularized environments, we comment on two natural directions for future work with respect to the unregularized Seignourel environments. First, instead of using the ‘two-scale’ approach, one might expect directly, say in a ‘diagonal’ argument as both  $N \uparrow \infty$  and  $\varepsilon \downarrow 0$ , that the equation (1.1), interpreted now in the ‘paracontrolled’ sense, would govern the annealed hydrodynamic limit of the zero-range process with respect to the unregularized Seignourel environment  $\{u_i^N = 1/2 + r_i/\sqrt{\sigma^2 N}\}$  as  $N \uparrow \infty$ .

Indeed, one can make some computations, which give a form, at least with respect to the annealed mean hydrodynamics, when the zero-range process consists of independent motions on say  $\mathbb{Z}$ , that is when  $g(k) \equiv k$  and so  $\Phi(\rho) \equiv \rho$ , and the initial distribution of particles on  $\mathbb{Z}$  is a product measure with Poisson marginals with parameters  $\{\beta(i/N)\}$  where  $\beta : \mathbb{R} \rightarrow [0, 1]$  is a bounded, continuous function. In part (I) of Theorem 3.6, we state that the law of the configuration  $\{\eta(i) : i \in \mathbb{Z}\}$  at time  $N^2 t$  is a product of Poisson marginals with quenched mean parameters  $\{\sum_{x \in \mathbb{Z}} \beta(x/N) P_N^\omega(R_{N^2 t}^{N,x} = i) : i \in \mathbb{Z}\}$  where  $R_{N^2 t}^{N,i}$  is the position at time  $N^2 t$  of a single particle, starting at  $i$ , moving in a fixed Seignourel environment  $\{u_i^N(\omega) : i \in \mathbb{Z}\}$ . In part (II), we show that the annealed mean with respect to  $\{u_i^N\}$  of the hydrodynamic limit mass density equals  $\mathcal{E}[\beta(u - X_t)]$  where  $X_t$  is the standard 1–Brox diffusion starting from the origin and say  $\mathcal{E}$  is the associated annealed expectation.

Moreover, by the proof of Theorem 3.6, one can also compute the annealed hydrodynamic limit mean density with respect to the regularized environments  $\{\bar{u}_i^N\}$  on  $\mathbb{T}_N$  with  $\varepsilon > 0$ . Note that the quenched diffusively scaled path of a single particle  $R_{N^2 t}^{N,\varepsilon}/N$ , starting at 0, in the regularized environment converges weakly to a diffusion  $X_t^\varepsilon$  with generator  $(1/2)\Delta + 2W_\varepsilon' \nabla$  (cf. Ch. 7 of Ethier and Kurtz [5]). Then, the annealed hydrodynamic limit mean density equals  $\mathcal{E}[\beta(u - X_t^\varepsilon)]$  which by our Theorem 3.5, equals the annealed mean of  $\rho^{(\varepsilon)}(t, u)$ . Since we expect, in the annealed sense, that  $X^\varepsilon \Rightarrow X$  as  $\varepsilon \downarrow 0$ , one can link formally the annealed mean limit  $\mathcal{E}[\beta(u - X_t)]$  to the  $\varepsilon \downarrow 0$  limit of  $\rho^{(\varepsilon)}$ .

For the second direction, one might follow a tagged particle in the interacting system to understand relations with Brox diffusion. Let  $x^N(t)$  be the position at time  $t$  of a tagged particle initially at the origin. Then, from Ito’s formula,

$$x^N(t) = \frac{2}{\sqrt{N}} \int_0^t \bar{u}_{x^N(s)}^N \frac{g(\eta_s(x^N(s)))}{\eta_s(x^N(s))} ds + M^N(t) .$$

One can compute that the quadratic variation of  $M^N(t)$  equals  $\int_0^t \frac{g(\eta_s(x^N(s)))}{\eta_s(x^N(s))} ds$ . Formally, given the hydrodynamic limit (1.2) and following the scheme in [18], one might expect the

quenched limit points  $v_t$  of  $v_t^N = x_{N^2 t}^N / N$  to satisfy the equation

$$v_t = 2 \int_0^t W'_\varepsilon(v_s) \frac{\Phi(\rho^{(\varepsilon)}(s, v_s))}{\rho^{(\varepsilon)}(s, v_s)} ds + M(t),$$

where  $M$  is a time-changed Brownian motion  $B(T^{-1}(t))$  and  $T(t) = \int_0^t \frac{\Phi(\rho^{(\varepsilon)}(s, v_s))}{\rho^{(\varepsilon)}(s, v_s)} ds$ . Again, considering independent motions, when  $g(k) \equiv k$  and so  $\Phi(u) \equiv u$  and  $M = B$ , one would recover the diffusion discussed earlier with the drift  $2W'_\varepsilon$ . Of course, it would be of interest, in this direction also, to understand the limits as  $\varepsilon \downarrow 0$ .

**Organization of the paper.** We define the microscopic model precisely in section 2, before stating results, Theorems 3.3, 3.5 and Theorem 3.6, in section 3. After preliminaries on basic martingales in section 4, we give the main proof outline in section 5. In sections 6 and 7, hydrodynamic 1 and 2-block replacement estimates are proved. Tightness and useful properties of limit measures are given in sections 8 and 9. Finally, uniqueness of weak solutions to the limit equations in a certain class, Theorem 10.1, is shown in section 10.

## 2. MODEL DESCRIPTION

We first introduce the random environments considered, before specifying the zero-range process in this random environment. Let  $\{r_x\}_{x \in \mathbb{N}}$  be a sequence of i.i.d. random variables with mean 0 and variance  $0 < \sigma^2 < \infty$ .

Let  $s_0 = 0$  and for  $n \geq 1$ ,  $s_n = \sum_{k=1}^n r_k$ . For  $0 \leq u \leq 1$ , let

$$X_u^N = \frac{1}{\sigma\sqrt{N}} s_{\lfloor Nu \rfloor} + \frac{Nu - \lfloor Nu \rfloor}{\sigma\sqrt{N}} r_{\lfloor Nu \rfloor + 1}, \quad (2.1)$$

where, recall,  $\lfloor a \rfloor$ ,  $a \in \mathbb{R}$ , stands for the integer part of  $a$ . It is standard that the random functions  $\{X_u^N : 0 \leq u \leq 1\}$  converge in distribution as  $N \uparrow \infty$  to the Brownian motion on  $[0, 1]$ . By Skorokhod's Representation Theorem, we may find a probability space  $(\Omega, \mathcal{P})$  and  $\{W_u^N : 0 \leq u \leq 1\}$ ,  $N \in \mathbb{N}$ , mappings from  $\Omega$  to  $C[0, 1]$ , such that, for all  $N \in \mathbb{N}$ ,

$$\{X_u^N : 0 \leq u \leq 1\} = \{W_u^N : 0 \leq u \leq 1\}$$

in distribution and moreover,  $\{W_u^N : 0 \leq u \leq 1\}$  converges uniformly almost surely to the standard Brownian motion  $\{W_u, 0 \leq u \leq 1\}$ .

*Quenched formulation.* We will now fix an  $\omega \in \Omega$  such that  $\{W_u^N(\omega) : 0 \leq u \leq 1\}$  converges (uniformly) to a Brownian path  $\{W_u(\omega) : 0 \leq u \leq 1\}$ .

Recall the discrete torus  $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$  for  $N \in \mathbb{N}$ . Throughout this article, we will identify  $\mathbb{T}_N$  with  $\{1, 2, \dots, N\}$  and also identify the unit torus  $\mathbb{T}$  with  $(0, 1]$ .

It will be convenient to extend  $W_t^N$  as well as  $W_t$  to  $t \in [-1, 2]$ : with  $\tilde{W}_u$  representing either  $W_u^N$  or  $W_u$ , define

$$\tilde{W}_u = \begin{cases} \tilde{W}_{u+1} - \tilde{W}_1 & u \in [-1, 0), \\ \tilde{W}_{u-1} + \tilde{W}_1 & u \in (1, 2]. \end{cases}$$

Here, in the time intervals  $[-1, 0)$  and  $(1, 2]$ , the trajectory  $\tilde{W}$  starts respectively from  $-\tilde{W}_1$  and  $\tilde{W}_1$ , then displacing according to the trajectory  $\tilde{W}$  in  $[0, 1]$ . Importantly, the increments of  $\tilde{W}$  are periodic in  $\mathbb{T}$ . Throughout, to simplify notation, we will drop the tilde in the notation for  $\tilde{W}$ .

Let  $\varepsilon > 0$  be a parameter, fixed throughout the paper. Let also  $\psi : [-1, 1] \mapsto \mathbb{R}$  be a continuously differentiable function with  $\int_{-1}^1 \psi(x) dx = 1$ . For each  $N \in \mathbb{N}$  and  $k \in \mathbb{T}_N$ , consider an  $\varepsilon$ -regularization of local environments such that

$$q_k^N \stackrel{d}{=} \frac{1}{\sigma N} \sum_{|j-k| \leq \lfloor N\varepsilon \rfloor} r_j \left[ \frac{1}{\varepsilon} \psi \left( \frac{j-k}{N\varepsilon} \right) \right]$$

where

$$\begin{aligned} q_k^N &:= \frac{1}{\sqrt{N}\varepsilon} \sum_{j=k-\lfloor N\varepsilon \rfloor}^{k+\lfloor N\varepsilon \rfloor-1} W_j^N \left[ \psi \left( \frac{j-k}{N\varepsilon} \right) - \psi \left( \frac{j-k+1}{N\varepsilon} \right) \right] \\ &\quad + \frac{1}{\sqrt{N}\varepsilon} W_{k+N\varepsilon}^N \psi \left( \frac{\lfloor N\varepsilon \rfloor}{N\varepsilon} \right) - \frac{1}{\sqrt{N}\varepsilon} W_{k-1-N\varepsilon}^N \psi \left( -\frac{\lfloor N\varepsilon \rfloor}{N\varepsilon} \right). \end{aligned} \quad (2.2)$$

In particular, when  $k/N \rightarrow x \in \mathbb{T}$  as  $N \uparrow \infty$ , we have  $\sqrt{N}q_k^N$  converges to

$$-\frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} W(u) \psi' \left( \frac{u-x}{\varepsilon} \right) du + \frac{1}{\varepsilon} \{W(x+\varepsilon)\psi(1) - W(x-\varepsilon)\psi(-1)\} := W'_\varepsilon(x). \quad (2.3)$$

As  $W^N$  converges uniformly to  $W$ , by the continuity of  $W$ , and the properties of  $\psi$ , in view of (2.2), there exists a constant  $C = C(\omega) < \infty$  such that

$$\limsup_{N \rightarrow \infty} \max_{1 \leq k \leq N} \left\{ \sqrt{N} |q_k^N| \right\} \leq C. \quad (2.4)$$

As remarked in the introduction,  $W'_\varepsilon$  is a smoothing of  $W'$ . In particular,  $W'_\varepsilon \in C^\gamma$  when  $\psi' \in C^\gamma$  for  $\gamma \geq 1/2$  and  $\psi(\pm 1) = 0$ . When one of  $\psi(1)$  or  $\psi(-1)$  does not vanish, then  $W'_\varepsilon \in \cap_{0 < \epsilon < 1/2} C^{1/2-\epsilon}$ . Also, when  $\psi' \in C^\gamma$  for  $\gamma < 1/2$ , then  $W'_\varepsilon \in \cap_{0 < \epsilon < 1/2} C^{1/2-\epsilon}$ . A natural case is when  $\psi(x) = (1/2)\mathbb{1}_{[-1,1]}(x)$  for which  $W'_\varepsilon(x) = (2\varepsilon)^{-1}[W(x+\varepsilon) - W(x-\varepsilon)]$ .

*Zero-range process in an abstract inhomogeneous environment.* We now introduce the zero-range process in an abstract deterministic environment  $\{\alpha_k^N : k \in \mathbb{T}_N\}$  where the linear interpolations for  $u \in \mathbb{T}$ ,

$$Y_u^N = \alpha_{\lfloor Nu \rfloor}^N + (Nu - \lfloor Nu \rfloor) \alpha_{\lfloor Nu \rfloor + 1}^N,$$

converge uniformly to  $\alpha(u)$  where  $\alpha(\cdot)$  is a continuous function on  $\mathbb{T}$ . In the above quenched setting, an example is  $\alpha_k^N \equiv \sqrt{N}q_k^N$  and  $\alpha(u) \equiv W'_\varepsilon(u)$ .

Set  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ , and let  $\Sigma_N = \mathbb{N}_0^{\mathbb{T}_N}$  be the configuration space. Elements of  $\Sigma_N$  are represented by the Greek letter  $\xi$ . Thus,  $\xi(k)$ ,  $k \in \mathbb{T}_N$ , stands for the number of particles at site  $k$  for the configuration  $\xi$ .

Fix a function  $g : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ . Denote by  $(\xi_t : t \geq 0)$  the continuous-time Markov chain whose evolution can be informally described as follows. Since  $\max_{1 \leq k \leq N} |\alpha_k^N|$  is uniformly bounded in  $N$ , take  $N$  sufficiently large so that  $|\alpha_k^N|/N < 1/2$  for all  $k \in \mathbb{T}_N$ . At rate  $g(\xi(k))[(1/2) \pm (\alpha_k^N/N)]$  a particle jumps from  $k$  to  $k \pm 1$ .

More precisely, the process  $\{\xi_t : t \geq 0\}$  is the Markov process, well defined on the countable space  $\Sigma_N$ , with generator  $L$  given by

$$\begin{aligned} Lf(\xi) &= \sum_{k=1}^N \left\{ g(\xi(k)) \left( \frac{1}{2} + \frac{\alpha_k^N}{N} \right) (f(\xi^{k,k+1}) - f(\xi)) \right. \\ &\quad \left. + g(\xi(k)) \left( \frac{1}{2} - \frac{\alpha_k^N}{N} \right) (f(\xi^{k,k-1}) - f(\xi)) \right\}. \end{aligned} \quad (2.5)$$

Here,  $\xi^{j,k}$  is the configuration obtained from  $\xi$  by moving a particle from  $j$  to  $k$ , that is,

$$\xi^{j,k}(\ell) = \begin{cases} \xi(j) - 1 & \ell = j, \\ \xi(k) + 1 & \ell = k, \\ \xi(\ell) & \ell \neq j, k. \end{cases}$$

To avoid degeneracies, we suppose that  $g(0) = 0$  and  $g(k) > 0$  for  $k \geq 1$ . We assume, further, that  $g$  satisfies

- (1)  $\sup_{k \in \mathbb{N}} |g(k+1) - g(k)| \leq g^* < \infty$ ;
- (2) There exists  $k_0 \in \mathbb{N}$  and  $c_1 > 0$  such that  $g(k) - g(j) \geq c_1$  for all  $k \geq j + k_0$ .

These properties guarantee that the zero-range process has good mixing properties, useful in the proof of the ‘Replacement Lemma’, stated in Lemma 5.1. This specification also implies that  $g_*k \leq g(k) \leq g^*k$  for some  $g_* > 0$ , more than enough to satisfy the technical condition ‘FEM’ (cf. [23][p.69]) used for particle truncation in the 1 and 2-blocks estimates presented in Lemmas 6.3 and 7.2.

We note that the case  $g(k) \equiv k$  is when all the particles in the system are independent, and construction of the process even on  $\mathbb{Z}$  is immediate.

**2.1. Invariant measure.** The building block for the invariant measures are  $\{\mathcal{P}_\phi\}$ , a family of Poisson-like distributions indexed by ‘fugacities’  $\phi \geq 0$ . For each  $\phi$ ,  $\mathcal{P}_\phi$  is defined by

$$\mathcal{P}_\phi(n) = \frac{1}{Z(\phi)} \frac{\phi^n}{g(n)!}, \quad \text{for } n \geq 0.$$

Here,  $g(0)! := 1$  and  $g(n)! := \prod_{j=1}^n g(j)$  for  $n \geq 1$ ;  $Z(\cdot)$  is the partition function:

$$Z(\phi) := \sum_{n=0}^{\infty} \frac{\phi^n}{g(n)!}.$$

Since  $g(k) \geq g_*k$ ,  $Z(\phi) < \infty$  for all  $0 \leq \phi < \infty$ .

Let  $R(\phi) = E_{\mathcal{P}_\phi}[X]$ , where  $X(n) = n$ , be the mean of the distribution  $\mathcal{P}_\phi$ . A direct computation yields that  $R'(\phi) > 0$ ,  $R(0) = 0$ ,  $\lim_{\phi \rightarrow \infty} R(\phi) = \infty$ . Since  $R$  is strictly increasing, it has an inverse, denoted by  $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , and we may parametrize the family of distributions  $\mathcal{P}_\phi$  by its mean. For  $\rho \geq 0$ , let  $\mathcal{Q}_\rho = \mathcal{P}_{\Phi(\rho)}$ , so that  $E_{\mathcal{Q}_\rho}[X] = E_{\mathcal{P}_{\Phi(\rho)}}[X] = R(\Phi(\rho)) = \rho$ .

A straightforward computation yields that  $E_{\mathcal{P}_\phi}[g(X)] = \phi$  for  $\phi \geq 0$ . Thus,

$$\Phi(\rho) = E_{\mathcal{P}_{\Phi(\rho)}}[g(X)] = E_{\mathcal{Q}_\rho}[g(X)] \quad , \quad \rho \geq 0. \quad (2.6)$$

As  $g_*k \leq g(k) \leq g^*k$ , we have that  $g_*\rho \leq \Phi(\rho) \leq g^*\rho$ . Moreover, a simple computation yields that  $\Phi'(\rho) = \Phi(\rho)/\sigma^2(\rho)$  where  $\sigma^2(\rho)$  is the variance of  $X$  under  $\mathcal{Q}_\rho$ . Under our assumptions on  $g$ , there exist constants  $0 < C_1 < C_2 < \infty$  such that  $0 < C_1 \leq \Phi'(\rho) \leq C_2 < \infty$  for all  $\rho \geq 0$  (cf. [24][equation (5.2)]). In particular,  $\Phi \in C^1[0, \infty)$  is a strictly increasing function of  $\rho$ .

We note, in the case  $g(k) \equiv k$ , that  $\Phi(\rho) \equiv \rho$  and  $\mathcal{P}_\phi$  is a Poisson measure with mean  $\phi$ .

Fix a vector  $(\phi_{k,N} : k \in \mathbb{T}_N)$  of non-negative real numbers. Denote by  $\mathcal{R}_N$  the product measure on  $\mathbb{N}_0^{\mathbb{T}_N}$  whose marginals are given by

$$\mathcal{R}_N(\xi(k) = n) = \mathcal{P}_{\phi_{k,N}}(n), \quad \text{for } k \in \mathbb{T}_N, n \geq 0. \quad (2.7)$$

It is straightforward (cf. [1], [7]) to check that  $\mathcal{R}_N$  is invariant with respect to the generator  $L$  in (2.5) as long as the fugacities  $\{\phi_{k,N}\}_{k \in \mathbb{T}_N}$  satisfy:

$$\left(\frac{1}{2} + \frac{\alpha_{k-1}^N}{N}\right)\phi_{k-1,N} + \left(\frac{1}{2} - \frac{\alpha_{k+1}^N}{N}\right)\phi_{k+1,N} = \phi_{k,N}, \quad k = 1, 2, \dots, N. \quad (2.8)$$

Notice that  $\{c\phi_{k,N}\}_{k \in \mathbb{T}_N}$ ,  $c \in \mathbb{R}$ , is a solution of (2.8) if  $\{\phi_{k,N}\}_{k \in \mathbb{T}_N}$  is a solution. In particular, any solution gives rise to a one-parameter family of solutions. Although standard, to introduce useful notation, we show that (2.8) admits a solution.

**Lemma 2.1.** *The equation (2.8) admits a solution, unique up to a multiplicative constant. Moreover, the solution is either strictly positive or strictly negative or identically equal to 0.*

*Proof.* Let  $\mathfrak{r}_k = (1/2) + (\alpha_k^N/N)$ ,  $\mathfrak{l}_k = (1/2) - (\alpha_k^N/N)$ ,  $k \in \mathbb{T}_N$ . With this notation, equation (2.8) becomes

$$\mathfrak{r}_{k-1}\phi_{k-1,N} + \mathfrak{l}_{k+1}\phi_{k+1,N} = \phi_{k,N}.$$

Since  $\mathfrak{r}_k + \mathfrak{l}_k = 1$ , we have that

$$\mathfrak{r}_{k-1}\phi_{k-1,N} - \mathfrak{l}_k\phi_{k,N} = \mathfrak{r}_k\phi_{k,N} - \mathfrak{l}_{k+1}\phi_{k+1,N}, \quad k \in \mathbb{T}_N.$$

Denote by  $\gamma$  be the common value of  $\mathfrak{r}_k\phi_{k,N} - \mathfrak{l}_{k+1}\phi_{k+1,N}$ , to get the recursive equation

$$\phi_{k+1,N} = \frac{\mathfrak{r}_k\phi_{k,N} - \gamma}{\mathfrak{l}_{k+1}}. \quad (2.9)$$

With the convention that  $\mathfrak{r}_{N+1} = \mathfrak{r}_1$  and  $\mathfrak{l}_{N+1} = \mathfrak{l}_1$ , let

$$\mathfrak{R}_{j,k} = \prod_{i=j}^k \mathfrak{r}_i, \quad \mathfrak{L}_{j,k} = \prod_{i=j}^k \mathfrak{l}_i, \quad 1 \leq j \leq k \leq N+1.$$

We extend the definition to indices  $j > k$  by setting  $\mathfrak{R}_{j,k} = \mathfrak{L}_{j,k} = 1$  if  $j > k$ . Solving the recursive equation yields that

$$\phi_{k,N} = \frac{\mathfrak{R}_{1,k-1}}{\mathfrak{L}_{2,k}} \phi_{1,N} - \frac{\mathfrak{S}_k}{\mathfrak{L}_{2,k}} \gamma, \quad 2 \leq k \leq N+1.$$

In this formula,

$$\mathfrak{S}_k = \sum_{j=2}^k \mathfrak{L}_{2,j-1} \mathfrak{R}_{j,k-1},$$

with the convention, adopted above, that  $\mathfrak{L}_{2,1} = \mathfrak{R}_{k,k-1} = 1$ .

Since  $\phi_{N+1,N} = \phi_{1,N}$ , we have that

$$\gamma = \frac{\mathfrak{R}_{1,N} - \mathfrak{L}_{2,N+1}}{\mathfrak{S}_{N+1}} \phi_{1,N}. \quad (2.10)$$

Reporting this value in the equation for  $\phi_{k,N}$  yields that

$$\phi_{k,N} = \left\{ \frac{\mathfrak{R}_{1,k-1}}{\mathfrak{L}_{2,k}} - \frac{\mathfrak{S}_k}{\mathfrak{S}_{N+1}} \frac{\mathfrak{R}_{1,N} - \mathfrak{L}_{2,N+1}}{\mathfrak{L}_{2,k}} \right\} \phi_{1,N}, \quad 2 \leq k \leq N. \quad (2.11)$$

Therefore, for each  $\phi_{1,N} \in \mathbb{R}$ , the solution of the difference equation (2.8) is given by (2.11). This proves existence and uniqueness up to a multiplicative constant. Moreover, it is not difficult to check that  $\mathfrak{S}_k \mathfrak{R}_{1,N} \leq \mathfrak{S}_{N+1} \mathfrak{R}_{1,k-1}$ . Therefore, as each variable  $\mathfrak{r}_i$ ,  $\mathfrak{l}_j$  is strictly positive for sufficiently large  $N$ , the solution is strictly positive if  $\phi_{1,N} > 0$ .  $\square$

Let  $\phi_{\max,N} = \max_{1 \leq k \leq N} \{\phi_{k,N}\}$  and  $\phi_{\min,N} = \min_{1 \leq k \leq N} \{\phi_{k,N}\}$ .

**Lemma 2.2.** *Let  $\phi_{k,N}$  be a solution of (2.8). Then, there exist constants  $C_1, C_2 < \infty$  such that for all  $N \in \mathbb{N}$*

$$1 \leq \frac{\phi_{\max,N}}{\phi_{\min,N}} \leq C_1 \quad \text{and} \quad \max_{1 \leq k \leq N} |\phi_{k,N} - \phi_{k+1,N}| \leq \frac{C_2}{N} \phi_{\max,N}.$$

*Proof.* By the definition of  $\alpha_k^N$ , there is a finite constant  $C$  such that  $\max_{1 \leq k \leq N} \alpha_k^N \leq C$ . Therefore, for each  $j, k \in \mathbb{T}_N$ , using the notation of Lemma 2.1  $|(\tau_j/\iota_k) - 1|$  and  $|(\iota_k/\tau_j) - 1|$  are bounded from above by

$$2 \frac{\max_\ell |\alpha_\ell^N|}{N} \frac{1}{\frac{1}{2} - \frac{\max_\ell |\alpha_\ell^N|}{N}} \leq \frac{8C}{N}, \quad \text{for } N \geq 4C. \quad (2.12)$$

Without loss of generality, we may assume that  $\phi_{1,N} = \phi_{\min,N}$ ,  $\phi_{m,N} = \phi_{\max,N}$ . Hence, by (2.11) and since all terms are positive and  $\mathfrak{L}_{2,N+1} = \mathfrak{L}_{2,m} \mathfrak{L}_{m+1,N+1}$ ,  $\phi_{\max,N}/\phi_{\min,N}$  is bounded by

$$\frac{\mathfrak{R}_{1,m-1}}{\mathfrak{L}_{2,m}} + \frac{\mathfrak{S}_m \mathfrak{L}_{m+1,N+1}}{\mathfrak{S}_{N+1}}.$$

By (2.12), the first term is bounded by  $[1 + (8C/N)]^N$ . We further show that the second term is also bounded by  $[1 + (8C/N)]^N$  by rewriting  $\mathfrak{L}_{m+1,N+1}$  as  $\mathfrak{R}_{m,N} [\mathfrak{L}_{m+1,N+1}/\mathfrak{R}_{m,N}]$  and using  $\mathfrak{S}_m \mathfrak{R}_{m,N} \leq \mathfrak{S}_{N+1}$ . This proves the first assertion of the lemma.

We turn to the second assertion. By (2.9),

$$\phi_{m+1,N} - \phi_{m,N} = \left( \frac{\tau_m}{\iota_{m+1}} - 1 \right) \phi_{m,N} - \frac{\gamma}{\iota_{m+1}}.$$

By (2.12), the absolute value of the first term on the right-hand side is bounded by  $(8C/N) \max_k \phi_{k,N}$ . By (2.10), and since  $\mathfrak{L}_{2,N+1} = \mathfrak{L}_{1,N}$ , the second one is equal to

$$\frac{\phi_{1,N}}{\iota_{m+1}} \frac{1}{\mathfrak{S}_{N+1}} \sum_{j=0}^{N-1} \{ \mathfrak{L}_{1,j} \mathfrak{R}_{j+1,N} - \mathfrak{L}_{1,j+1} \mathfrak{R}_{j+2,N} \},$$

where we used the convention that  $\mathfrak{L}_{i,j} = 1$  and  $\mathfrak{R}_{i,j} = 1$  if  $i > j$ . Changing variables this expression becomes

$$\frac{\iota_1}{\iota_{m+1}} \frac{\phi_{1,N}}{\mathfrak{S}_{N+1}} \sum_{j=2}^{N+1} \mathfrak{L}_{2,j-1} \mathfrak{R}_{j,N} \left( \frac{\tau_{j-1}}{\iota_{j-1}} - 1 \right),$$

There is  $C_0 > 0$  such that  $\iota_1/\iota_{m+1} \leq C_0$ . Also by (2.12), the absolute value of the expression inside the parenthesis is bounded by  $8C/N$ , uniformly over  $j$ . The remaining sum is bounded by  $\mathfrak{S}_{N+1}$ . This expression is, therefore, bounded by  $C_0(8C/N) \phi_{1,N}$ . To complete the proof of the second assertion of the lemma, it remains to recollect all previous estimates.  $\square$

### 3. RESULTS

We first specify the initial measures for the zero-range processes. These include the usual ‘local equilibrium’ measures as well as others. We then state the main results of this work.

In the following, with respect to a given probability measure  $\mu$ , we denote by  $E_\mu$  and  $\text{Var}_\mu$  its expectation and variance.

**3.1. Initial measures.** We consider an initial macroscopic density profile  $\rho_0(\cdot) \in L^1(\mathbb{T})$ , and an initial microscopic measure satisfying the following condition. Denote by  $\mathcal{R}_N$  (cf. (2.7)) the invariant measure chosen so that  $\max_{k \in \mathbb{T}_N} \{\phi_{k,N}\} = 1$ .

**Condition 3.1.** Let  $\{\mu^N\}_{N \in \mathbb{N}}$  be a sequence of probability measures on  $\mathbb{N}_0^{\mathbb{T}_N}$  such that

(a)  $\{\mu^N\}_{N \in \mathbb{N}}$  is associated with profile  $\rho_0$  in the sense that for any  $G \in C(\mathbb{T})$  and  $\delta > 0$

$$\lim_{N \rightarrow \infty} \mu^N \left[ \left| \frac{1}{N} \sum_{k=1}^N G\left(\frac{k}{N}\right) \xi(k) - \int_{\mathbb{T}} G(x) \rho_0(x) dx \right| > \delta \right] = 0.$$

(b) The relative entropy of  $\mu^N$  with respect to  $\mathcal{R}_N$  is of order  $N$ . That is, there exists a finite constant  $C_0$  such that  $H(\mu^N | \mathcal{R}_N) := \int f_0 \ln f_0 d\mathcal{R}_N \leq C_0 N$  for all  $N \geq 1$ , where  $f_0 = d\mu^N / d\mathcal{R}_N$ .

A useful consequence of the relative entropy bound in part (b) of Condition 3.1 and the bounds on the fugacities of the invariant measure  $\mathcal{R}_N$  in Lemma 2.2 is that the expected number of particles under  $\mu^N$  is of order  $N$ . Indeed, by the entropy inequality (cf. [23]),

$$E_{\mu^N} \left[ \sum_{k \in \mathbb{T}_N} \xi(k) \right] \leq \frac{1}{\gamma} H(\mu^N | \mathcal{R}_N) + \frac{1}{\gamma} \log E_{\mathcal{R}_N} \left[ e^{\gamma \sum_{x \in \mathbb{T}_N} \xi(k)} \right] \quad (3.1)$$

for all  $\gamma > 0$ . By Condition 3.1.b and by definition of  $\mathcal{R}_N$ , this expression is bounded by

$$\frac{C_0}{\gamma} N + \frac{1}{\gamma} \sum_{k=1}^N \log \frac{Z(e^\gamma \phi_{k,N})}{Z(\phi_{k,N})}.$$

Since  $\max_k \phi_{k,N} = 1$ ,  $\min_k \phi_{k,N} \geq c_0 > 0$  and  $Z$  is an increasing function defined on  $\mathbb{R}_+$ , choosing, say,  $\gamma = 1$ , yields that the previous expression is bounded by  $C_0 N$  for some finite constant  $C_0$ .

Condition 3.1 is satisfied, for example, by ‘local equilibrium’ measures  $\{\mu_{\text{le}}^N\}_{N \in \mathbb{N}}$  associated to macroscopic profiles  $\rho_0$  in  $L^\infty(\mathbb{T})$ . For each  $N \in \mathbb{N}$ , let  $\mu_{\text{le}}^N$  be the product measure on  $\mathbb{N}_0^{\mathbb{T}_N}$  with marginals given by

$$\mu_{\text{le}}^N(\xi(k) = n) = \mathcal{P}_{\tilde{\phi}_{k,N}}(n), \quad \text{for } k \in \mathbb{T}_N, n \geq 0,$$

where the parameters  $\{\tilde{\phi}_{k,N} : 1 \leq k \leq N\}$  are such that  $E_{\mu_{\text{le}}^N}[\xi(k)] = \rho_{k,N}$  for

$$\rho_{k,N} = N \int_{(k-1)/N}^{k/N} \rho_0(x) dx.$$

To prove Condition 3.1 (a), for  $\mu_{\text{le}}^N$ , approximate the integral by a Riemann sum and apply Chebyshev and Schwarz inequality, keeping in mind that the measure  $\mu_{\text{le}}^N$  is product. The next lemma asserts that Condition 3.1 (b) holds.

**Lemma 3.2.** *There exists  $C_0 > 0$  such that  $H(\mu_{\text{le}}^N | \mathcal{R}_N) \leq C_0 N$  for all  $N \in \mathbb{N}$ .*

*Proof.* Write

$$\begin{aligned} H(\mu_{\text{le}}^N | \mathcal{R}_N) &= \sum_{k=1}^N E_{\mu_{\text{le}}^N} \left\{ \ln \left[ \left( \frac{\tilde{\phi}_{k,N}}{\phi_{k,N}} \right)^{\xi(k)} \right] + \ln \frac{Z(\phi_{k,N})}{Z(\tilde{\phi}_{k,N})} \right\} \\ &= \sum_{k=1}^N \rho_{k,N} \ln \frac{\tilde{\phi}_{k,N}}{\phi_{k,N}} + \sum_{k=1}^N \ln \frac{Z(\phi_{k,N})}{Z(\tilde{\phi}_{k,N})}. \end{aligned}$$

Then, the lemma follows as (a)  $\tilde{\phi}_{k,N} \leq g^* \rho_{k,N}$  is uniformly bounded by the fugacity bounds after (2.6) as  $\|\rho_0\|_\infty < \infty$ , (b)  $Z(0) = 1$ , and (c)  $0 < c \leq \phi_{k,N} \leq 1$  for all  $1 \leq k \leq N$  by Lemma 2.2.  $\square$

**3.2. Main results.** For each  $N$ , we will observe the evolution of the zero-range process speeded up by  $N^2$ , and consider in the sequel the process  $\eta_t := \xi_{N^2 t}$ , generated by  $N^2 L$  (cf. (2.5)), for times  $0 \leq t \leq T$ , where  $T > 0$  refers to a fixed time horizon. We will access the space-time structure of the process through the scaled mass empirical measure:

$$\pi_t^N(dx) := \frac{1}{N} \sum_{k=1}^N \eta_t(k) \delta_{k/N}(dx),$$

where  $\delta_x$ ,  $x \in \mathbb{T}$ , stands for the Dirac mass at  $x$ .

Let  $\mathcal{M}$  be the space of finite nonnegative measures on  $\mathbb{T}$ , and observe that  $\pi_t^N \in \mathcal{M}$ . We will place a metric  $d(\cdot, \cdot)$  on  $\mathcal{M}$  which realizes the dual topology of  $C(\mathbb{T})$  (see [23][p. 49] for a definitive choice). Here, the trajectories  $\{\pi_t^N : 0 \leq t \leq T\}$  are elements of the Skorokhod space  $D([0, T], \mathcal{M})$ , endowed with the associated Skorokhod topology.

In the following, for  $G \in C(\mathbb{T})$  and  $\pi \in \mathcal{M}$ , denote  $\langle G, \pi \rangle = \int_0^1 G(u) \pi(du)$ . Also, the process measure and associated expectation governing  $\eta$  starting from  $\mu$  will be denoted by  $\mathbb{P}_\mu$  and  $\mathbb{E}_\mu$ . When the process starts from  $\{\mu^N\}_{N \in \mathbb{N}}$ , in the class satisfying Condition 3.1, we will denote by  $\mathbb{P}_N := \mathbb{P}_{\mu^N}$  and  $\mathbb{E}_N := \mathbb{E}_{\mu^N}$ , the associated process measure and expectation.

We first state a hydrodynamic limit (HDL) with respect to the abstract environments  $\{\alpha_k^N\}$ . The proof of Theorem 3.3 is outlined in Section 5.

**Theorem 3.3.** *For initial distributions  $\mu^N$  satisfying Condition 3.1, consider the speeded up process  $\eta_t$  as above with generator  $N^2 L$ . Then, for any  $t \geq 0$ , test function  $G \in C^\infty(\mathbb{T})$ , and  $\delta > 0$ ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_N \left[ \left| \langle G, \pi_t^N \rangle - \int_{\mathbb{T}} G(x) \rho(t, x) dx \right| > \delta \right] = 0,$$

where  $\rho(t, x)$  is the unique weak solution of

$$\begin{cases} \partial_t \rho(t, x) = \frac{1}{2} \partial_{xx} \Phi(\rho(t, x)) - 2 \partial_x (\alpha(x) \Phi(\rho(t, x))), \\ \rho(0, x) = \rho_0(x), \end{cases} \quad (3.2)$$

in the class of ‘good’ weak solutions given in Definition 3.4 below with  $\Psi = \Phi$  and  $K = \alpha$ .

Consider the PDE

$$\begin{cases} \partial_t \rho(t, x) = \partial_{xx} \Psi(\rho(t, x)) + \partial_x (K(x) \Psi(\rho(t, x))), & x \in \mathbb{T}, t \geq 0 \\ \rho(0, x) = \rho_0(x) \end{cases} \quad (3.3)$$

where  $\Psi(\cdot) \in C^1[0, \infty)$  satisfies  $0 \leq \Psi'(\cdot) \leq C_\Psi$ , the function  $K(\cdot)$  is measurable and in  $L^\infty(\mathbb{T})$ , and  $\rho_0 \in L^1(\mathbb{T})$  is nonnegative.

In the context of Theorem 3.3,  $\Phi$  is certainly continuously differentiable with  $\Phi'$  positive and bounded above (cf. below (2.6)), and  $\alpha$  is a continuous function.

**Definition 3.4.** *We say  $\rho(t, x) = \rho(t, x; \Psi, K) : [0, T] \times \mathbb{T} \mapsto [0, \infty)$  is a good weak solution to (3.3) if*

- (1)  $\int_{\mathbb{T}} \rho(t, x) dx = \int_{\mathbb{T}} \rho_0(x) dx$  for all  $t \in [0, T]$ .
- (2)  $\rho(t, \cdot)$  is weakly continuous, that is, for all  $G \in C(\mathbb{T})$ ,  $\int_{\mathbb{T}} G(x) \rho(t, x) dx$  is a continuous function in  $t$ .

- (3) *There exists an  $L^1([0, T] \times \mathbb{T})$  function denoted by  $\partial_x \Psi(\rho(s, x))$  such that for all  $G(s, x) \in C^{0,1}([0, T] \times \mathbb{T})$ , it holds*

$$\int_0^T \int_{\mathbb{T}} \partial_x G(s, x) \Psi(\rho(s, x)) dx ds = - \int_0^T \int_{\mathbb{T}} G(s, x) \partial_x \Psi(\rho(s, x)) dx ds. \quad (3.4)$$

- (4) *For all  $G(s, x) \in C_c^\infty([0, T] \times \mathbb{T})$*

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}} \partial_s G(s, x) \rho(s, x) dx ds + \int_{\mathbb{T}} G(0, x) \rho_0(x) dx \\ &= \int_0^T \int_{\mathbb{T}} [-\partial_{xx} G(s, x) \Psi(\rho(s, x)) + \partial_x G(s, x) [K(x) \Psi(\rho(s, x))]] dx ds. \end{aligned}$$

Later, in Theorem 10.1 in Section 10, we show that the PDE (3.3) has at most one good weak solution.

As a consequence of Theorem 3.3 and its proof, we observe that both quenched and annealed hydrodynamic limits follow with respect to random environment

$$\left\{ r_k^N := \frac{1}{\sigma N \varepsilon} \sum_{|j-k| \leq \lfloor N \varepsilon \rfloor} r_j \psi \left( \frac{j-k}{\varepsilon} \right) : k \in \mathbb{T}_N \right\}.$$

Let  $E_P \mathbb{P}_N^\omega$  be the annealed probability measure, where  $P(d\omega)$  governs the random environment  $\omega = \{r_x\}_{x \in \mathbb{N}}$  and  $\mathbb{P}_N^\omega = \mathbb{P}_N$  is the process measure of the speeded up zero-range process  $\eta$ , with single particle jump rates  $(1/2) \pm r_k^N / \sqrt{N}$  to the left and right from location  $k \in \mathbb{T}_N$ , conditioned on the environment. Recall that  $\{q_k^N\} \stackrel{d}{=} \{r_k^N\}$  and that a.s.  $\sqrt{N} q_k^N \rightarrow W'_\varepsilon(x)$  when  $k/N \rightarrow x$  (cf. (2.2), (2.3)).

**Theorem 3.5.** *With respect to the random environments, we have the hydrodynamic limits:*

- (I) *(Quenched HDL) For almost all realizations  $\omega$ , the statement of Theorem 3.3 holds with respect to  $\alpha^N = \sqrt{N} q^N$  and  $\alpha = W'_\varepsilon$ .*  
 (II) *(Annealed HDL) The law of  $\pi^N$ , under  $E_P \mathbb{P}_N^\omega$ , converges weakly to the law of  $\rho(\cdot, x) dx = \rho(\cdot, x; \Phi, W'_\varepsilon) dx$  with respect to the distribution of Brownian motion  $W$ .*

*Proof.* Part (I) follows directly from Theorem 3.3. For part (II), let  $Q_\omega^N$  be the distribution of  $\pi^N \in D([0, T], \mathcal{M})$  conditioned on the environment realization  $\omega$ . Then, averaging over the environment, the distribution of  $\pi^N$  is  $\int Q_\omega^N P(d\omega)$ . From part (I), we observe that  $\rho(\cdot, x; \Phi, W'_\varepsilon(\omega)) dx$ , noting the dependence on  $\omega$ , is the quenched hydrodynamic limit conditioned on the environment. Averaging over the environment, the distribution of  $\rho(\cdot, x) dx$  is  $\int \delta_{\rho(\cdot, dx; \Phi, W'_\varepsilon(\omega)) dx} P(d\omega)$ . Here, the law of  $W'_\varepsilon(\omega)$  under  $P(d\omega)$  is that of  $\frac{-1}{\varepsilon} \int W(u) \psi' \left( \frac{u-x}{\varepsilon} \right) du + \frac{1}{\varepsilon} \{W(x+\varepsilon) \psi(1) - W(x-\varepsilon) \psi(-1)\}$  (cf. (2.3)).

Let  $f$  be a bounded continuous function on  $D([0, T], \mathcal{M})$ . By the proof of Theorem 3.3 (see Step 5 in the proof outline Section 5), applied to the quenched environments, we have that  $Q_\omega^N$  converges weakly to  $\delta_{\rho(\cdot, x; \Phi, W'_\varepsilon(\omega)) dx}$  for almost all environments  $\omega$ . Hence,  $\int f(\pi) Q_\omega^N(d\pi)$  converges for almost all environments  $\omega$  to  $\int f(\pi) \delta_{\rho(\cdot, x; \Phi, W'_\varepsilon(\omega)) dx}(d\pi)$ . Then, by bounded convergence, as  $N \uparrow \infty$ ,

$$\int \int f(\pi) Q_\omega^N(d\pi) P(d\omega) \rightarrow \int \int f(\pi) \delta_{\rho(\cdot, x; \Phi, W'_\varepsilon(\omega)) dx}(d\pi) P(d\omega),$$

and Part (II) follows.  $\square$

Finally, we focus on a system of independent particles, seen in diffusive scale and starting from a local equilibrium measure, in a common unregularized Seignourel environment on  $\mathbb{Z}$ , that is when  $g(k) \equiv k$ , and a single particle has rates of moving from  $k$  to  $k \pm 1$  are  $\{1/2 \pm r_k/\sqrt{\sigma^2 N} : k \in \mathbb{Z}\}$ , where  $\omega = \{r_i\}$  are i.i.d. bounded, mean 0 random variables with variance  $\sigma^2 > 0$ . Let  $P_N^\omega$  and  $E_N^\omega$  refer to the quenched process measure and expectation with respect to a random walk moving in this environment.

**Theorem 3.6.** *Consider a system of independent particles on  $\mathbb{Z}$  moving in a common Seignourel environment, with initial product Poisson distribution  $\nu_{i_e}^N = \prod_{k \in \mathbb{Z}} \mathcal{P}_{\beta(k/N)}$  associated to a bounded, continuous profile  $\beta : \mathbb{R} \rightarrow \mathbb{R}$ .*

(I) (Quenched law at time  $t$ ) For fixed  $\omega$ , the law of the configuration  $\{\eta_k(t) : k \in \mathbb{Z}\}$  is a product Poisson measure  $\prod_{k \in \mathbb{Z}} \mathcal{P}_{\beta_t(k, N)}$  where  $\beta_t(k, N) = \sum_{x \in \mathbb{Z}} \beta(x/N) P_N^\omega(R_{N^2 t}^{x, N} = k)$  for  $k \in \mathbb{Z}$ , and  $R_s^{k, N}$  is the position at time  $s$  of a random walk starting at  $k$  in this environment.

(II) (Annealed mean HDL) Let  $G \in C^\infty(\mathbb{R})$  be a compactly supported test function. Then,

$$\lim_{N \rightarrow \infty} E_P \mathbb{E}_N^\omega [\langle G, \pi_t^N \rangle] = \int_{\mathbb{R}} G(u) \mathcal{E}[\beta(u - X_t)] du,$$

where  $\mathcal{E}$  is the annealed expectation with respect to the standard 1–Brox diffusion  $X$ .

*Proof.* We show part (I) by computing explicitly the moment generating function of the law of the configuration  $\{\eta_t(x) : x \in \mathbb{Z}\}$  via the method of Chapter 1 [23]. Let  $\{\lambda(i) : i \in \mathbb{Z}\}$  be real parameters, where  $\lambda(i) \neq 0$  only for a finite number of  $i \in \mathbb{Z}$ . With respect to a fixed random environment  $\omega$ , decompose  $\eta_t(x) = \sum_{z \in \mathbb{Z}} \sum_{j=1}^{\eta_0(z)} \mathbb{1}(R_{N^2 t}^{z, j, N} = x)$  in terms of the initial configuration and the positions  $R_{N^2 t}^{z, j, N}$  of the  $j$ th particle at time  $N^2 t$  which is initially at  $z$ . Here, we use the convention that empty sums vanish.

Hence,  $\sum_{x \in \mathbb{Z}} \lambda(x) \eta_t(x) = \sum_{z \in \mathbb{Z}} \sum_{j=1}^{\eta_0(z)} \lambda(R_{N^2 t}^{z, j, N})$ . Since, for each  $z$ ,  $\{R_{N^2 t}^{z, j, N} : 1 \leq j \leq \eta_0(z)\}$  are i.i.d., and  $\eta_0$  is distributed according to  $\nu_{i_e}^N$ , we have

$$\mathbb{E}_N^\omega \left[ \exp \left\{ \sum_{x \in \mathbb{Z}} \lambda(x) \eta_t(x) \right\} \right] = \prod_{z \in \mathbb{Z}} e^{\beta(z/N)} \{ E_N^\omega [e^{\lambda(R_{N^2 t}^{z, N})}] - 1 \}.$$

Since  $E_N^\omega [e^{\lambda(R_{N^2 t}^{z, N})}] = \sum_{x \in \mathbb{Z}} e^{\lambda(x)} P_N^\omega(R_{N^2 t}^{z, N} = x)$ , the last display equals, as desired,

$$\prod_{x \in \mathbb{Z}} \exp \left\{ \left[ \sum_{z \in \mathbb{Z}} \beta(z/N) P_N^\omega(R_{N^2 t}^{z, N} = x) \right] (e^{\lambda(x)} - 1) \right\}.$$

To show part (II), as a consequence of part (I), we find that the quenched mean of  $N^{-1} \sum_{k \in \mathbb{Z}} G(k/N) \eta_t(k)$  equals

$$\frac{1}{N} \sum_{k \in \mathbb{Z}} G(k/N) \sum_{z \in \mathbb{Z}} \beta(z/N) P_N^\omega(R_{N^2 t}^{z, N} = k).$$

By translation-invariance of  $\omega$  under  $P(d\omega)$ ,  $E_P [P_N^\omega(R_{N^2 t}^{z, N} = k)] = E_P [P_N^\omega(R_{N^2 t}^{0, N} + z = k)]$ . Hence, the annealed mean of  $N^{-1} \sum_{k \in \mathbb{Z}} G(k/N) \eta_{N^2 t}(k)$ , equals

$$\begin{aligned} & \int \frac{1}{N} \sum_{k \in \mathbb{Z}} G(k/N) \sum_{z \in \mathbb{Z}} \beta(z/N) P_N^\omega(R_{N^2 t}^{0, N} + z = k) P(d\omega) \\ &= \int \frac{1}{N} \sum_{k \in \mathbb{Z}} G(k/N) E_N^\omega [\beta(k/N - R_{N^2 t}^{0, N}/N)] P(d\omega). \end{aligned}$$

Now, from Seignourel's annealed limit, we have  $R_{N^2t}^{0,N}/N \Rightarrow X_t$ . Since  $\beta$  is bounded and continuous and  $G \in C^\infty(\mathbb{R})$  has compact support, we conclude that the previous display converges, as  $N \uparrow \infty$ , to  $\int_{\mathbb{R}} G(u) \mathcal{E}[\beta(u - X_t)] du$  as desired.  $\square$

#### 4. STOCHASTIC DIFFERENTIALS AND MARTINGALES

To analyze  $\langle G, \pi_t^N \rangle$ , we compute its stochastic differential in terms of certain martingales. Let  $G$  be a smooth function on  $[0, T] \times \mathbb{T}$ , and let us write  $G_t(x) := G(t, x)$ . Then,

$$M_t^{N,G} = \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \left\{ \langle \partial_s G_s, \pi_s^N \rangle + N^2 L \langle G_s, \pi_s^N \rangle \right\} ds$$

is a mean zero martingale. Denote the discrete Laplacian  $\Delta_N$  and discrete gradient  $\nabla_N$  by

$$\begin{aligned} \Delta_N G\left(\frac{k}{N}\right) &:= N^2 \left( G\left(\frac{k+1}{N}\right) + G\left(\frac{k-1}{N}\right) - 2G\left(\frac{k}{N}\right) \right), \\ \nabla_N G\left(\frac{k}{N}\right) &:= \frac{N}{2} \left( G\left(\frac{k+1}{N}\right) - G\left(\frac{k-1}{N}\right) \right), \end{aligned}$$

and write

$$N^2 L \langle G, \pi_s^N \rangle = \frac{1}{N} \sum_{1 \leq k \leq N} \left( \frac{1}{2} \Delta_N G_s\left(\frac{k}{N}\right) g(\eta_s(k)) + 2 \nabla_N G_s\left(\frac{k}{N}\right) g(\eta_s(k)) \alpha_k^N \right). \quad (4.1)$$

We will define

$$D_{N,k}^{G,s} := \frac{1}{2} \Delta_N G_s\left(\frac{k}{N}\right) + 2 \nabla_N G_s\left(\frac{k}{N}\right) \alpha_k^N. \quad (4.2)$$

As  $|\alpha_k^N|$  is uniformly bounded from above by a constant  $C$ , given uniform convergence to  $\alpha$  on the torus  $\mathbb{T}$ , we have

$$\left| D_{N,k}^{G,s} \right| \leq \|\partial_{xx} G\|_\infty + 2C \|\partial_x G\|_\infty. \quad (4.3)$$

The quadratic variation of  $M_t^{N,G}$  is given by

$$\langle M^{N,G} \rangle_t = \int_0^t \left\{ N^2 L \left( \langle G_s, \pi_s^N \rangle^2 \right) - 2 \langle G_s, \pi_s^N \rangle N^2 L \langle G_s, \pi_s^N \rangle \right\} ds.$$

Standard calculation shows that

$$\begin{aligned} \langle M^{N,G} \rangle_t &= \int_0^t \sum_{k=1}^N \left\{ g(\eta_s(k)) \left( \frac{1}{2} + \frac{\alpha_k^N}{N} \right) \left( G_s\left(\frac{k+1}{N}\right) - G_s\left(\frac{k}{N}\right) \right)^2 \right. \\ &\quad \left. + g(\eta_s(k)) \left( \frac{1}{2} - \frac{\alpha_k^N}{N} \right) \left( G_s\left(\frac{k-1}{N}\right) - G_s\left(\frac{k}{N}\right) \right)^2 \right\} ds. \end{aligned}$$

This variation may be bounded as follows.

**Lemma 4.1.** *For smooth functions  $G$  on  $[0, T] \times \mathbb{T}$ , there is a constant  $C_G$  such that for large  $N$ ,*

$$\sup_{0 \leq t \leq T} \mathbb{E}_N \langle M^{N,G} \rangle_t \leq g^* C_G T N^{-1}.$$

*Proof.* For  $N$  large, we may assume that  $1/2 + |\alpha_k^N/N| \leq 1$ . Also since  $G$  is smooth and  $g(\cdot)$  grows at most linearly, we obtain

$$\begin{aligned} \mathbb{E}_N \langle M^{N,G} \rangle_t &\leq 2g^* \|\partial_x G\|_\infty N^{-1} \mathbb{E}_N \left[ \int_0^t \frac{1}{N} \sum_{k=1}^N \eta_s(k) ds \right] \\ &= 2g^* \|\partial_x G\|_\infty N^{-1} t \mathbb{E}_N \left[ \frac{1}{N} \sum_{k=1}^N \eta_0(k) \right]. \end{aligned}$$

We have used that total number of particles is conserved in the last equality. By (3.1), we have that  $\sup_N \mathbb{E}_N [N^{-1} \sum_{k=1}^N \eta_0(k)] < \infty$ , and the result follows.  $\square$

## 5. PROOF OUTLINE

We now outline the proof of Theorem 3.3. Let  $Q^N$  be the probability measure on the trajectory space  $D([0, T], \mathcal{M})$  governing  $\pi_t^N$  when the process starts from  $\mu^N$ . By Lemma 8.1, the family of measures  $\{Q^N\}_{N \in \mathbb{N}}$  is tight with respect to the uniform topology, stronger than the Skorokhod topology.

Let now  $Q$  be any limit measure. We will show that  $Q$  is supported on a class of weak solutions to the nonlinear PDE (3.2).

*Step 1.* Let  $G$  be smooth on  $[0, T] \times \mathbb{T}$ . Recall the martingale  $M_t^{N,G}$  and its quadratic variation  $\langle M^{N,G} \rangle_t$  in the last section. By Lemma 4.1, we have  $\mathbb{E}_N (M_T^{N,G})^2 = \mathbb{E}_N \langle M^{N,G} \rangle_T$  vanishes as  $N \rightarrow \infty$ . By Doob's inequality, for each  $\delta > 0$ ,

$$\begin{aligned} &\mathbb{P}_N \left[ \sup_{0 \leq t \leq T} \left| \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t (\langle \partial_s G_s, \pi_s^N \rangle + N^2 L \langle G_s, \pi_s^N \rangle) ds \right| > \delta \right] \\ &\leq \frac{4}{\delta^2} \mathbb{E}_N \langle M^{N,G} \rangle_T \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Recall the evaluation of  $N^2 L \langle G_s, \pi_s^N \rangle$  in (4.1). Then,

$$\begin{aligned} &\lim_{N \rightarrow \infty} \mathbb{P}_N \left[ \sup_{0 \leq t \leq T} \left| \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \left\{ \langle \partial_s G_s, \pi_s^N \rangle \right. \right. \right. \quad (5.1) \\ &\quad \left. \left. \left. \frac{1}{N} \sum_{1 \leq k \leq N} \left( \frac{1}{2} \Delta_N G_s \left( \frac{k}{N} \right) g(\eta_s(k)) + 2 \nabla_N G_s \left( \frac{k}{N} \right) g(\eta_s(k)) \alpha_k^N \right) \right\} ds \right| > \delta \right] = 0. \end{aligned}$$

*Step 2.* We now replace the nonlinear term  $g(\eta_s(k))$  by a function of the empirical density of particles. To be precise, let  $\eta^l(x) = \frac{1}{2l+1} \sum_{|y-x| \leq l} \eta(y)$ , that is the average density of particles in the box centered at  $x$  with length  $2l+1$ .

Recall the coefficient  $D_{N,k}^{G,s}$  in (4.2). By the triangle inequality, the 1 and 2-block estimates (Lemmas 6.3 and 7.2) imply the following replacement lemma.

**Lemma 5.1** (Replacement Lemma). *For each  $\delta > 0$ , we have*

$$\limsup_{\theta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_N \left[ \left| \frac{1}{N} \sum_{1 \leq k \leq N} \int_0^T D_{N,k}^{G,t} \left( g(\eta_t(k)) - \Phi(\eta_t^{\theta N}(k)) \right) dt \right| \geq \delta \right] = 0.$$

*Step 3.* For each  $\theta > 0$ , take  $\iota_\theta = (2\theta)^{-1} \mathbb{1}_{[-\theta, \theta]}$ . The average density  $\eta_t^{\theta N}(k)$  is written as a function of the empirical measure  $\pi_t^N$

$$\eta_t^{\theta N}(k) = \frac{2\theta N}{2\theta N + 1} \langle \iota_\theta(\cdot - k/N), \pi_t^N \rangle.$$

Then, noting the form of  $D_{N,k}^{G,s}$  and the uniform convergence of  $\alpha^N$  to  $\bar{\alpha}$ , we may replace  $\nabla_N$ ,  $\Delta_N$ , and  $\alpha_k^N$  by  $\partial_x$ ,  $\partial_{xx}$ , and  $\alpha(\frac{k}{N})$  respectively, and also the sum by an integral. Hence, we get from (5.1) in terms of the induced distribution  $Q^N$  that

$$\begin{aligned} & \limsup_{\theta \rightarrow 0} \limsup_{N \rightarrow \infty} Q^N \left[ \left| \langle G_T, \pi_T \rangle - \langle G_0, \pi_0 \rangle - \int_0^T \left\{ \langle \partial_s G_s, \pi_s \rangle \right. \right. \right. \\ & \left. \left. \left. + \int_{\mathbb{T}} \left( \frac{1}{2} \partial_{xx} G_s(x) + 2\partial_x G_s(x) \alpha(x) \right) \Phi(\langle \iota_\theta(\cdot - x), \pi_s^N \rangle) dx \right\} ds \right| > \delta \right] = 0. \end{aligned} \quad (5.2)$$

Taking  $N \rightarrow \infty$ , along a subsequence, as the set of trajectories in (5.2) is open with respect to the uniform topology, we obtain

$$\begin{aligned} & \limsup_{\theta \rightarrow 0} Q \left[ \left| \langle G_T, \pi_T \rangle - \langle G_0, \pi_0 \rangle - \int_0^T \left\{ \langle \partial_s G_s, \pi_s \rangle \right. \right. \right. \\ & \left. \left. \left. + \int_{\mathbb{T}} \left( \frac{1}{2} \partial_{xx} G_s(x) + 2\partial_x G_s(x) \alpha(x) \right) \Phi(\langle \iota_\theta(\cdot - x), \pi_s \rangle) dx \right\} ds \right| > \delta \right] = 0. \end{aligned}$$

*Step 4.* We show in Lemma 9.1 that  $Q$  is supported on trajectories  $\pi_s(dx) = \rho(s, x)dx$  where  $\rho \in L^1([0, T] \times \mathbb{T})$ . To replace  $\langle \iota_\theta(\cdot - x), \pi_s \rangle$  by  $\rho(s, x)$ , it is enough to show, for all  $\delta > 0$ , that

$$\limsup_{\theta \rightarrow 0} Q \left[ \left| \int_0^T \int_{\mathbb{T}} D_{G,s} (\Phi(\langle \iota_\theta(\cdot - x), \pi_s \rangle) - \Phi(\rho(s, x))) dx ds \right| > \delta \right] = 0$$

where  $D_{G,s} = \frac{1}{2} \partial_{xx} G_s(x) + 2\partial_x G_s(x) \alpha(x)$ . By, considering the Lebesgue points of  $\rho$ , almost surely with respect to  $Q$  (cf. [23]), we have

$$\lim_{\theta \rightarrow 0} \int_0^T \int_{\mathbb{T}} D_{G,s} \Phi(\iota_\theta(\cdot - x), \pi_s) dx ds = \int_0^T \int_{\mathbb{T}} D_{G,s} \Phi(\rho(s, x)) dx ds.$$

Now, we have

$$\begin{aligned} & Q \left[ \left| \langle G_T, \rho(T, x) \rangle - \langle G_0, \rho(0, x) \rangle - \int_0^T \left\{ \langle \partial_s G_s, \rho(s, x) \rangle \right. \right. \right. \\ & \left. \left. \left. + \int_{\mathbb{T}} \left( \frac{1}{2} \partial_{xx} G_s(x) + 2\partial_x G_s(x) \alpha(x) \right) \Phi(\rho(s, x)) dx \right\} ds \right| = 0 \right] = 1. \end{aligned}$$

*Step 5.* Now, each  $\rho(t, x)$  solves weakly the equation

$$\partial_t \rho = \frac{1}{2} \partial_{xx} \Phi(\rho) - 2\partial_x (\alpha(x) \Phi(\rho)).$$

As a consequence of the weak formulation,  $\rho$  satisfies conservation of mass (cf. Lemma 9.1):  $\int_{\mathbb{T}} \rho(t, x) dx = \int_{\mathbb{T}} \rho_0(x) dx$ . Moreover, the initial condition  $\rho(0, x) = \rho_0(x)$  holds by Condition 3.1. From convergence of  $Q^N$  to  $Q$  with respect to the uniform topology,  $\rho$  is weakly continuous in time: Namely, for each test function  $G \in C(\mathbb{T})$ , the map  $t \mapsto \int_{\mathbb{T}} G(x) \rho(t, x) dx$  is continuous. In addition, in Proposition 9.2, we show an energy estimate which defines a weak spatial derivative of  $\Phi(\rho(t, x))$ . Therefore,  $\rho$  is a good weak solution to (3.2).

As mentioned after Theorem 3.3, in Subsection 10, we show that there is at most one good weak solution  $\rho$  to (3.2) (cf. Definition 3.4). We conclude then that the sequence of  $Q^N$  converges weakly to the Dirac measure on  $\rho(\cdot, x)dx$ . Finally, as  $Q^N$  converges to  $Q$  with respect to the uniform topology, we have for each  $0 \leq t \leq T$  that  $\langle G, \pi_t^N \rangle$  weakly converges to the constant  $\int_{\mathbb{T}} G(x)\rho(t, x)dx$ , and therefore convergence in probability as stated in Theorem 3.3.  $\square$

## 6. 1-BLOCK ESTIMATE

The ‘1-block’ estimate is obtained by using a Rayleigh-type estimation of a variational eigenvalue expression derived from a Feynman-Kac bound. A spectral gap bound plays an important role in this step. There are differences here, with the scheme of [12] and [18], in the context of the inhomogeneous environment, that we detail.

Recall the generator  $L$ , cf. (2.5), and the invariant measure  $\mathcal{R}_N$  (cf. (2.7), where  $\phi_{k,N}$  is taken so that  $\max_k \phi_{k,N} = 1$ ). As  $\mathcal{R}_N$  is not reversible with respect to  $L$ , we will work with  $S$ , the symmetric part of  $L$ :

$$Sf(\eta) = \frac{1}{2} \sum_{k=1}^N \left\{ g(\eta(k)) \mathfrak{p}_{k,+}^N (f(\eta^{k,k+1}) - f(\eta)) + g(\eta(k)) \mathfrak{p}_{k,-}^N (f(\eta^{k,k-1}) - f(\eta)) \right\}$$

where

$$\mathfrak{p}_{k,+}^N := \left( \frac{1}{2} + \frac{\alpha_k^N}{N} \right) + \frac{\phi_{k+1,N}}{\phi_{k,N}} \left( \frac{1}{2} - \frac{\alpha_{k+1}^N}{N} \right), \quad \mathfrak{p}_{k,-}^N := \left( \frac{1}{2} - \frac{\alpha_k^N}{N} \right) + \frac{\phi_{k-1,N}}{\phi_{k,N}} \left( \frac{1}{2} + \frac{\alpha_{k-1}^N}{N} \right). \quad (6.1)$$

Then,  $\mathcal{R}_N$  is reversible under the generator  $S$ . The Dirichlet form is

$$E_{\mathcal{R}_N} [f(-Sf)] = E_{\mathcal{R}_N} [f(-Lf)] = \frac{1}{2} \sum_{k=1}^N E_{\mathcal{R}_N} \left[ g(\eta(k)) \mathfrak{p}_{k,+}^N (f(\eta^{k,k+1}) - f(\eta))^2 \right]. \quad (6.2)$$

**6.1. Spectral gap bound for 1-block estimate.** For  $k \in \mathbb{T}_N$  and  $l \geq 1$ , define the set  $\Lambda_{k,l} = \{k-l, k-l+1, \dots, k+l\} \subset \mathbb{T}_N$ . Consider the process restricted to  $\Lambda_{k,l}$  generated by  $S_{k,l}$  where

$$S_{k,l}f(\eta) = \frac{1}{2} \sum_{x, x+1 \in \Lambda_{k,l}} \left\{ g(\eta(x)) \mathfrak{p}_{x,+}^N (f(\eta^{x,x+1}) - f(\eta)) + g(\eta(x+1)) \mathfrak{p}_{x+1,-}^N (f(\eta^{x+1,x}) - f(\eta)) \right\}. \quad (6.3)$$

Let  $\Omega_{k,l} = \mathbb{N}_0^{\Lambda_{k,l}}$  be the state space of configurations restricted on sites  $\Lambda_{k,l}$ . For each  $\eta \in \Omega_{k,l}$ , define  $\kappa_{k,l}(\eta) = \prod_{x \in \Lambda_{k,l}} \mathcal{P}_{\phi_{x,N}}(\eta(x))$ , that is,  $\kappa_{k,l}$  is the product measure  $\mathcal{R}_N$  restricted to  $\Omega_{k,l}$ . Define the state space of configurations with exactly  $j$  particles on the sites  $\Lambda_{k,l}$ :

$$\Omega_{k,l,j} = \left\{ \eta \in \Omega_{k,l} : \sum_{x \in \Lambda_{k,l}} \eta(x) = j \right\}.$$

Let  $\kappa_{k,l,j}$  be the associated reversible canonical measure obtained by conditioning  $\kappa_{k,l}$  on  $\Omega_{k,l,j}$ . The corresponding Dirichlet form is

$$E_{\kappa_{k,l,j}} [f(-S_{k,l}f)] = \frac{1}{2} \sum_{x, x+1 \in \Lambda_{k,l}} E_{\kappa_{k,l,j}} \left[ g(\eta(x)) \mathfrak{p}_{x,+}^N (f(\eta^{x,x+1}) - f(\eta))^2 \right]. \quad (6.4)$$

We will obtain the spectral gap estimate corresponding to the localized inhomogeneous process by comparison with the spectral gap for the standard translation-invariant localized process. Consider the generator  $L_l$  on  $\Omega_{k,l}$  given by

$$L_l f(\eta) = \sum_{x, x+1 \in \Lambda_{k,l}} \frac{1}{2} \{g(\eta(x)) [f(\eta^{x, x+1}) - f(\eta)] + g(\eta(x+1)) [f(\eta^{x+1, x}) - f(\eta)]\}.$$

For any  $\rho > 0$ , let  $\nu_\rho$  be the product measure on  $\Omega = \mathbb{N}_0^{\mathbb{T}^N}$  with common marginal  $\mathcal{P}_\rho = \mathcal{P}_{\Phi(\phi)}$  on each site  $k \in \mathbb{N}$  with mean  $\rho$ , and let  $\nu_l^\rho$  be its restriction to  $\Omega_{k,l}$ .

Consider  $\nu_{l,j}$ , the associated canonical measure on  $\Omega_{k,l,j}$ , with respect to  $j$  particles in  $\Lambda_{k,l}$ . Notice that  $\nu_{l,j}$  does not depend on  $\rho$ . It is well-known that both  $\nu_l^\rho$  and  $\nu_{l,j}$  are invariant measures with respect to the localized generator  $L_l$  (cf. [1]). The corresponding Dirichlet form is given by

$$E_{\nu_{l,j}} [f(-L_l f)] = \frac{1}{2} \sum_{x, x+1 \in \Lambda_{k,l}} E_{\nu_{l,j}} \left[ g(\eta(x)) (f(\eta^{x, x+1}) - f(\eta))^2 \right]. \quad (6.5)$$

We are now ready to state the lemma for the spectral gap bounds. Recall  $\mathbf{p}_{k,+}^N$  from (6.1). Let  $r_{k,l,N}^{-1} := \min_{x \in \Lambda_{k,l}} \{\mathbf{p}_{x,+}^N\}$ .

**Lemma 6.1.** *We have the following estimates:*

- (1) *Uniform bound: For all  $\eta \in \Omega_{k,l,j}$ , we have*

$$\left( \frac{\phi_{\min, k, l}}{\phi_{\max, k, l}} \right)^j \leq \frac{\kappa_{k,l,j}(\eta)}{\nu_{l,j}(\eta)} \leq \left( \frac{\phi_{\max, k, l}}{\phi_{\min, k, l}} \right)^j \quad (6.6)$$

where  $\phi_{\min, k, l} = \min_{x \in \Lambda_{k,l}} \phi_{x,N}$  and  $\phi_{\max, k, l} = \max_{x \in \Lambda_{k,l}} \phi_{x,N}$ .

- (2) *Poincaré inequality: We have*

$$\text{Var}_{\kappa_{k,l,j}}(f) \leq C_{k,l,j} E_{\kappa_{k,l,j}} [f(-S_{k,l} f)]$$

where  $C_{k,l,j} := C(2l+1)^2 r_{k,l,N} \left( \frac{\phi_{\max, k, l}}{\phi_{\min, k, l}} \right)^{2j}$  bounds the inverse of the spectral gap of  $-S_{k,l}$  on  $\Omega_{k,l,j}$  and  $C$  is a universal constant.

- (3) *For each  $l$  fixed, we have*

$$\lim_{N \uparrow \infty} \sup_{1 \leq k \leq N} \frac{\phi_{\max, k, l}}{\phi_{\min, k, l}} = 1, \quad \lim_{N \uparrow \infty} \sup_{1 \leq k \leq N} r_{k,l,N} = 1$$

and hence, for fixed  $l$  and  $j$ ,  $\sup_{N \geq 1} \sup_{1 \leq k \leq N} C_{k,l,j} < \infty$ .

*Proof.* Fix an arbitrary  $\rho > 0$ . By the definitions of conditioned measures  $\kappa_{k,l,j}$  and  $\nu_{l,j}$ , we have, for  $\eta \in \Omega_{k,l,j}$ ,

$$\frac{\kappa_{k,l,j}(\eta)}{\nu_{l,j}(\eta)} = \frac{\kappa_{k,l}(\eta)}{\nu_l^\rho(\eta)} \frac{\nu_l^\rho(\Omega_{k,l,j})}{\kappa_{k,l}(\Omega_{k,l,j})}. \quad (6.7)$$

The product structure of  $\kappa_{k,l}$  and  $\nu_l^\rho$  allows a direct computation

$$\frac{\kappa_{k,l}(\eta)}{\nu_l^\rho(\eta)} = \frac{\prod_{x \in \Lambda_{k,l}} \{(\phi_{x,N})^{\eta(x)} / Z(\phi_{x,N})\}}{\prod_{x \in \Lambda_{k,l}} \{(\phi_0)^{\eta(x)} / Z(\phi_0)\}}, \quad (6.8)$$

where  $\phi_0$  is the common fugacity for (the marginals of)  $\nu_\rho$ . Recalling that  $\phi_{\min,k,l} = \min_{x \in \Lambda_{k,l}} \phi_{x,N}$  and  $\phi_{\max,k,l} = \max_{x \in \Lambda_{k,l}} \phi_{x,N}$ , for  $\eta \in \Omega_{k,l,j}$ , we can estimate  $\kappa_{k,l}(\eta)/\nu_l^\rho(\eta)$  by

$$\left(\frac{\phi_{\min,k,l}}{\phi_0}\right)^j \prod_{x \in \Lambda_{k,l}} \frac{Z(\phi_{x,N})}{Z(\phi_0)} \leq \frac{\kappa_{k,l}(\eta)}{\nu_l^\rho(\eta)} \leq \left(\frac{\phi_{\max,k,l}}{\phi_0}\right)^j \prod_{x \in \Lambda_{k,l}} \frac{Z(\phi_{x,N})}{Z(\phi_0)}. \quad (6.9)$$

Note that  $\kappa_{k,l}(\Omega_{k,l,j}) = \sum_{\eta \in \Omega_{k,l,j}} [\kappa_{k,l}(\eta)/\nu_l^\rho(\eta)] \nu_l^\rho(\eta)$ . Then,  $\kappa_{k,l}(\Omega_{k,l,j})/\nu_l^\rho(\Omega_{k,l,j})$  is estimated by the same bounds as in (6.9). Then, rearranging these estimates, (6.6) follows from (6.7).

Turning now to the Poincaré inequality, the proof relies on the well known spectral gap estimate for one dimensional localized symmetric zero range process (cf. [24]): For all  $j$ , with respect to a universal constant  $C$ ,

$$\text{Var}_{\nu_{l,j}}(f) \leq C(2l+1)^2 E_{\nu_{l,j}}[f(-L_l f)]. \quad (6.10)$$

To get an estimate with respect to  $-S_{k,l}$ , from (6.5) and (6.4), using (6.6), we have

$$E_{\nu_{l,j}}[f(-L_l f)] \leq r_{k,l,N} \left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j E_{\kappa_{k,l,j}}[f(-S_{k,l} f)]. \quad (6.11)$$

Now, since

$$\begin{aligned} \text{Var}_{\kappa_{k,l,j}}(f) &= \inf_a E_{\kappa_{k,l,j}}[(f-a)^2] \\ &\leq \left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j \inf_a E_{\nu_{l,j}}[(f-a)^2] = \left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j \text{Var}_{\nu_{l,j}}(f), \end{aligned}$$

the desired Poincaré inequality follows from (6.10) and (6.11).

The last item (3) follows from Lemma 2.2 and that  $\sup_N \sup_{1 \leq x \leq N} \alpha_x^N < C$ .  $\square$

**6.2. Relative entropy.** For  $t > 0$ , let  $\mu_t^N$  be the distribution of  $\eta_t$ . As the entropy production is negative, cf. p. 340, [23], we have  $H(\mu_t^N | \mathcal{R}_N) \leq H(\mu^N | \mathcal{R}_N) \leq C_0 N$ . Furthermore, the relative entropy of  $\mu_t^N$  with respect to the homogeneous invariant measures  $\nu_\rho$  is also of order  $O(N)$  which will be useful in the sequel.

**Lemma 6.2.** *For any fixed  $\rho > 0$ , there is a constant  $C = C(\omega)$  such that  $H(\mu_t^N | \nu_\rho) \leq CN$ .*

*Proof.* Write

$$H(\mu_t^N | \nu_\rho) = \int \ln \left( \frac{d\mu_t^N}{d\nu_\rho} \right) d\mu_t^N = \int \ln \left( \frac{d\mu_t^N}{d\mathcal{R}_N} \right) d\mu_t^N + \int \ln \left( \frac{d\mathcal{R}_N}{d\nu_\rho} \right) d\mu_t^N.$$

The first term on the right-hand side is exactly  $H(\mu_t^N | \mathcal{R}_N) = O(N)$  by part (2) of Condition 3.1. The integrand in the second term equals,

$$\ln \frac{d\mathcal{R}_N}{d\nu_\rho}(\eta) = \ln \frac{\mathcal{R}_N(\eta)}{\nu_\rho(\eta)} = \sum_{k=1}^N \eta(k) \ln \frac{\phi_{k,N}}{\phi_0} + \sum_{k=1}^N \ln \frac{Z(\phi_0)}{Z(\phi_{k,N})}.$$

The desired estimate now follows these observations:  $0 < c \leq \phi_{k,N} \leq 1$  by Lemma 2.2, and the mean expected number of particles,  $\int \sum_{k \in \mathbb{T}_N} \eta(k) d\mu_t^N = O(N)$  by (3.1).  $\square$

**6.3. 1-block estimate.** We prove the 1-block estimate:

**Lemma 6.3** (1-block estimate). *For every  $T > 0$ ,*

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \frac{1}{N} \sum_{1 \leq k \leq N} \left| \int_0^T V_{k,l}(s, \eta_s) ds \right| \right] = 0$$

where  $V_{k,l}(s, \eta) := D_{N,k}^{G,s} (g(\eta(k)) - \Phi(\eta^l(k)))$  and  $D_{N,k}^{G,s}$  is as in (4.2).

*Proof.* We first introduce a cutoff of large densities. Fix  $A > 0$ , and let

$$\tilde{V}_{k,l,A}(s, \eta) := V_{k,l}(s, \eta) \mathbb{1}_{\{\eta^l(k) \leq A\}}.$$

Notice that  $g(\cdot)$  considered here satisfies the ‘FEM’ assumption in [23]. By Lemma 6.2, one can replace  $V_{k,l}$  by  $\tilde{V}_{k,l,A}$ , following exactly the argument in Lemma 4.2 in p.90, [23].

It now remains to prove that for every  $A > 0$ ,  $T > 0$ ,

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{1 \leq k \leq N} \mathbb{E}_N \left[ \left| \int_0^T \tilde{V}_{k,l,A}(s, \eta_s) ds \right| \right] = 0.$$

Define  $\Lambda_{k,l}(\eta)$  as the number of particles in  $\Lambda_{k,l}$ , that is  $\Lambda_{k,l}(\eta) := (2l+1)\eta^l(k)$ . As in [12], we will replace  $\tilde{V}_{k,l,A}(s, \eta)$  by its ‘centering’:

$$V_{k,l,A}(s, \eta) := D_{N,k}^{G,s} \left\{ g(\eta(k)) - E_{\kappa_{k,l,\Lambda_{k,l}(\eta)}} [g(\eta(k))] \right\} \mathbb{1}_{\{\eta^l(k) \leq A\}}.$$

Note that  $E_{\kappa_{k,l,j}} V_{k,l,A} = 0$  for all  $k, l, j$  which will be used in the Rayleigh-type estimation. The error introduced by such a replacement, noting (4.3), is less than or equal to

$$C(G) \mathbb{E}_N \left[ \int_0^T \mathbb{1}_{\{0 < \eta_s^l(k) \leq A\}} \left| E_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}} [g(\eta(k))] - \Phi(\eta_s^l(k)) \right| ds \right]. \quad (6.12)$$

Note that  $\Phi(\eta_s^l(k)) = E_{\nu_{\eta_s^l(k)}} [g]$ . By the triangle inequality, the expectation in (6.12) is bounded by

$$\begin{aligned} & \mathbb{E}_N \left[ \int_0^T \mathbb{1}_{\{0 < \eta_s^l(k) \leq A\}} \left| E_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}} [g(\eta(k))] - E_{\nu_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}}} [g(\eta(k))] \right| ds \right] \\ & + \mathbb{E}_N \left[ \int_0^T \mathbb{1}_{\{0 < \eta_s^l(k) \leq A\}} \left| E_{\nu_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}}} [g(\eta(k))] - E_{\nu_{\eta_s^l(k)}} [g] \right| ds \right] =: I_1 + I_2. \end{aligned}$$

Using (6.6) and then  $g(k) \leq g^* k$ , the term  $I_1$  is bounded by

$$\begin{aligned} & \mathbb{E}_N \left[ \int_0^T \mathbb{1}_{\{0 < \eta_s^l(k) \leq A\}} E_{\nu_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}}} [g(\eta(k))] \left[ \left( \frac{\phi_{\max,k,l}}{\phi_{\min,k,l}} \right)^{\Lambda_{k,l}(\eta_s)} - 1 \right] ds \right] \\ & \leq T g^* (2l+1) A \left[ \left( \frac{\phi_{\max,k,l}}{\phi_{\min,k,l}} \right)^{(2l+1)A} - 1 \right]. \end{aligned}$$

Notice that  $\sup_{1 \leq k \leq N} \frac{\phi_{\max,k,l}}{\phi_{\min,k,l}} \rightarrow 1$  by Lemma 6.1. Then, for each fixed  $l$  and  $A$ , the term  $\sup_{1 \leq k \leq N} I_1$  vanishes as  $N \uparrow \infty$ .

Now, we turn to estimate the term  $I_2$ . By the equivalence of ensembles (cf. p.355, [23]), the absolute value in  $I_2$  vanishes as  $l \uparrow \infty$ , uniformly in  $k$ . Therefore, the term  $I_2$  vanishes as soon as we take  $N \uparrow \infty$ ,  $l \uparrow \infty$  in order.

To prove the lemma, it now remains to show

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{0 \leq k \leq N} \mathbb{E}_N \left[ \left| \int_0^T V_{k,l,A}(s, \eta_s) ds \right| \right] = 0.$$

By the entropy inequality (cf. p.338 [23]) and the assumption  $H(\mu^N | \mathcal{R}_N) \leq C_0 N$ , we have, for any  $\gamma > 0$

$$\mathbb{E}_N \left[ \left| \int_0^T V_{k,l,A}(s, \eta_s) ds \right| \right] \leq \frac{C_0}{\gamma} + \frac{1}{\gamma N} \ln \mathbb{E}_{\mathcal{R}_N} \left[ \exp \left\{ \gamma N \left| \int_0^T V_{k,l,A}(s, \eta_s) ds \right| \right\} \right]. \quad (6.13)$$

By the Feynman-Kac formula (cf. Lemma 7.20 [23]), since  $e^{|x|} \leq e^x + e^{-x}$ , to estimate (6.13), it will be enough to bound

$$\frac{C_0}{\gamma} + \frac{1}{\gamma N} \int_0^T \lambda_{N,l}(s) ds \quad (6.14)$$

where  $\lambda_{N,l}(s)$  is the largest eigenvalue of  $N^2 S + \gamma N V_{k,l,A}(s, \eta)$ , and a similar expression where  $V_{k,l,A}$  is replaced by  $-V_{k,l,A}$ . As the argument to bound (6.14) will also hold for the expression with respect to  $-V_{k,l,A}$ , we concentrate in the following on the display (6.14).

Now, fix  $s \in [0, T]$  and omit the argument  $s$  to simplify notation. To estimate the eigenvalue  $\lambda_{N,l}(s)$ , we make use of the variational formula:

$$(\gamma N)^{-1} \lambda_{N,l} = \sup_f \left\{ E_{\mathcal{R}_N} [V_{k,l,A} f] - \gamma^{-1} N E_{\mathcal{R}_N} \left[ \sqrt{f} (-S \sqrt{f}) \right] \right\}$$

where the supremum is over all  $f$  which are densities with respect to  $\mathcal{R}_N$ .

Recall that  $\kappa_{k,l}$  is the restriction of  $\mathcal{R}_N$  to  $\Lambda_{k,l}$ , and that  $S_{k,l}$  is the localized generator. For any density  $f$ , we consider its restriction with respect to configurations sites  $\Lambda_{k,l}$ , i.e. we define  $f_{k,l} = E_{\mathcal{R}_N} [f | \Omega_{k,l}]$ . Notice that  $E_{\mathcal{R}_N} [\sqrt{f} (-S_{k,l} \sqrt{f})] \leq E_{\mathcal{R}_N} [\sqrt{f} (-S \sqrt{f})]$ . By convexity of the Dirichlet form, we have

$$(\gamma N)^{-1} \lambda_{N,l} \leq \sup_{f_{k,l}} \left\{ E_{\kappa_{k,l}} [V_{k,l,A} f_{k,l}] - \gamma^{-1} N E_{\kappa_{k,l}} \left[ \sqrt{f_{k,l}} (-S_{k,l} \sqrt{f_{k,l}}) \right] \right\}.$$

We now write  $f_{k,l} d\kappa_{k,l}$  with respect to sets  $\Omega_{k,l,j}$  of configurations with total particle number  $j$  on  $\Lambda_{k,l}$ :

$$E_{\kappa_{k,l}} [V_{k,l,A} f_{k,l}] = \sum_{j \geq 0} c_{k,l,j}(f) \int V_{k,l,A} f_{k,l,j} d\kappa_{k,l,j}, \quad (6.15)$$

where  $c_{k,l,j}(f) = \int_{\Omega_{k,l,j}} f_{k,l} d\kappa_{k,l}$ , and  $f_{k,l,j} = c_{k,l,j}(f)^{-1} \kappa_{k,l}(\Omega_{k,l,j}) f_{k,l}$ . Here,  $\sum_{j \geq 0} c_{k,l,j} = 1$  and  $f_{k,l,j}$  is a density with respect to  $\kappa_{k,l,j}$ .

Then, on  $\Omega_{k,l,j}$ , we have  $\frac{S_{k,l} \sqrt{f_{k,l}}}{\sqrt{f_{k,l}}} = \frac{S_{k,l} \sqrt{f_{k,l,j}}}{\sqrt{f_{k,l,j}}}$ . By (6.15), we write

$$E_{\kappa_{k,l}} \left[ \sqrt{f_{k,l}} (-S_{k,l} \sqrt{f_{k,l}}) \right] = \sum_{j \geq 0} c_{k,l,j}(f) E_{\kappa_{k,l,j}} \left[ \sqrt{f_{k,l,j}} (-S_{k,l} \sqrt{f_{k,l,j}}) \right].$$

Then, we have

$$(\gamma N)^{-1} \lambda_{N,l} \leq \sup_{0 \leq j \leq A(2l+1)} \sup_f \left\{ E_{\kappa_{k,l,j}} [V_{k,l,A} f] - \gamma^{-1} N E_{\kappa_{k,l,j}} \left[ \sqrt{f} (-S_{k,l} \sqrt{f}) \right] \right\},$$

where the inside supremum is on densities  $f$  with respect to  $\kappa_{k,l,j}$ .

By Lemma 6.1, we have  $C_{k,l,j}$  as the inverse spectral gap estimate of  $S_{k,l}$ . Note also that  $\|V_{k,l,A}\|_\infty \leq C(A,G)$ . Using the Rayleigh estimate (cf. p. 377, [23]), we have

$$\begin{aligned} & E_{\kappa_{k,l,j}} [V_{k,l,A}f] - \gamma^{-1} N E_{\kappa_{k,l,j}} \left[ \sqrt{f} (-S_{k,l} \sqrt{f}) \right] \\ & \leq \frac{\gamma N^{-1}}{1 - 2C(A,G)C_{k,l,j} \gamma N^{-1}} E_{\kappa_{k,l,j}} [V_{k,l,A} (-S_{k,l})^{-1} V_{k,l,A}]. \end{aligned}$$

As remarked in the beginning of the proof,  $E_{\kappa_{k,l,j}} V_{k,l,A} = 0$ . Observe that the spectral gap estimate of  $S_{k,l}$  in Lemma 6.1 also implies that  $\|S_{k,l}^{-1}\|_2$ , the  $L^2(\kappa_{k,l,j})$  norm of the operator  $S_{k,l}^{-1}$  on mean zero functions, is less than or equal to  $C_{k,l,j}$ . Thus, by Cauchy-Schwarz, we have

$$E_{\kappa_{k,l,j}} [V_{k,l,A} (-S_{k,l})^{-1} V_{k,l,A}] \leq C_{k,l,j} E_{\kappa_{k,l,j}} [V_{k,l,A}^2].$$

Retracing the steps, we obtain

$$\mathbb{E}_N \left[ \int_0^T V_{k,l,A}(\eta_s) ds \right] \leq \frac{C_0}{\gamma} + \sup_{0 \leq j \leq A(2l+1)} \frac{T \gamma N^{-1} C_{k,l,j}}{1 - 2C(A,G)C_{k,l,j} \gamma N^{-1}} E_{\kappa_{k,l,j}} [V_{k,l,A}^2].$$

The second term in the last expression vanishes uniformly as  $N \rightarrow \infty$  for  $1 \leq k \leq N$  and  $j \leq A(2l+1)$ . The lemma now is proved by letting  $\gamma \rightarrow \infty$ .  $\square$

## 7. 2-BLOCK ESTIMATE

We now detail the 2-block estimate, in the context of the inhomogeneous environment, following the outline of the 1-block estimate.

Recall the notation  $\Lambda_{k,l}$  from the 1-block estimate. For  $l \geq 1$  and  $k, k'$  such that  $|k - k'| > 2l$  and  $k + l \leq k' - l$ , let  $\Lambda_{k,k',l} = \Lambda_{k,l} \cup \Lambda_{k',l}$ . We introduce the following localized generator  $S_{k,k',l}$  governing a process on  $\Omega_{k,k',l} = \mathbb{N}_0^{\Lambda_{k,k',l}}$ . Inside each block, the process moves as before, but we add an extra bond interaction between sites  $k + l$  and  $k' - l$ :

$$\begin{aligned} S_{k,k',l} f(\eta) &= S_{k,l} f(\eta) + S_{k',l} f(\eta) + \frac{1}{2} g(\eta(k+l)) \mathbf{p}_{k+l,k'-l}^N (f(\eta^{k+l,k'-l}) - f(\eta)) \\ &\quad + \frac{1}{2} g(\eta(k'-l)) \mathbf{p}_{k'-l,k+l}^N (f(\eta^{k'-l,k+l}) - f(\eta)) \end{aligned}$$

where

$$\mathbf{p}_{k+l,k'-l}^N = \frac{1}{2} + \frac{\alpha_{k+l}^N}{N} + \frac{\phi_{k'-l,N}}{\phi_{k+l,N}} \left( \frac{1}{2} - \frac{\alpha_{k'-l}^N}{N} \right), \quad \mathbf{p}_{k'-l,k+l}^N = \frac{1}{2} - \frac{\alpha_{k'-l}^N}{N} + \frac{\phi_{k+l,N}}{\phi_{k'-l,N}} \left( \frac{1}{2} + \frac{\alpha_{k+l}^N}{N} \right).$$

As before, consider the localized product measure  $\kappa_{k,k',l} = \prod_{x \in \Lambda_{k,k',l}} \mathcal{P}_{\phi_{x,N}}$ , namely  $\kappa = \mathcal{R}_N$  limited to sites in  $\Lambda_{k,k',l}$ . Define as well as the canonical measure  $\kappa_{k,k',l,j}$  on  $\Omega_{k,k',l,j} := \{\eta \in \Omega_{k,k',l} : \sum_{x \in \Lambda_{k,k',l}} \eta(x) = j\}$ , that is  $\kappa_{k,k',l}$  is conditioned so that there are exactly  $j$  particles counted in  $\Omega_{k,k',l}$ . With the choice of extra rates  $\mathbf{p}_{k+l,k'-l}^N$  and  $\mathbf{p}_{k'-l,k+l}^N$ , both measures are invariant and reversible for the dynamics with Markov generator  $S_{k,k',l}$ .

The corresponding Dirichlet form, with measure  $\tilde{\kappa}$  given by  $\kappa_{k,k',l}$  or  $\kappa_{k,k',l,j}$ , is given by

$$\begin{aligned} E_{\tilde{\kappa}} [f(-S_{k,k',l} f)] &= \frac{1}{2} \sum_{x,x+1 \in \Lambda_{k,k',l}} E_{\tilde{\kappa}} \left[ g(\eta(x)) \mathbf{p}_{x,x+1}^N (f(\eta^{x,x+1}) - f(\eta))^2 \right] \\ &\quad + \frac{1}{2} E_{\tilde{\kappa}} \left[ g(\eta(k+l)) \mathbf{p}_{k+l,k'-l}^N (f(\eta^{k+l,k'-l}) - f(\eta))^2 \right]. \end{aligned} \tag{7.1}$$

Recall also the generator of symmetric zero-range  $L_l$  with respect to  $\Lambda_{k,l}$  (cf. below (6.4)). Let  $L'_l$  be the corresponding generator with respect to  $\Lambda_{k',l}$ . Define the generator  $L_{l,l}$  with respect to  $\Lambda_{k,k',l}$  by

$$\begin{aligned} L_{l,l}f(\eta) &= L_l f(\eta) + L'_l f(\eta) + \frac{1}{2} \left[ f(\eta^{k+l,k'-l}) - f(\eta) \right] g(\eta(k+l)) \\ &\quad + \frac{1}{2} \left[ f(\eta^{k'-l,k+l}) - f(\eta) \right] g(\eta(k'-l)). \end{aligned}$$

When  $|k-k'|$  is large, the process governed by  $L_{l,l}$  is the same as if the blocks were adjacent, with a connecting bond, and so no longer depends on  $k, k'$  but only on the width  $l$ .

Corresponding to the set-up of the gap bound Lemma 6.1, let  $\nu_{l,l}^\rho$  be the product of  $4l+2$  distributions with common marginal  $\rho$ . One may inspect that  $\nu_{l,l}^\rho$  is invariant to the dynamics generated by  $L_{l,l}$ . Let now  $\nu_{l,l,j}$  be  $\nu_{l,l}^\rho$  conditioned on that the total number of particles in the  $4l+2$  sites is  $j$ . Note that  $\nu_{l,l,j}$  is independent of  $\rho$ . This canonical measure  $\nu_{l,l,j}$  is also invariant to the dynamics. The corresponding Dirichlet form is given by

$$\begin{aligned} E_{\nu_{l,l,j}} [f(-L_{l,l}f)] &= \sum_{x,x+1 \in \Lambda_{k,k',l}} E_{\nu_{l,l,j}} \left[ \frac{1}{2} g(\eta(x)) [f(\eta^{x,x+1}) - f(\eta)]^2 \right] \\ &\quad + E_{\nu_{l,l,j}} \left[ \frac{1}{2} g(\eta(k'-l)) \left[ f(\eta^{k'-l,k+l}) - f(\eta) \right]^2 \right]. \end{aligned}$$

Let  $r_{k,k',l,N}^{-1} := \min \{ \mathfrak{p}_{k+l,k'-l}^N, \min_{x,x+1 \in \Lambda_{k,k',l}} \{ \mathfrak{p}_{x,+}^N \} \}$ .

**Lemma 7.1.** *We have the following estimates:*

(1) *Uniform bound: For all  $\eta \in \Omega_{k,k',l,j}$ , we have*

$$\left( \frac{\phi_{\min,k,k',l}}{\phi_{\max,k,k',l}} \right)^j \leq \frac{\kappa_{k,k',l,j}(\eta)}{\nu_{l,l,j}(\eta)} \leq \left( \frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}} \right)^j \quad (7.2)$$

where  $\phi_{\min,k,k',l} = \min_{x \in \Lambda_{k,k',l}} \phi_{x,N}$  and  $\phi_{\max,k,k',l} = \max_{x \in \Lambda_{k,k',l}} \phi_{x,N}$ .

(2) *Poincaré inequality: For fixed  $j \geq 0$  and  $k, k'$  such that  $|k-k'| > 2l+1$ , we have*

$$\text{Var}_{\kappa_{k,k',l,j}}(f) \leq C_{k,k',l,j} E_{\kappa_{k,k',l,j}} [f(-S_{k,k',l}f)] \quad (7.3)$$

where  $C_{k,k',l,j} = C(4l+2)^2 r_{k,k',l,N} \left( \frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}} \right)^{2j}$  for a universal constant  $C$ .

(3) *For each  $l$  fixed, there exists a constant  $C_0$  such that*

$$\sup_{k,k',N} \frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}} \leq C_0 \quad \limsup_{N \uparrow \infty} \sup_{k',k} r_{k,k',l,N} \leq C_0.$$

Hence, for fixed  $j$  and  $l$ , we have  $\limsup_{\theta \downarrow 0} \limsup_{N \uparrow \infty} \sup_{2l+1 \leq |k'-k| \leq \theta N} C_{k,k',l,j} < \infty$ .

*Proof.* We will compare  $\kappa_{k,k',l,j}$  with  $\nu_{l,l,j}$  and make use of the known Poincaré bound, as in the proof of Lemma 6.1:

$$\text{Var}_{\nu_{l,l,j}}(f) \leq C(4l+2)^2 E_{\nu_{l,l,j}} [f(-L_{l,l}f)], \quad (7.4)$$

where  $C$  is some universal constant.

For  $\eta \in \Omega_{k,k',l,j}$ , we have

$$\frac{\kappa_{k,k',l,j}(\eta)}{\nu_{l,l,j}(\eta)} = \frac{\kappa_{k,k',l}(\eta)}{\nu_{l,l}^\rho(\eta)} \frac{\nu_{l,l}^\rho(\Omega_{k,k',l,j})}{\kappa_{k,k',l}(\Omega_{k,k',l,j})}.$$

Since  $\kappa_{k,k',l}$  and  $\nu_{l,l}^\rho$  are product measures, that is,

$$\begin{aligned}\kappa_{k,k',l}(\eta) &= \prod_{x \in \Lambda_{k,k',l}} (\phi_{x,N})^{\eta(x)} / [Z(\phi_{x,N})g(\eta(x))!] \\ \nu_{l,l}^\rho(\eta) &= \prod_{x \in \Lambda_{k,k',l}} (\phi_0)^{\eta(x)} / [Z(\phi_0)g(\eta(x))!],\end{aligned}\tag{7.5}$$

we have for  $\eta \in \Omega_{k,l,j}$  that

$$\left(\frac{\phi_{\min,k,k',l}}{\phi_0}\right)^j \prod_{x \in \Lambda_{k,k',l}} \frac{Z(\phi_{x,N})}{Z(\phi_0)} \leq \frac{\kappa_{k,k',l}(\eta)}{\nu_{l,l}^\rho(\eta)} \leq \left(\frac{\phi_{\max,k,k',l}}{\phi_0}\right)^j \prod_{x \in \Lambda_{k,k',l}} \frac{Z(\phi_{x,N})}{Z(\phi_0)}.$$

Consequently,

$$\left(\frac{\phi_{\min,k,k',l}}{\phi_0}\right)^j \prod_{x \in \Lambda_{k,k',l}} \frac{Z(\phi_{x,N})}{Z(\phi_0)} \leq \frac{\kappa_{k,k',l}(\Omega_{k,k',l,j})}{\nu_{l,l}^\rho(\Omega_{k,k',l,j})} \leq \left(\frac{\phi_{\max,k,k',l}}{\phi_0}\right)^j \prod_{x \in \Lambda_{k,k',l}} \frac{Z(\phi_{x,N})}{Z(\phi_0)}.$$

Therefore, with respect to the conditioned measures, we obtain (7.2).

Then, as in the proof of Lemma 6.1, we have

$$E_{\nu_{l,l,j}} [f(-L_{l,l}f)] \leq r_{k,k',l,N} \left(\frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}}\right)^j E_{\kappa_{k,k',l,j}} [f(-S_{k,k',l}f)].\tag{7.6}$$

Also, in turn,

$$\begin{aligned}\text{Var}_{\kappa_{k,k',l,j}}(f) &= \inf_a E_{\kappa_{k,k',l,j}} [(f-a)^2] \leq \left(\frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}}\right)^j \inf_a E_{\nu_{l,l,j}} [(f-a)^2] \\ &= \left(\frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}}\right)^j \text{Var}_{\nu_{l,l,j}}(f).\end{aligned}$$

The spectral gap estimate (7.3) now follows from (7.4) and (7.6).

Finally, as in the proof of Lemma 6.1, item (3) follows from Lemma 2.2 and that  $\sup_N \sup_{1 \leq x \leq N} \alpha_x^N < C$ .  $\square$

We now state a 2-block estimate.

**Lemma 7.2** (2-block estimate). *We have*

$$\limsup_{l \rightarrow \infty} \limsup_{\theta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \frac{1}{N} \sum_{1 \leq k \leq N} \int_0^T D_{N,k}^{G,s} \left( \Phi(\eta_s^l(k)) - \Phi(\eta_s^{\theta N}(k)) \right) ds \right] = 0.\tag{7.7}$$

*Proof.* We separate the argument into steps.

*Step 1.* Since  $\Phi(\cdot)$  is Lipschitz on  $\mathbb{R}^+$  and  $D_{N,k}^{G,s}$  is bounded (cf. (4.3)), it is enough to show

$$\limsup_{l \rightarrow \infty} \limsup_{\theta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \frac{1}{N} \sum_{1 \leq k \leq N} \int_0^T |\eta_s^{\theta N}(k) - \eta_s^l(k)| ds \right] = 0.$$

By the triangle inequality, it will be enough to show that, as  $N \uparrow \infty$ ,  $\theta \downarrow 0$ , and  $l \uparrow \infty$ ,

$$\begin{aligned}\mathbb{E}_N \left[ \int_0^T \frac{1}{N} \sum_{1 \leq k \leq N} \left| \eta_s^{\theta N}(k) - \frac{1}{2\theta N + 1} \sum_{|x-k| \leq \theta N} \eta_s^l(x) \right| ds \right] &\rightarrow 0 \quad \text{and} \\ \mathbb{E}_N \left[ \int_0^T \frac{1}{N} \sum_{1 \leq k \leq N} \left| \frac{1}{2\theta N + 1} \sum_{|x-k| \leq \theta N} \eta_s^l(x) - \eta_s^l(k) \right| ds \right] &\rightarrow 0.\end{aligned}\tag{7.8}$$

*Step 2.* Note that

$$\begin{aligned} \left| \eta^{\theta N}(k) - \frac{1}{2\theta N + 1} \sum_{|x-k| \leq \theta N} \eta^l(x) \right| &\leq \frac{1}{2\theta N + 1} \sum_{\substack{|x-k-\theta N| \leq l \\ \text{or } |x-k+\theta N| \leq l}} \eta(x) \\ &= \frac{2l+1}{2\theta N + 1} (\eta^l(k - \theta N) + \eta^l(k + \theta N)). \end{aligned}$$

Then, the first limit in (7.8), as  $\sum_k \eta_s^l(k \pm \theta N) = \sum_k \eta_0(k)$ , vanishes as  $N \rightarrow \infty$  for fixed  $l$ , by the estimate (3.1). By a similar argument, we can restrain the  $x$  in the summation of the second limit in (7.8) to  $k'$  such that  $2l+1 \leq |k' - k| \leq \theta N$ . Then, the second limit will follow if we show that

$$\limsup_{l \rightarrow \infty} \limsup_{\theta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \int_0^T \frac{1}{N} \sum_{1 \leq k \leq N} \left| \frac{1}{2\theta N + 1} \sum_{2l+1 \leq |x-k| \leq \theta N} \eta_s^l(x) - \eta_s^l(k) \right| ds \right] = 0.$$

*Step 3.* We will apply a cutoff of large densities first. Let

$$\eta_s^l(k, k') = \eta_s^l(k) + \eta_s^l(k').$$

The same argument as for the cutoff in Lemma 6.3 (cf. p. 92, [23]) gives, as  $A \uparrow \infty$ ,

$$\limsup_{l \uparrow \infty} \limsup_{\theta \downarrow 0} \limsup_{N \uparrow \infty} \sup_{2l+1 \leq |y| \leq \theta N} \mathbb{E}_N \left[ \int_0^T \frac{1}{N} \sum_{1 \leq k \leq N} \eta_s^l(k, k+y) \mathbb{1}_{\eta_s^l(k, k+y) > A} ds \right] \rightarrow 0.$$

Hence, it remains to show that, for any fixed  $A$ ,

$$\sup_{\substack{1 \leq k < k' \leq N \\ 2l+1 \leq |k' - k| \leq \theta N}} \mathbb{E}_N \left[ \int_0^T \left| \eta_s^l(k) - \eta_s^l(k') \right| \mathbb{1}_{\{\eta_s^l(k, k') \leq A\}} ds \right]$$

vanishes as we take  $N \rightarrow \infty$ ,  $\theta \rightarrow 0$ , and then  $l \rightarrow \infty$ .

*Step 4.* Let

$$V_{k, k', l, A}(\eta) := \left| \eta^l(k) - \eta^l(k') \right| \mathbb{1}_{\{\eta^l(k, k') \leq A\}}.$$

Following the proof of Lemma 6.3, for fixed  $l, \theta, N, k, k'$ , in order to estimate

$$\mathbb{E}_N \left[ \int_0^T V_{k, k', l, A}(\eta_s) ds \right]$$

it suffices to bound

$$(\gamma N)^{-1} \lambda_{N, l} = \sup_f \left\{ E_{\mathcal{R}_N} [V_{k, k', l, A} f] - \gamma^{-1} N E_{\mathcal{R}_N} \left[ \sqrt{f} (-S \sqrt{f}) \right] \right\}. \quad (7.9)$$

where the supremum is over all  $f$  which are densities with respect to  $\mathcal{R}_N$ .

*Step 5.* Recall the generator  $S_{k, k', l}$  and its Dirichlet form defined in the beginning of this subsection. Recall also  $\kappa_{k, k', l}$  is the restriction of  $\kappa = \mathcal{R}_N$  to  $\Lambda_{k, k', l}$ . The Dirichlet form with respect to the full generator  $S$  under  $\mathcal{R}_N$  is given by

$$\begin{aligned} E_{\mathcal{R}_N} [f(-Sf)] \\ = \sum_{1 \leq k \leq N} E_{\mathcal{R}_N} \left[ g(\eta(k)) \frac{1}{2} \left[ \left( \frac{1}{2} + \frac{\alpha_k^N}{N} \right) + \frac{\phi_{k+1, N}}{\phi_{k, N}} \left( \frac{1}{2} - \frac{\alpha_{k+1}^N}{N} \right) \right] (f(\eta^{k, k+1}) - f(\eta))^2 \right]. \end{aligned}$$

We now argue the following Dirichlet form inequality, for some  $C$ ,

$$E_{\kappa_{k, k', l}} \left[ \sqrt{f} (-S_{k, k', l} \sqrt{f}) \right] \leq C(1 + \theta N) E_{\kappa_{k, k', l}} \left[ \sqrt{f} (-S \sqrt{f}) \right]. \quad (7.10)$$

First, writing out the Dirichlet form in (7.1), in terms of the product measure  $\mathcal{R}_N$ , we have

$$\begin{aligned} & E_{\kappa_{k,k',l}} [f(-S_{k,k',l}f)] \\ &= \sum_{x,x+1 \in \Lambda_{k,k',l}} E_{\mathcal{R}_N} \left[ g(\eta(x)) \frac{1}{2} \left[ \left( \frac{1}{2} + \frac{\alpha_x^N}{N} \right) + \frac{\phi_{x+1,N}}{\phi_{x,N}} \left( \frac{1}{2} - \frac{\alpha_{x+1}^N}{N} \right) \right] (f(\eta^{x,x+1}) - f(\eta))^2 \right] \\ & \quad + E_{\mathcal{R}_N} \left[ g(\eta(k+l)) \frac{1}{2} \left[ \left( \frac{1}{2} + \frac{\alpha_{k+l}^N}{N} \right) + \frac{\phi_{k'+l,N}}{\phi_{k+l,N}} \left( \frac{1}{2} - \frac{\alpha_{k'+l}^N}{N} \right) \right] (f(\eta^{k+l,k'+l}) - f(\eta))^2 \right]. \end{aligned}$$

Next, by adding and subtracting at most  $\theta N$  terms, we have

$$\begin{aligned} & \left[ f(\eta^{k+l,k'+l}) - f(\eta) \right]^2 \\ & \leq (k' - k - 2l) \sum_{q=0}^{k'-k-2l-1} \left[ f(\eta^{k+l,k+l+q+1}) - f(\eta^{k+l,k+l+q}) \right]^2. \end{aligned}$$

By the change of variables  $\xi = \eta^{k+l,k+l+q}$ , which takes away a particle at  $k+l$  and adds one at  $k+l+q$ , we have (cf. (7.5)) that

$$\mathcal{R}_N(\eta) = \frac{\phi_{k+l,N}}{\phi_{k+l+q,N}} \frac{g(\eta(k+l+q)+1)}{g(\eta(k+l))} \mathcal{R}_N(\xi).$$

Also as  $\frac{\phi_{\max,N}}{\phi_{\min,N}}$  is uniformly bounded from above and below, and  $\sup_N \sup_{1 \leq x \leq N} \alpha_x^N < \infty$ , we may find a constant  $C$  such that

$$\begin{aligned} \mathfrak{p}_{k+l,k'-l}^N &= \left( \frac{1}{2} + \frac{\alpha_{k+l}^N}{N} \right) + \frac{\phi_{k'+l,N}}{\phi_{k+l,N}} \left( \frac{1}{2} - \frac{\alpha_{k'+l}^N}{N} \right) \\ &\leq C \left\{ \left( \frac{1}{2} + \frac{\alpha_{k+l+q}^N}{N} \right) + \frac{\phi_{k+l+q+1,N}}{\phi_{k+l+q,N}} \left( \frac{1}{2} - \frac{\alpha_{k+l+q+1}^N}{N} \right) \right\}. \end{aligned}$$

Then, taking into account the above observations, we have

$$\begin{aligned} & E_{\mathcal{R}_N} \left[ g(\eta(k+l)) \frac{1}{2} \mathfrak{p}_{k+l,k'-l}^N \left[ f(\eta^{k+l,k+l+q+1}) - f(\eta^{k+l,k+l+q}) \right]^2 \right] \\ &= \sum_{\xi} \mathcal{R}_N(\eta) g(\eta(k+l)) \frac{1}{2} \mathfrak{p}_{k+l,k'-l}^N \left[ f(\xi^{k+l+q,k+l+q+1}) - f(\xi) \right]^2 \\ &\leq C E_{\mathcal{R}_N} \left[ g(\eta(k+l+q)) \frac{1}{2} \left( \frac{1}{2} + \frac{\alpha_{k+l+q}^N}{N} \right) + \frac{\phi_{k+l+q+1}}{\phi_{k+l+q}} \left( \frac{1}{2} - \frac{\alpha_{k+l+q+1}^N}{N} \right) \right. \\ & \quad \left. \times \left[ f(\eta^{k+l+q,k+l+q+1}) - f(\eta) \right]^2 \right]. \end{aligned}$$

From these observations, (7.10) follows.

*Step 6.* Inputting (7.10) into (7.9), and considering the conditional expectation of  $f$  with respect to  $\Omega_{k,k',l}$  as in the 1-block estimate proof, for  $N$  large, we have

$$\begin{aligned} & (\gamma N)^{-1} \lambda_{N,l} \\ & \leq \sup_{f_{k,k',l}} \left\{ E_{\kappa_{k,k',l}} [V_{k,k',l,A} f_{k,k',l}] - \frac{1}{2\theta\gamma} E_{\kappa_{k,k',l}} \left[ \sqrt{f_{k,k',l}} (-S_{k,k',l} \sqrt{f_{k,k',l}}) \right] \right\}, \end{aligned}$$

where the supremum is over densities  $f_{k,k',l}$  with respect to  $\kappa_{k,k',l}$ .

Again, as in the proof of the 1-block estimate, decomposing  $f_{k,k',l}d\mu_{k,k',l}$  along configurations with common total number  $j$ , we need only to bound

$$\sup_{0 \leq j \leq A(2l+1)} \sup_f \left\{ E_{\kappa_{k,k',l,j}} [V_{k,k',l,A} f] - \frac{1}{2\theta\gamma} E_{\kappa_{k,k',l,j}} \left[ \sqrt{f}(-S_{k,k',l}\sqrt{f}) \right] \right\},$$

where the supremum is over densities  $f$  with respect to  $\kappa_{k,k',l,j}$ .

*Step 7.* Consider the centered object

$$\widehat{V}_{k,k',l,A} = V_{k,k',l,A} - E_{\kappa_{k,k',l,j}} [V_{k,k',l,A}].$$

Using the Rayleigh expansion (cf. p.375, [23]) where the inverse spectral gap is bounded by  $C_{k,k',l,j}$  of  $S_{k,k',l}$  (Lemma 7.1), and  $\|\widehat{V}_{k,k',l,A}\|_\infty \leq A$ , we have

$$\begin{aligned} & E_{\kappa_{k,k',l,j}} \left[ \widehat{V}_{k,k',l,A} f \right] - \frac{1}{2\theta\gamma} E_{\kappa_{k,k',l,j}} \left[ \sqrt{f}(-S_{k,k',l}\sqrt{f}) \right] \\ & \leq \frac{2\theta\gamma}{1 - 4AC_{k,k',l,j}\theta\gamma} E_{\mu_{k,k',l,j}} \left[ \widehat{V}_{k,k',l,A} (-S_{k,k',l})^{-1} \widehat{V}_{k,k',l,A} \right] \\ & \leq \frac{2\theta\gamma C_{k,k',l,j}}{1 - 4AC_{k,k',l,j}\theta\gamma} E_{\kappa_{k,k',l,j}} \left[ \widehat{V}_{k,k',l,A}^2 \right] \rightarrow 0 \text{ as } \theta \rightarrow 0. \end{aligned}$$

*Step 8.* To finish, we still need to estimate the centering term  $E_{\kappa_{k,k',l,j}} [V_{k,k',l,A}]$ . By Lemma 7.1,

$$E_{\kappa_{k,k',l,j}} [V_{k,k',l,A}] \leq C_0^j E_{\nu_{l,j}} [V_{k,k',l,A}].$$

Note that this bound of  $E_{\nu_{l,j}} [V_{k,k',l,A}]$  does not depend on  $N$  or  $\theta$ . By adding and subtracting  $j/(2(2l+1))$ , we need only bound  $E_{\nu_{l,j}} [|\eta^l(k) - j/(2(2l+1))|]$ . By exchangeability and an equivalence of ensemble estimate (cf. p. 355 [23]), the canonical variance

$$\begin{aligned} E_{\nu_{l,j}} \left[ \left| \eta^l(k) - j/(2(2l+1)) \right|^2 \right] &= O(l^{-1}) E_{\nu_{l,j}} \left[ (\eta(k) - j/(2(2l+1)))^2 \right] \\ &+ O(1) E_{\nu_{l,j}} \left[ (\eta(k) - j/(2(2l+1)))(\eta(k+1) - j/(2(2l+1))) \right] \end{aligned}$$

and further bounded by  $C(A) \text{Var}_{\nu_{l,l}^{j/(2(2l+1))}} (\eta^l(k))$  (recall  $\nu_{l,l}^o$  defined before Lemma 7.1). This variance is of order  $O(l^{-1})$ , since the single site variance  $\text{Var}_{\nu_{l,l}^{j/(2(2l+1))}} (\eta(k))$  is uniformly bounded for  $j/(2(2l+1)) \leq A$ . Hence,  $\limsup_{N \uparrow \infty, \theta \downarrow 0} E_{\nu_{l,j}} [V_{k,k',l,A}]$  is of order  $O(l^{-1/2})$ , vanishing as  $l \uparrow \infty$ . This finishes the proof.  $\square$

## 8. TIGHTNESS OF LIMIT MEASURES

In this section, we obtain tightness of the family of probability measures  $\{Q^N\}_{N \in \mathbb{N}}$  on the trajectory space  $D([0, T], \mathcal{M})$ . We show that  $\{Q^N\}$  is tight with respect to the uniform topology, stronger than the Skorokhod topology on  $D([0, T], \mathcal{M})$ .

**Lemma 8.1.**  *$\{Q^N\}_{N \in \mathbb{N}}$  is relatively compact with respect to the uniform topology. As a consequence, all limit points  $Q$  are supported on weakly continuous trajectories  $\pi$ , that is for  $G \in C^\infty(\mathbb{T})$  we have  $t \in [0, T] \mapsto \langle G, \pi_t \rangle$  is continuous.*

*Proof.* To deduce that  $\{Q^N\}$  is relatively compact with respect to uniform topology, we show the following items (cf. Theorem 15.5 in [3]).

- (1) For each  $t \in [0, T]$ ,  $\epsilon > 0$ , there exists a compact set  $K_{t,\epsilon} \subset \mathcal{M}$  such that

$$\sup_N Q^N \left[ \pi_t^N : \pi_t^N \notin K_{t,\epsilon} \right] \leq \epsilon. \quad (8.1)$$

(2) For every  $\epsilon > 0$ ,

$$\lim_{\gamma \rightarrow 0} \limsup_{N \rightarrow \infty} Q^N \left[ \pi^N : \sup_{|t-s| < \gamma} d(\pi_t^N, \pi_s^N) > \epsilon \right] = 0. \quad (8.2)$$

We now consider (8.1). Notice that, for any  $A > 0$ , the set  $\{\mu \in \mathcal{M} : \langle 1, \mu \rangle \leq A\}$  is compact in  $\mathcal{M}$ . Since the total number of particles is conserved, we have  $Q^N[\langle 1, \pi_t^N \rangle > A] = Q^N[\langle 1, \pi_0^N \rangle > A] \leq \frac{1}{A} \mathbb{E}_N \left[ \frac{1}{N} \sum_{k=1}^N \eta_0(k) \right]$ . By (3.1), we have  $\mathbb{E}_N \left[ \frac{1}{N} \sum_{k=1}^N \eta_0(k) \right] \leq C$  for some constant  $C < \infty$  independent of  $N$  and  $A$ . Then, the first condition (8.1) is checked by taking  $A$  large.

To verify the second condition (8.2), it is enough to show a counterpart of the condition for the distributions of  $\langle G, \pi_t^N \rangle$  where  $G$  is any smooth test function on  $\mathbb{T}$  (cf. p. 54, [23]). In other words, we need to show, for every  $\epsilon > 0$ ,

$$\lim_{\gamma \rightarrow 0} \lim_{N \rightarrow \infty} Q^N \left[ \pi^N : \sup_{|t-s| < \gamma} \left| \langle G, \pi_t^N \rangle - \langle G, \pi_s^N \rangle \right| > \epsilon \right] = 0. \quad (8.3)$$

To this end, notice that  $\langle G, \pi_t^N \rangle = \langle G, \pi_0^N \rangle + \int_0^t N^2 L \langle G, \pi_s^N \rangle ds + M_t^{N,G}$ , then we only need to consider the oscillations of  $\int_0^t N^2 L \langle G, \pi_s^N \rangle ds$  and  $M_t^{N,G}$  respectively.

Recall the generator computation (4.1) and the notation  $D_{N,k}^{G,s}$  in (4.2). As  $g$  grows at most linearly and  $D_{N,k}^{G,s}$  is bounded (cf. (4.3)), we have

$$\begin{aligned} \sup_{|t-s| < \gamma} \left| \int_s^t N^2 L \langle G, \pi_\tau^N \rangle d\tau \right| &= \sup_{|t-s| < \gamma} \left| \int_s^t \frac{1}{N} \sum_{1 \leq k \leq N} D_{N,k}^{G,s} g(\eta_\tau(k)) d\tau \right| \\ &\leq g^* C_G \sup_{|t-s| < \gamma} \int_s^t \left\{ \frac{1}{N} \sum_{1 \leq k \leq N} \eta_\tau(k) \right\} d\tau = g^* C_G \gamma \frac{1}{N} \sum_{1 \leq k \leq N} \eta_0(k). \end{aligned}$$

Recall that  $\mathbb{E}_N \left[ \frac{1}{N} \sum_{k=1}^N \eta_0(k) \right]$  is uniformly bounded in  $N$ . Then, by Markov inequality, we conclude that  $Q^N \left[ \sup_{|t-s| < \gamma} \left| \int_s^t N^2 L \langle G, \pi_\tau^N \rangle d\tau \right| > \epsilon \right]$  vanishes as  $N \uparrow \infty$  and  $\gamma \downarrow 0$ .

We turn to the martingale  $M_t^{N,G}$ . By  $|M_t^{N,G} - M_s^{N,G}| \leq |M_t^{N,G}| + |M_s^{N,G}|$ , we have  $\mathbb{P}_N \left[ \sup_{|t-s| < \gamma} |M_t^{N,G} - M_s^{N,G}| > \epsilon \right] \leq 2\mathbb{P}_N \left[ \sup_{0 \leq t \leq T} |M_t^{N,G}| > \epsilon/2 \right]$ . Using Chebyshev and Doob's inequality, we further bound it by

$$\frac{8}{\epsilon^2} \mathbb{E}_N \left[ \left( \sup_{0 \leq t \leq T} |M_t^{N,G}| \right)^2 \right] \leq \frac{32}{\epsilon^2} \mathbb{E}_N \left[ (M_T^{N,G})^2 \right] = \frac{32}{\epsilon^2} \mathbb{E}_N \langle M^{N,G} \rangle_T.$$

By Lemma 4.1,  $\mathbb{E}_N \langle M^{N,G} \rangle_T = O(N^{-1})$ . Then, we conclude

$$\lim_{\gamma \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{P}_N \left[ \sup_{|t-s| < \gamma} |M_t^{N,G} - M_s^{N,G}| > \epsilon \right] = 0. \quad \square$$

## 9. PROPERTIES OF LIMIT MEASURES.

By Lemma 8.1, the sequence  $\{Q^N\}$  is relatively compact with respect to the uniform topology. Consider any convergent subsequence of  $Q^N$  and relabel so that  $Q^N \Rightarrow Q$ . We now consider absolute continuity and an energy estimate for trajectories under  $Q$ .

**9.1. Absolute continuity.** We now address absolute continuity and conservation of mass properties under  $Q$ .

**Lemma 9.1.**  *$Q$  is supported on absolutely continuous trajectories: We have  $Q$ -a.s., for all  $0 \leq t \leq T$ , that  $\pi_t(dx) = \rho(t, x)dx$  where  $\int_{\mathbb{T}} \rho(t, x)dx = \int_{\mathbb{T}} \rho(0, x)dx$  with respect to a measurable, nonnegative  $\rho$ .*

*Proof.* A standard proof, namely that of Lemma 1.6, p. 73, [23], shows the first statement. The second follows directly from the weak convergence of  $Q^N$  to  $Q$  and the conservation of mass  $\sum_{x \in \mathbb{T}_N} \eta_t(x) = \sum_{x \in \mathbb{T}_N} \eta_0(x)$ .  $\square$

**9.2. Energy estimate.** We now state an important ‘energy estimate’ for the paths on which  $Q$  is supported. We follow the framework presented in Section 5.7 of [23], however, there are major differences due to the inhomogeneous random environment. Previous bounds on the random environment developed in Section 2.1 will be useful in the argument.

**Proposition 9.2.**  *$Q$  is supported on paths  $\rho(t, x)dx$  with the property that there exists an  $L^1([0, T] \times \mathbb{T})$  function denoted by  $\partial_x \Phi(\rho(s, x))$  such that*

$$\int_0^T \int_{\mathbb{T}} \partial_x G(s, x) \Phi(\rho(s, x)) dx ds = - \int_0^T \int_{\mathbb{T}} G(s, x) \partial_x \Phi(\rho(s, x)) dx ds$$

for all  $G$  smooth on  $[0, T] \times \mathbb{T}$ .

A main ingredient for the proof of Proposition 9.2 is the following lemma. For  $\epsilon > 0$ ,  $\delta > 0$ ,  $H(\cdot) \in C^1(\mathbb{T})$  and  $N \in \mathbb{N}$ , we define

$$\begin{aligned} W_N(\epsilon, \delta, H, \eta) &:= \sum_{1 \leq x \leq N} \frac{H(x/N)}{\epsilon N} [\Phi(\eta^{\delta N}(x)) - \Phi(\eta^{\delta N}(x + \epsilon N))] \\ &- \frac{4}{c^2 N} \sum_{1 \leq x \leq N} \frac{H^2(x/N)}{\epsilon N} \sum_{0 \leq k \leq \epsilon N} \Phi(\eta^{\delta N}(x + k)) - \sum_{1 \leq x \leq N} \frac{CH(x/N)}{cN} \Phi(\eta^{\delta N}(x)). \end{aligned}$$

Here, the constants  $c$  and  $C$ , as we recall from Lemma 2.2, come from the inequalities

$$0 < c \leq \min_k \phi_{k, N} \leq \max_k \phi_{k, N} \leq 1, \quad \text{and} \quad \max_k |\phi_{k, N} - \phi_{k+1, N}| \leq \frac{C}{N}.$$

**Lemma 9.3.** *Let  $\{H_j\}_{j \in \mathbb{N}}$  be a dense sequence in  $C^{0,1}([0, T] \times \mathbb{T})$ . Then, there exists a constant  $K_0$  such that for any  $m \geq 1$ , and  $\epsilon > 0$ ,*

$$\limsup_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \max_{1 \leq j \leq m} \left\{ \int_0^T W_N(\epsilon, \delta, H_j(s, \cdot), \eta_s) ds \right\} \right] \leq K_0.$$

Before going to the proof of the lemma, we turn to Proposition 9.2.

*Proof of Proposition 9.2.* It follows from Lemma 9.3 that

$$\begin{aligned} E_Q \left[ \sup_{H \in C^{0,1}([0, T] \times \mathbb{T})} \left\{ \int_0^T \int_{\mathbb{T}} \partial_x H(s, x) \Phi(\rho(s, x)) dx ds \right. \right. \\ \left. \left. - \frac{4}{c^2} \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds - \frac{C}{c} \int_0^T \int_{\mathbb{T}} H(s, x) \Phi(\rho(s, x)) dx ds \right\} \right] \leq K_0, \end{aligned}$$

cf. p. 103, Lemma 7.2 in [23]. As a result, for  $Q$ -a.e. path  $\rho(s, u)du$ , there exists  $B = B(\rho)$  such that, for all  $H \in C^{0,1}([0, T] \times \mathbb{T})$ ,

$$\begin{aligned} \int_0^T \int_{\mathbb{T}} \partial_x H(s, x) \Phi(\rho(s, x)) dx ds - \frac{4}{c^2} \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds \\ - \frac{C}{c} \int_0^T \int_{\mathbb{T}} H(s, x) \Phi(\rho(s, x)) dx ds \leq B, \end{aligned}$$

Notice that

$$\begin{aligned} \int_0^T \int_{\mathbb{T}} H(s, x) \Phi(\rho(s, x)) dx ds &\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds + \frac{1}{2} \int_0^T \int_{\mathbb{T}} \Phi(\rho(s, x)) dx ds \\ &\leq \frac{1}{2} \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds + \frac{g^*}{2} \int_0^T \int_{\mathbb{T}} \rho(s, x) dx ds \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds + \frac{g^* T}{2} \int_{\mathbb{T}} \rho(0, x) dx. \end{aligned}$$

We obtain

$$\int_0^T \int_{\mathbb{T}} \partial_x H(s, x) \Phi(\rho(s, x)) dx ds - C' \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds \leq B',$$

where  $C' = \frac{4}{c^2} + \frac{C}{2c}$  and  $B' = \frac{Cg^*T \int_{\mathbb{T}} \rho(0, x) dx}{2c} + B$ . Now, the proof follows exactly from proof of Theorem 7.1, p. 105, [23].  $\square$

We now return to the proof of Lemma 9.3.

*Proof of Lemma 9.3.* By the replacement lemma (Lemma 5.1, and notice that  $D_{N,k}^{G,t}$  can be replaced by any bounded function), it suffices to show that there exists constant  $K_0$  such that for any  $m \geq 1$  and  $\epsilon > 0$

$$\limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \max_{1 \leq j \leq m} \left\{ \int_0^T \tilde{W}_N(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \right] \leq K_0 \quad (9.1)$$

where

$$\begin{aligned} \tilde{W}_N(\epsilon, H(\cdot), \eta) &:= \sum_{1 \leq x \leq N} \frac{H(x/N)}{\epsilon N} (g(\eta(x)) - g(\eta(x + \epsilon N))) \\ &- \frac{4}{c^2 N} \sum_{1 \leq x \leq N} \frac{H^2(x/N)}{\epsilon N} \sum_{0 \leq k \leq \epsilon N} g(\eta(x + k)) - \sum_{1 \leq x \leq N} \frac{CH(x/N)}{cN} g(\eta(x)). \end{aligned}$$

Let  $\mathbb{1}_{A,N}(\eta) := \mathbb{1}_{\sum_{k=1}^N \eta(k) \leq AN}$ . Define  $W_{A,N}(\epsilon, H(\cdot), \eta) := \tilde{W}_N(\epsilon, H(\cdot), \eta) \mathbb{1}_{A,N}(\eta)$ .

As in the beginning of proof of the 1-block estimate, stated in Lemma 6.3, where we cut off high densities, assertion (9.1) holds provided we prove that

$$\limsup_{N \rightarrow \infty} \mathbb{E}_N \left[ \max_{1 \leq j \leq m} \left\{ \int_0^T W_{A,N}(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \right] \leq K_0. \quad (9.2)$$

To this end, by the entropy inequality, the expectation in (9.2) is bounded from above by

$$\frac{1}{N} H(\mu^N | \mathcal{R}_N) + \frac{1}{N} \ln \mathbb{E}_{\mathcal{R}_N} \left[ \exp \left\{ \max_{1 \leq j \leq m} \left\{ N \int_0^T W_{A,N}(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \right\} \right].$$

Since the relative entropy  $H(\mu^N|\mathcal{R}_N) \leq C_0N$ , we obtain the left hand side of (9.2) is bounded from above by

$$C_0 + \max_{1 \leq j \leq m} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}_{\mathcal{R}_N} \left[ \exp \left\{ N \int_0^T W_{A,N}(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \right].$$

By Feynman-Kac formula, for any fixed index  $j$ , the limsup term in previous expression is bounded from above by

$$\limsup_{N \rightarrow \infty} \int_0^T \sup_f \left\{ E_{\mathcal{R}_N} [W_{A,N}(\epsilon, H_j(s, \cdot), \eta) f(\eta)] - N E_{\mathcal{R}_N} [\sqrt{f}(-S\sqrt{f})] \right\} ds$$

where the supremum is over all  $f$  which are densities with respect to  $\mathcal{R}_N$ . As  $c \leq \min_k \phi_{k,N}$  and  $\alpha_k^N$  is bounded, the Dirichlet form  $E_{\mathcal{R}_N} [\sqrt{f}(-S\sqrt{f})]$  (cf. (6.2)) is estimated as

$$\begin{aligned} & \sum_{1 \leq x \leq N} E_{\mathcal{R}_N} \left[ \frac{1}{2} \left( \phi_{x,N} \left( \frac{1}{2} + \frac{\alpha_x^N}{N} \right) + \phi_{x+1,N} \left( \frac{1}{2} - \frac{\alpha_{x+1}^N}{N} \right) \right) \left( \sqrt{f(\eta + \delta_x)} - \sqrt{f(\eta + \delta_{x+1})} \right)^2 \right] \\ & \geq \sum_{1 \leq x \leq N} E_{\mathcal{R}_N} \left[ \frac{c}{4} \left( \sqrt{f(\eta + \delta_x)} - \sqrt{f(\eta + \delta_{x+1})} \right)^2 \right]. \end{aligned} \quad (9.3)$$

Here, we used, for each  $x$ ,  $E_{\mathcal{R}_N} [g(\eta(x)) f(\eta)] = E_{\mathcal{R}_N} [\phi_{x,N} f(\eta + \delta_x)]$ , where  $\delta_x$  stands for the configuration with the only particle at  $x$ ;  $\eta + \delta_x$  is the configuration obtaining from adding one particle at  $x$  to  $\eta$ .

It now remains to show, for all  $H$  in  $C^{0,1}([0, T] \times \mathbb{T})$ , that

$$E_{\mathcal{R}_N} [W_{A,N}(\epsilon, H(s, \cdot), \eta) f(\eta)] - N E_{\mathcal{R}_N} [\sqrt{f}(-S\sqrt{f})] \leq 0. \quad (9.4)$$

We first compute that  $E_{\mathcal{R}_N} [W_{A,N}(\epsilon, H(s, \cdot), \eta) f(\eta)]$  equals

$$\begin{aligned} & E_{\mathcal{R}_N} \left[ \sum_{1 \leq x \leq N} \frac{H(x/N)}{\epsilon N} (g(\eta(x)) - g(\eta(x + \epsilon N))) \mathbb{1}_{A,N}(\eta) f(\eta) \right] \\ & - \frac{4}{c^2 N} E_{\mathcal{R}_N} \left[ \sum_{1 \leq x \leq N} \frac{H^2(x/N)}{\epsilon N} \sum_{0 \leq k \leq \epsilon N} g(\eta(x+k)) \mathbb{1}_{A,N}(\eta) \right] \\ & - E_{\mathcal{R}_N} \left[ \sum_{1 \leq x \leq N} \frac{CH(x/N)}{cN} g(\eta(x)) \mathbb{1}_{A,N}(\eta) \right]. \end{aligned} \quad (9.5)$$

Notice that  $E_{\mathcal{R}_N} [g(\eta(x)) \mathbb{1}_{A,N}(\eta) f(\eta)]$  may be written as

$$E_{\mathcal{R}_N} [\phi_{x,N} \mathbb{1}_{A,N}(\eta + \delta_x) f(\eta + \delta_x)] = E_{\mathcal{R}_N} [\phi_{x,N} \mathbb{1}_{A-1/N,N}(\eta) f(\eta + \delta_x)].$$

Then, the first expectation in (9.5) is written as

$$\begin{aligned} & \sum_{x=1}^N \frac{H(x/N)}{\epsilon N} E_{\mathcal{R}_N} \left[ (\phi_{x,N} f(\eta + \delta_x) - \phi_{x+\epsilon N,N} f(\eta + \delta_{x+\epsilon N})) \mathbb{1}_{A-1/N,N}(\eta) \right] \\ & \leq \sum_{x=1}^N \frac{H(x/N)}{\epsilon N} E_{\mathcal{R}_N} \left[ (f(\eta + \delta_x) (\phi_{x,N} - \phi_{x+\epsilon N,N})) \mathbb{1}_{A-1/N,N}(\eta) \right] \\ & \quad + \sum_{x=1}^N \frac{H(x/N)}{\epsilon N} E_{\mathcal{R}_N} \left[ (\phi_{x+\epsilon N,N} (f(\eta + \delta_x) - f(\eta + \delta_{x+\epsilon N}))) \mathbb{1}_{A-1/N,N}(\eta) \right]. \end{aligned} \quad (9.6)$$

Using that  $0 < c \leq \min_{1 \leq k \leq N} \phi_{k,N} \leq \max_k \phi_{k,N} \leq 1$  and  $\max_{1 \leq k \leq N} |\phi_{k,N} - \phi_{k+1,N}| \leq CN^{-1}$ , the first sum in the right hand side of the inequality (9.6) is bounded from above by

$$\begin{aligned} & \sum_{x=1}^N \frac{H(x/N)}{N} E_{\mathcal{R}_N} \left[ f(\eta + \delta_x) \frac{|\phi_{x,N} - \phi_{x+\epsilon N,N}|}{\epsilon} \mathbb{1}_{A-1/N,N}(\eta) \right] \\ & \leq \sum_{x=1}^N \frac{CH(x/N)}{N} E_{\mathcal{R}_N} [f(\eta + \delta_x) \mathbb{1}_{A-1/N,N}(\eta)] \\ & = \sum_{x=1}^N \frac{CH(x/N)}{N\phi_{x,N}} E_{\mathcal{R}_N} [g(\eta(x)) \mathbb{1}_{A,N}(\eta) f(\eta)] \leq \sum_{x=1}^N \frac{CH(x/N)}{cN} E_{\mathcal{R}_N} [g(\eta(x)) \mathbb{1}_{A,N}(\eta) f(\eta)]. \end{aligned} \quad (9.7)$$

Now, we proceed to the second sum in the right hand side of (9.6). Using  $0 < c \leq \min_k \phi_{k,N} \leq \max_k \phi_{k,N} \leq 1$ , the sum is bounded from above by

$$\sum_{x=1}^N \frac{H(x/N)}{\epsilon N} E_{\mathcal{R}_N} \left[ (f(\eta + \delta_x) - f(\eta + \delta_{x+\epsilon N})) \mathbb{1}_{A-1/N,N}(\eta) \right]$$

which is rewritten as

$$\begin{aligned} & E_{\mathcal{R}_N} \left[ \sum_{x=1}^N \sum_{k=0}^{\epsilon N-1} \frac{H(x/N)}{\epsilon N} (f(\eta + \delta_{x+k}) - f(\eta + \delta_{x+k+1})) \mathbb{1}_{A-1/N,N}(\eta) \right] \\ & = E_{\mathcal{R}_N} \left[ \sum_{x=1}^N \sum_{k=0}^{\epsilon N-1} \frac{H(x/N)}{\epsilon N} \mathbb{1}_{A-1/N,N}(\eta) (\sqrt{f(\eta + \delta_{x+k})} + \sqrt{f(\eta + \delta_{x+k+1})}) \right. \\ & \quad \left. \times (\sqrt{f(\eta + \delta_{x+k})} - \sqrt{f(\eta + \delta_{x+k+1})}) \right]. \end{aligned} \quad (9.8)$$

Using  $2ab \leq a^2 + b^2$ , for any  $\beta > 0$ , (9.8) is bounded from above by

$$\begin{aligned} & E_{\mathcal{R}_N} \left[ \sum_{x=1}^N \sum_{k=0}^{\epsilon N-1} \frac{H^2(x/N)}{2\epsilon N\beta} \mathbb{1}_{A-1/N,N}(\eta) (\sqrt{f(\eta + \delta_{x+k})} + \sqrt{f(\eta + \delta_{x+k+1})})^2 \right] \\ & \quad + E_{\mathcal{R}_N} \left[ \sum_{x=1}^N \sum_{k=0}^{\epsilon N-1} \frac{\beta}{2\epsilon N} (\sqrt{f(\eta + \delta_{x+k})} - \sqrt{f(\eta + \delta_{x+k+1})})^2 \right]. \end{aligned} \quad (9.9)$$

The first expectation in (9.9) is bounded from above by

$$\begin{aligned} & E_{\mathcal{R}_N} \left[ \sum_{x=1}^N \sum_{k=0}^{\epsilon N-1} \frac{H^2(x/N)}{2\epsilon N\beta} 2(f(\eta + \delta_{x+k}) + f(\eta + \delta_{x+k+1})) \mathbb{1}_{A-1/N,N}(\eta) \right] \\ & = \sum_{x=1}^N \frac{H^2(x/N)}{\epsilon N\beta} \sum_{k=0}^{\epsilon N-1} E_{\mathcal{R}_N} \left[ \left( \frac{g(x+k)}{\phi_{x+k,N}} + \frac{g(x+k+1)}{\phi_{x+k+1,N}} \right) \mathbb{1}_{A,N}(\eta) f(\eta) \right] \\ & \leq \sum_{x=1}^N \frac{2H^2(x/N)}{c\epsilon N\beta} \sum_{k=0}^{\epsilon N-1} E_{\mathcal{R}_N} [g(x+k) \mathbb{1}_{A,N}(\eta) f(\eta)]. \end{aligned} \quad (9.10)$$

The second expectation in (9.9) is rewritten and bounded, noting (9.3), as

$$E_{\mathcal{R}_N} \left[ \sum_{x=1}^N \frac{\beta}{2} (\sqrt{f(\eta + \delta_x)} - \sqrt{f(\eta + \delta_{x+1})})^2 \right] \leq \frac{2\beta}{c} E_{\mathcal{R}_N} [\sqrt{f}(-S\sqrt{f})].$$

Now, we set  $\beta = cN/2$ . Putting together (9.5), (9.7), and (9.10), we obtain (9.4).  $\square$

## 10. UNIQUENESS OF WEAK SOLUTIONS

In this section, we present results on uniqueness of good weak solutions to the PDE (3.3). Recall Definition 3.4 of a ‘good’ weak solution to (3.3).

**Theorem 10.1.** *There exists at most one good weak solution to (3.3).*

*Proof.* Let  $\rho$  be a good weak solution to (3.3). Since  $\partial_x \Psi(\rho(s, x))$  exists, for all  $G(s, x) \in C_c^\infty((0, T) \times \mathbb{T})$ , we have

$$\int_0^T \int_{\mathbb{T}} \partial_s G(s, x) \rho(s, x) dx ds = \int_0^T \int_{\mathbb{T}} \partial_x G(s, x) [\partial_x \Psi(\rho(s, x)) + K(x) \Psi(\rho(s, x))] dx ds. \quad (10.1)$$

Define  $\varphi(s, x) = \int_0^x \rho(s, u) du$ . For any  $G(s, x) \in C_c^\infty((0, T) \times \mathbb{T})$ , let

$$F(s, x) := \int_0^x G(s, x) dx - x \int_{\mathbb{T}} G(s, x) dx.$$

Note that  $F(s, x)$  is also in the space  $C_c^\infty((0, T) \times \mathbb{T})$ . Therefore we may apply (10.1) for  $F(s, x)$ . The left hand side, after integration by parts, becomes

$$\int_0^T [\partial_s F(s, x) \varphi(s, x)] \Big|_0^1 ds - \int_0^T \int_0^1 \partial_s \left( G(s, x) - \int_0^1 G(s, x) dx \right) \varphi(s, x) dx ds. \quad (10.2)$$

The term  $\int_0^T [\partial_s F(s, x) \varphi(s, x)] \Big|_0^1 ds$  vanishes as the total mass  $\int_{\mathbb{T}} \rho(s, x) dx$  is conserved. Then, (10.2) can be rewritten as

$$\begin{aligned} & - \int_0^T \int_0^1 \partial_s \left( G(s, x) - \int_0^1 G(s, x) dx \right) \varphi(s, x) dx ds \\ & = - \int_0^T \int_0^1 \partial_s G(s, x) \left( \varphi(s, x) - \int_0^1 \varphi(s, x) dx \right) dx ds. \end{aligned}$$

Define  $\phi(s, x) = \varphi(s, x) - \int_0^1 \varphi(s, u) du$ . We now have

$$\int_0^T \int_0^1 \partial_s G(s, x) \phi(s, x) dx ds = - \int_0^T \int_0^1 G(s, x) h(s, x) dx ds \quad (10.3)$$

where

$$h(s, x) = \partial_x \Psi(\rho(s, x)) + K(x) \Psi(\rho(s, x)) - \int_0^1 K(x) \Psi(\rho(s, x)) dx.$$

By straightforward approximation, we obtain from (10.3), for any  $G(\cdot) \in C_c^\infty(0, T)$  and  $q(\cdot) \in L^\infty[0, 1]$ ,

$$\int_0^T \partial_s G(s) \left[ \int_0^1 q(x) \phi(s, x) dx \right] ds = - \int_0^T G(s) \left[ \int_0^1 q(x) h(s, x) dx \right] ds \quad (10.4)$$

As  $\phi(s, \cdot)$  and  $h(s, \cdot)$  are both in  $L^1([0, T]; L^1(\mathbb{T}))$ , (10.4) implies that  $\frac{d}{ds} \phi(s, \cdot)$ , the weak derivative of  $\phi(s, \cdot)$ , exists and  $\frac{d}{ds} \phi(s, \cdot) = h(s, \cdot)$ . Moreover, in terms of the Bochner integral (cf. [6][p. 302]),

$$\phi(t, \cdot) = \int_0^t \frac{d}{ds} \phi(s, \cdot) ds + \phi(0, \cdot). \quad (10.5)$$

Now, assume there are two good weak solutions  $\rho_1, \rho_2$  and corresponding quantities  $\phi_1, \phi_2$ . If we show that  $\phi_1 = \phi_2$ , then it follows  $\varphi_1(s, x) - \varphi_2(s, x) = \int_0^1 \varphi_1(s, u) - \varphi_2(s, u) du$  for all  $s, x$ . By conservation of mass, it holds  $\varphi_1(s, 1) - \varphi_2(s, 1) = 0$  for all  $s$ . Then, we conclude  $\varphi_1 = \varphi_2$ , hence,  $\rho_1 = \rho_2$  a.e..

To this end, let  $\bar{\phi} = \phi_1 - \phi_2$  and  $\bar{\Psi}_{s,x} = \Psi(\rho_1(s, x)) - \Psi(\rho_2(s, x))$ . Therefore, by Lemma 10.2, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^1 (\bar{\phi}(t, x))^2 dx - \frac{1}{2} \int_0^1 (\bar{\phi}(0, x))^2 dx \\ &= \int_0^t \int_0^1 \bar{\phi}(s, x) [\partial_x \bar{\Psi}_{s,x} + K(x) \bar{\Psi}_{s,x}] dx ds - \int_0^t \int_0^1 \bar{\phi}(s, x) \left[ \int_0^1 K(u) \bar{\Psi}_{s,u} du \right] dx ds. \end{aligned} \quad (10.6)$$

Notice, from Lemma 10.3, that

$$\int_0^t \int_0^1 \bar{\phi}(s, x) \partial_x \bar{\Psi}_{s,x} dx ds = - \int_0^t \int_0^1 \partial_x \bar{\phi}(s, x) \bar{\Psi}_{s,x} dx ds = - \int_0^t \int_0^1 (\partial_x \bar{\phi}(s, x))^2 \Psi'_{s,x} dx ds.$$

Here, we have applied the mean value theorem so that  $\bar{\Psi}_{s,x} = \Psi'_{s,x} \partial_x \bar{\phi}(s, x)$ .

Let  $A$  be such that  $|K(x)| \leq A < \infty$ . Note that  $\Psi(\cdot)$  is an increasing function and  $0 \leq \Psi'(\cdot) \leq C_\Psi$ . We have,

$$\begin{aligned} & \int_0^t \int_0^1 \bar{\phi}(s, x) K(x) \bar{\Psi}_{s,x} dx ds \leq \int_0^t \int_0^1 A |\bar{\phi}(s, x)| |\partial_x \bar{\phi}(s, x)| \Psi'_{s,x} dx ds \\ & \leq \int_0^t \int_0^1 \left[ \frac{A^2}{2} (\bar{\phi}(s, x))^2 \Psi'_{s,x} + \frac{1}{2} (\partial_x \bar{\phi}(s, x))^2 \Psi'_{s,x} \right] dx ds \\ & \leq \frac{A^2 C_\Psi}{2} \int_0^t \int_0^1 (\bar{\phi}(s, x))^2 dx ds + \frac{1}{2} \int_0^t \int_0^1 (\partial_x \bar{\phi}(s, x))^2 \Psi'_{s,x} dx ds. \end{aligned}$$

We also have

$$\begin{aligned} & - \int_0^t \int_0^1 \bar{\phi}(s, x) \left[ \int_0^1 K(u) \bar{\Psi}_{s,u} du \right] dx ds \\ & \leq A \sqrt{C_\Psi} \int_0^t \left( \int_0^1 |\bar{\phi}(s, x)| dx \right) \cdot \left( \int_0^1 |\sqrt{\Psi'_{s,x}} \partial_x \bar{\phi}(s, x)| dx \right) ds \\ & \leq \frac{A^2 C_\Psi}{2} \int_0^t \left( \int_0^1 |\bar{\phi}(s, x)| dx \right)^2 ds + \frac{1}{2} \int_0^t \left( \int_0^1 |\sqrt{\Psi'_{s,x}} \partial_x \bar{\phi}(s, x)| dx \right)^2 ds \\ & \leq \frac{A^2 C_\Psi}{2} \int_0^t \int_0^1 (\bar{\phi}(s, x))^2 dx ds + \frac{1}{2} \int_0^t \int_0^1 \Psi'_{s,x} (\partial_x \bar{\phi}(s, x))^2 dx ds. \end{aligned}$$

Putting together the above, from equation (10.6), we get

$$\int_0^1 (\bar{\phi}(t, x))^2 dx - \int_0^1 (\bar{\phi}(0, x))^2 dx \leq 2A^2 C_\Psi \int_0^t \int_0^1 (\bar{\phi}(s, x))^2 dx ds.$$

Notice that  $\bar{\phi}(0, x) = 0$ . The desired result now follows from Gronwall's inequality.  $\square$

**Lemma 10.2.** Let  $\bar{h}(s, \cdot) := \frac{d}{ds} \bar{\phi}(s, \cdot)$ . We have

$$\frac{1}{2} \int_0^1 (\bar{\phi}(t, x))^2 dx - \frac{1}{2} \int_0^1 (\bar{\phi}(0, x))^2 dx = \int_0^t \int_0^1 \bar{\phi}(s, x) \bar{h}(s, x) dx ds.$$

*Proof.* For each  $n \in \mathbb{N}$ , let  $t_{k,n} = \frac{kt}{n}$ ,  $k = 0, 1, \dots, n-1$ . Then

$$\begin{aligned} & \frac{1}{2} \int_0^1 [(\bar{\phi}(t, x))^2 - (\bar{\phi}(0, x))^2] dx = \frac{1}{2} \int_0^1 \sum_{k=0}^{n-1} [(\bar{\phi}(t_{k+1,n}, x))^2 - (\bar{\phi}(t_{k,n}, x))^2] dx \\ &= \int_0^1 \sum_{k=0}^{n-1} \left[ \frac{\bar{\phi}(t_{k+1,n}, x) + \bar{\phi}(t_{k,n}, x)}{2} \int_{t_k}^{t_{k+1}} \bar{h}(s, x) ds \right] dx \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_0^1 \frac{\bar{\phi}(t_{k+1,n}, x) + \bar{\phi}(t_{k,n}, x)}{2} \bar{h}(s, x) dx ds \end{aligned}$$

Define  $\bar{\phi}_n(s, x) := \frac{\bar{\phi}(t_{k+1,n}, x) + \bar{\phi}(t_{k,n}, x)}{2}$  if  $s \in [t_{k,n}, t_{k+1,n})$ . Then, by weak continuity of  $\rho(t, x)$ , we have that  $\bar{\phi}_n(s, x)$  converges a.e. to  $\bar{\phi}(s, x)$  on  $[0, T] \times \mathbb{T}$ . By dominated convergence, noting that  $\bar{h}(s, x)$  belongs to  $L^1([0, T], L^1(\mathbb{T}))$ , we have, as  $n \uparrow \infty$ ,

$$\begin{aligned} \frac{1}{2} \int_0^1 [(\bar{\phi}(t, x))^2 - (\bar{\phi}(0, x))^2] dx &= \int_0^t \int_0^1 \bar{\phi}_n(s, x) \bar{h}(s, x) dx ds \\ &\rightarrow \int_0^t \int_0^1 \bar{\phi}(s, x) \bar{h}(s, x) dx ds. \end{aligned} \tag{10.7}$$

Since the left hand side of (10.7) is independent of  $n$ , the lemma is proved.  $\square$

**Lemma 10.3.** *We have  $\int_0^t \int_0^1 \bar{\phi}(s, x) \partial_x \bar{\Psi}_{s,x} dx ds = - \int_0^t \int_0^1 \partial_x \bar{\phi}(s, x) \bar{\Psi}_{s,x} dx ds$ .*

*Proof.* We first extend the class of test functions in (3.4). Assume that  $F : [0, T] \times [0, 1] \mapsto \mathbb{R}$  satisfies the following: (1)  $F$  is measurable; (2) for any fixed  $s$ ,  $F(s, \cdot)$  is absolutely continuous; (3) there is a constant  $C < \infty$  such that  $|\partial_x F(s, x)| \leq C$  for almost all  $s, x$ ; (4)  $\int_0^1 \partial_x F(s, x) dx = 0$  for all  $s$ .

Let  $\tau_\epsilon(x)$  be the standard mollifier supported on  $[-\epsilon, \epsilon]$ . Define

$$F_\epsilon(s, x) = \int_0^T \int_{\mathbb{T}} F(s, u) \tau_\epsilon(x - u) \tau_\epsilon(s - q) du dq$$

with  $F$  extended to be 0 for  $s \notin [0, T]$ . By (3.4),

$$\int_0^T \int_{\mathbb{T}} F_\epsilon(s, x) \partial_x \bar{\Psi}_{s,x} dx ds = - \int_0^T \int_{\mathbb{T}} \partial_x F_\epsilon(s, x) \bar{\Psi}_{s,x} dx ds.$$

Taking  $\epsilon \rightarrow 0$ , as  $\partial_x F$  (and therefore  $F$ ) is bounded and  $\bar{\Psi}_{s,x}$  is integrable, dominated convergence gives

$$\int_0^T \int_{\mathbb{T}} F(s, x) \partial_x \bar{\Psi}_{s,x} dx ds = - \int_0^T \int_{\mathbb{T}} \partial_x F(s, x) \bar{\Psi}_{s,x} dx ds. \tag{10.8}$$

We now extend the admissible test functions further from  $F$  to  $\bar{\phi}$  as claimed in the lemma. Introduce a truncation on  $\partial_x \bar{\phi}$ :

$$(\partial_x \bar{\phi})_{A,s,u} = \begin{cases} \partial_x \bar{\phi}(s, u) & -A \leq \partial_x \bar{\phi}(s, u) \leq A \\ A & \partial_x \bar{\phi}(s, u) > A \\ -A & \partial_x \bar{\phi}(s, u) < -A. \end{cases}$$

Apply (10.8) with  $F(s, x) = [\int_0^x (\partial_x \bar{\phi})_{A,s,u} du - x \int_0^1 (\partial_x \bar{\phi})_{A,s,u} du] \mathbb{1}_{[0,t]}(s)$  to get

$$\begin{aligned} & \int_0^t \int_0^1 \left[ \int_0^x (\partial_x \bar{\phi})_{A,s,u} du - x \int_0^1 (\partial_x \bar{\phi})_{A,s,u} du \right] \partial_x \bar{\Psi}_{s,x} dx ds \\ &= - \int_0^t \int_0^1 \left[ (\partial_x \bar{\phi})_{A,s,x} - \int_0^1 (\partial_x \bar{\phi})_{A,s,u} du \right] \bar{\Psi}_{s,x} dx ds. \end{aligned}$$

As  $|(\partial_x \bar{\phi})_{A,s,x}| \leq |\partial_x \bar{\phi}(s, x)| = |\rho_1(s, x) - \rho_2(s, x)|$ , by conservation of mass, and that  $\partial_x \bar{\Psi}_{s,x}$  is integrable, we have by dominated convergence that

$$\begin{aligned} & \lim_{A \rightarrow \infty} \int_0^t \int_0^1 \left[ \int_0^x (\partial_x \bar{\phi})_{A,s,u} du - x \int_0^1 (\partial_x \bar{\phi})_{A,s,u} du \right] \partial_x \bar{\Psi}_{s,x} dx ds \\ &= \int_0^t \int_0^1 \left[ \int_0^x \partial_x \bar{\phi}(s, u) du - x \int_0^1 \partial_x \bar{\phi}(s, u) du \right] \partial_x \bar{\Psi}_{s,x} dx ds = \int_0^t \int_0^1 \bar{\phi}(s, x) \partial_x \bar{\Psi}_{s,x} dx ds. \end{aligned}$$

Here, we used  $\int_0^1 \partial_x \bar{\phi}(s, u) du = \int_0^1 (\rho_1(s, u) - \rho_2(s, u)) du = 0$ . Similarly,

$$\lim_{A \rightarrow \infty} \int_0^t \int_0^1 \left[ \int_0^1 (\partial_x \bar{\phi})_{A,s,u} du \right] \bar{\Psi}_{s,x} dx ds = 0.$$

Finally, notice that  $(\partial_x \bar{\phi})_{A,s,x} \bar{\Psi}_{s,x} = (\partial_x \bar{\phi})_{A,s,x} \partial_x \bar{\phi}(s, x) \Psi'_{s,x}$  increases in  $A$  since  $\Psi'_{s,x} \geq 0$  (cf. (2.6)). By monotone convergence, we have

$$\lim_{A \rightarrow \infty} \int_0^t \int_0^1 (\partial_x \bar{\phi})_{A,s,x} \bar{\Psi}_{s,x} dx ds = \int_0^t \int_0^1 \partial_x \bar{\phi}(s, x) \bar{\Psi}_{s,x} dx ds,$$

finishing the proof.  $\square$

**Acknowledgements.** C.L. has been partly supported by FAPERJ CNE E-26/201.207/2014, by CNPq Bolsa de Produtividade em Pesquisa PQ 303538/2014-7, by ANR-15-CE40-0020-01 LSD of the French National Research Agency. S.S. was partly supported by ARO-W911NF-18-1-0311 and a Simons Foundations Sabbatical grant.

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