Crystal Growth on Locally Finite Partially Ordered Sets

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Abstract

We consider a Markovian growth process on a partially ordered set Λ, equivalent to last passage percolation (LPP) with independent (not necessarily identical) exponentially distributed weights on the elements of Λ. Such a process includes inhomogeneous exponential LPP on the Euclidean lattice $\mathbb{N}_0^d$. We give non-asymptotic bounds on the mean and variance, as well as higher, central, and exponential moments of the passage time τ_A to grow any set $A \subseteq \Lambda$ in terms of characteristics of A. We also give a limit shape theorem when Λ is equipped with a monoid structure. Methods involve making use of the backward equation associated to the Markovian evolution and comparison inequalities with respect to the time-reversed generator.

Keywords— last passage percolation, partially ordered set, monoid, shape, corner growth, variance, moment, passage time
MSC— 60K35, 06A06

1 Introduction

We study a 'crystal growth' process on locally finite partially ordered sets (posets), which includes Euclidean lattices. From a broader viewpoint, as is well-known, such growths are of interest in material science and other applications. The model we consider maps to a 'last passage percolation' model with independent (inhomogeneous) exponential weights on the poset.

Our aim in this article is to give non-asymptotic bounds on the moments and variances of the passage times τ_A for the formation of sets A, in terms of characteristics of A. We also consider a law of large numbers (LLN) shape limit theorem for τ_{A^n}/n when the poset is a monoid (see below for definitions), allowing one to define the iterates A^n. In the Euclidean lattice context, even when $d = 2$ and the weights are independent and exponential with homogeneous rates, in which case, there is a wealth of celebrated results, our bounds on the mean, variance, higher moments, and moment generating function of τ_A, for arbitrary A, appear new.

Although we describe our results for an arbitrary poset Λ, one may like to keep in mind the standard d-dimensional lattice context where $\Lambda = \mathbb{N}_0^d \subseteq \mathbb{Z}^d$ (where $\mathbb{N}_0 = \{0, 1, \ldots\}$) and for all $\alpha, \beta \in \mathbb{N}_0^d$ with $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$,

$$\alpha \leq \beta \text{ if and only if } \alpha_i \leq \beta_i \text{ for all } 1 \leq i \leq d.$$

We will consider a Markov process X_t that describes a crystal growth in Λ. In this process, more carefully defined and generalized to locally finite posets in Section 2, we require that all of the sites 'below' $\alpha \in \Lambda$ must already be present before the process can add α to its growth with a certain rate λ_α.

For any subset $S \subseteq \Lambda$, we say S is a **lower set** when for all $\beta \in S$, if $\alpha \in \Lambda$ with $\alpha \leq \beta$ then $\alpha \in S$ (see Figure 3 for an example). We denote by $L(\Lambda)$ the set of finite lower sets in Λ. Note that $L(\Lambda)$ is also a poset under inclusion $\subseteq$. For any $\beta \in \Lambda$ and $B \in L(\Lambda)$, we say

$$\langle \beta \rangle := \{\alpha \in \Lambda \ : \ \alpha \leq \beta\} \in L(\Lambda) \quad \text{and} \quad \langle B \rangle := \{A \in L(\Lambda) \ : \ A \subseteq B\} \subseteq L(\Lambda).$$

Because of the specifications of the growth, for all $t \geq 0$, we have $X_t \in L(\Lambda)$. Since $L(\Lambda)$ is a poset, we may define the stopping times of X_t: For any $A \in L(\Lambda)$, we say

$$\tau_A := \inf\{t \in [0, \infty) \ : \ A \subseteq X_t\} = \sup\{t \in [0, \infty) \ : \ A \not\subseteq X_t\} \tag{1}$$

and by abuse of notation, $\tau_\alpha = \tau_{\langle \alpha \rangle}$ for $\alpha \in \Lambda$. Because X_t is increasing with respect to the partial order, for any $t \in [0, \infty)$ and $A \in L(\Lambda)$, the following two events are equivalent:

$$\{A \subseteq X_t\} = \{\tau_A \leq t\}. \tag{2}$$

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Importantly, such a set equivalence will allow us later to use the backward equation of the process X_t to characterize the distribution function $\mathbb{P}(\tau_A \le t)$.

When Λ is a general poset, the associated process X_t is a natural generalization of the crystal growth model on $\Lambda = \mathbb{N}_0^d$. Note, when $\Lambda = \mathbb{N}_0^d$, the group structure of $\mathbb{Z}^d$ allows us to define $A + B$ for any $A, B \in L(\mathbb{Z}^d)$. Then, for any integer $n \ge 0$, we have $n \cdot A = \{\alpha_1 + \ldots + \alpha_n : \alpha_1, \ldots, \alpha_n \in A\}$ and a limiting shape function $g(\alpha) = \lim_{n\to\infty} \tau_{n\cdot\langle\alpha\rangle}/n$ can be formulated. In a general poset Λ, one needs a way to generalize the operation $+$ in $\mathbb{Z}^d$ to Λ. A natural way to do this is to assume there exists a monoid structure on Λ. More specifically, there is an associative binary operation $(x, y) \mapsto xy : \Lambda \times \Lambda \to \Lambda$ along with an identity $1_\Lambda \in \Lambda$. These notions are developed and stated in Section 3.5. General references to posets and monoids include [36] and [47].

We mention a couple of examples to keep in mind, beyond the Euclidean case $\Lambda = \mathbb{N}_0^d$.

Example 1.1. A large class of partially ordered monoids that directly generalize $\mathbb{N}_0^d$ are the positive cones C of finite dimensional vector spaces intersected with lattices $\overline{\Lambda}$. Suppose $\le_C$ is a partial order on $\mathbb{R}^d$ defined by a cone $C \subseteq \mathbb{R}^d$ (i.e. $x \le_C y$ when $y - x \in C$) and let $\overline{\Lambda} \subseteq \mathbb{R}^d$ be a discrete subgroup. Then, we define $\Lambda = \overline{\Lambda} \cap C$ which is the non-negative cone of $\ge_C$ when restricted to $\overline{\Lambda}$. When C is the non-negative octant in d-dimensions and $\overline{\Lambda} = \mathbb{Z}^d$, we recover the simple lattice model $\mathbb{N}_0^d$.

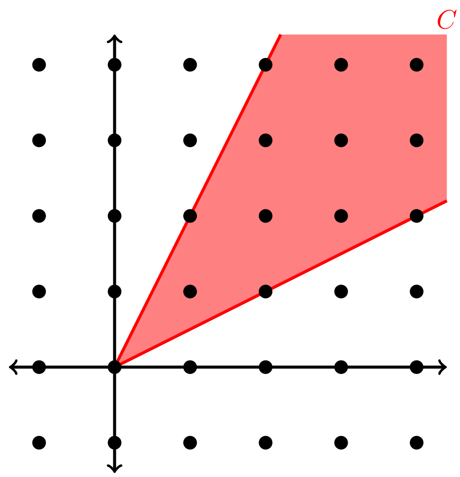

Figure 1: The positive cone C in $\overline{\Lambda} = \mathbb{Z}^2$

To give another example, we can take $\overline{\Lambda}$ to be the free group on d generators S. Define

$$\Lambda := \{a_1 \cdots a_n \in \overline{\Lambda} : a_1, \ldots, a_n \in S\}$$

so that Λ are the elements in $\overline{\Lambda}$ corresponding to words not containing inverses of any of the generators. Then, one may think of Λ as a directed tree where each element has exactly d elements directly above it, and ordering is defined by the existence of directed paths in the tree.

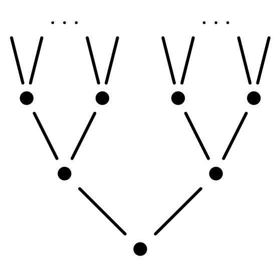

Figure 2: Λ is a tree order with two elements immediately above each element

1.1 Discussion of Previous Literature

When $\Lambda = \mathbb{N}_0^d$ and $d = 2$, the process X_t is the well-known 'corner growth' model with independent exponential weights $\{G_\alpha\}_{\alpha\in\Lambda}$ with rates $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ (cf. [24], [40], [46]). One may also view the times τ_A as passage times in an inhomogeneous 'last passage percolation' (LPP) model. That is, τ_A corresponds to the maximal cost $\max_\pi \sum_{i=1}^{\ell(\pi)} G_{\pi_i}$ over 'maximal increasing' paths π, where $\ell(\pi)$ is the length of the path, π_i is the i-th element in the path. See Section 2.2 which makes this correspondence precise on a general poset Λ.

With respect to $\Lambda = \mathbb{N}_0^d$, for independent, identically distributed (i.i.d.) exponential weights with parameter λ (i.e. $\lambda_\alpha \equiv \lambda$ is constant), the process in $d = 2$ is part of an 'exactly solvable' class. There are many connections with combinatorics, random matrices, polymer models, and totally asymmetric exclusion interacting particle systems through which the limiting scaled, centered statistics of τ_{ξ_n} have been found, among other limits. In particular, when $\xi_n = (n, \lfloor \gamma n \rfloor) \in \mathbb{N}_0^2$ for $\gamma \in (0, 1)$, the exact orders of the mean and variance of τ_{ξ_n} are n and $n^{2/3}$, respectively, the latter corresponding to the well-known 'KPZ' relation. A survey of these and other celebrated results and their histories can be seen for instance in [2], [3], [4], [5], [15], [18], [38], [41], and [46].

Further, in $d \geq 2$, when the distribution of $\{G_\alpha\}_{\alpha\in\Lambda}$ is more arbitrary, there are several results with respect to the law of τ_{ξ_n}, although not as complete as when the LPP model is 'exactly solvable'. In $d \geq 2$, for a general class of i.i.d. 'weights' $\{G_\alpha\}_{\alpha\in\Lambda}$ satisfying a moment condition, the a.s. convergence of an abstract LLN limit shape $g(x) = \lim_{n\to\infty} \tau_{\langle \xi_n \rangle}/n$ is known (cf. [22], [34], [43]). In $d = 2$, close to an axis, when $\xi_n = (n, \lfloor \gamma n \rfloor)$, the shape has 'universal' asymptotics $g(1, \gamma) = \mu + 2\sigma\sqrt{\gamma} + o(\sqrt{\gamma})$, as $\gamma \to 0^+$, where μ and σ^2 are the mean and variance of G_α (cf. [40], [43], and [41]). Moreover, when $\xi_n = (n, \lfloor n^a \rfloor)$, the time τ_{ξ_n}, centered by $n\mu + 2\sigma n^{\frac{1+a}{2}}$ and scaled by $\sigma n^{1/2-a/6}$, converges to F_2, the GUE Tracy-Widom distribution (cf. [16]). For heavy-tailed weights, convergence of the scaled passage times to those in a continuum last passage model has been shown (cf. [33], [31]).

In $d = 2$, for types of inhomogeneous independent exponential weights, the limit shape $g(\cdot)$ may also be computed (cf. [24], [25], [28], or [40]). Also, with respect to types of inhomogeneous independent exponential weights, among other results, variance and central moment bounds of τ_{ξ_n} with orders matching those in the i.i.d. 'exactly solvable' model are shown (cf. [29], [26], [27]). We comment also with respect to the homogeneous growth process X_t in $d = 3$ (or LPP with i.i.d. exponential weights), there are conjectures for the explicit form of the limit shape $g(\cdot)$ in [44]; see also [48] for a mapping of the growth process to coupled totally asymmetric exclusions, a 'zigzag' process.

Also, in $d \geq 2$, for a class of i.i.d. weights, including gamma distributions (and so exponential distributions), the variance of τ_{ξ_n}, where $\xi_n = n\big(e_1 + \ldots + e_d\big)$ and $\{e_i\}_{i=1}^d$ are the standard basis vectors of $\mathbb{Z}^d$, is bounded $\mathrm{Var}\,(\tau_{\xi_n}) \leq Cn/\log(n)$ (cf. [32]). See also [9] and [8] in the 'first-passage percolation' (FPP) context as well as [50] for related variance bounds in a dependent FPP model.

In $d = 2$, the bound $C\sqrt{\log(n)} \leq \mathrm{Var}\,(\tau_{\xi_n})$ has been proven for a general class of i.i.d. weights $\{G_\alpha\}_{\alpha\in\Lambda}$ (cf. [7] and, for the FPP context, [20]). Given that a positive constant lower bound $C \leq \mathrm{Var}\,(\tau_{\xi_n})$ in $d \geq 3$ in FPP is known (cf. [37]), one might formulate a similar bound in LPP for a class of i.i.d. weights, although it seems such a statement is not extant.

In $d \geq 2$, these upper and lower bounds for the variance $\mathrm{Var}\,(\tau_{\xi_n})$ in non-exactly solvable i.i.d. weights $\{G_\alpha\}_{\alpha\in\Lambda}$ LPP models are the best known, the conjectured order being $n^{2/3}$ when $d = 2$ and the weights have sufficient moments as in exactly solvable models. However, in $d \geq 3$, the order n^β (if a power β exists) of the variance is unresolved, although there are numerical simulations, including [49].

Although LPP models have been studied mostly in the $\Lambda = \mathbb{N}_0^d$ setting, the work [1] formulates the process on posets and discusses a geometrical perspective of passage times. Also, a few other settings with different schemes have been considered. The work [30] considers Barak-Erdős graphs on $\mathbb{Z}$ and related 'slabs' on posets, while the work [51] features the complete graph with N vertices. Also, the work [13] considers a continuous space model. In the first two works, weights are put on the edges of the graphs and last passage times are defined via maximal length of (directed or self-avoiding) paths. The third work considers a generalization of Hammersley's last passage percolation problem on $[0, 1]^2$. We note also in passing that there are several works on FPP in settings beyond the standard $\mathbb{Z}^d$ model, including [2], [6], [10], [11], [12], [14], [17], [21], [23], [35], and [39].

1.2 Sketch of Results and Methods

In view of the previous discussion, our results are summarized as follows. In a general poset Λ, including the i.i.d. exponential weights Euclidean LPP model $\Lambda = \mathbb{N}_0^d$, we provide several non-asymptotic bounds in Section 3 with respect to the passage times τ_A when $A \in L(\Lambda)$ is an arbitrary set. These include bounds for $\mathrm{Var}\,(\tau_A)$, the moments $\mathbb{E}\,[\tau_A^n]$, the central moments $\mathbb{E}\,[(\tau_A - \mathbb{E}\,[\tau_A])^n]$, and also the moment generating function $u \mapsto \mathbb{E}\,[e^{u\tau_A}]$. These estimates are specified in terms of characteristics of the set $A \in L(\Lambda)$ and the inhomogeneous rates $\{\lambda_\alpha\}_{\alpha\in\Lambda}$. We also give a LLN shape limit theorem when the poset Λ is a monoid (Theorem 3.18).

More concretely, we show that for each $A \in L(\Lambda)$ that $\lambda_-(A)\mathrm{Var}\,(\tau_A) \leq \mathbb{E}\,[\tau_A]$ (see Theorem 3.1) where $\lambda_-(A)$ is the minimal rate λ_α for $\alpha \in A$. We also bound $\lambda_-(A)\mathbb{E}\,[\tau_A]$ by $\left(\sqrt{\ell(A)} + \sqrt{\kappa(A) + \eta(A)}\right)^2$ (see Theorem 3.10) where $\ell(A)$ is a 'length', $\kappa(A)$ is a 'width', and $\eta(A)$ measures a 'spread of the rates' for A.

In the Euclidean context $\Lambda = \mathbb{N}_0^d$, one can show that $\kappa(A) \leq \log(d)\ell(A)$ (Lemma 3.11). Further, refinements of the upper bound when $A = \langle \xi_n \rangle$ in terms of the 'entropies' of ξ_n are given in Theorem 3.12. Then, $\mathrm{Var}\,(\tau_A)$ is of linear order in $\ell(A)$, which gives a 'diffusive' variance bound. This is sharp in that it holds for each $A \in L(\Lambda)$, including when $A = \langle \xi_n \rangle$ and ξ_n lies on an axis. Recall in this case, $\mathrm{Var}\,(\tau_A)$ is the variance of a sum of $\ell(A)$ independent exponential random variables.

We also give a lower bound for $\mathrm{Var}\,(\tau_A)$ in terms of the structure of A (see Proposition 3.4). In the Euclidean context, the bound for $A = \langle \xi_n \rangle$ reduces to a $\log(n)$ lower bound in $d = 2$ (see [7]). Additionally, in $d \geq 3$, this estimate gives a positive constant lower bound which, although understood in the FPP context [37], seems not to be written in the LPP literature (see Corollary 3.5 and Corollary 3.6). We also give an example of a poset Λ where $\mathrm{Var}(\tau_A)$ is bounded below in order by $|A|$ (Example 3.8).

As previously commented, in the setting of independent exponential weights LPP models in $d \geq 2$, when $A \in L(\Lambda)$ is an arbitrary set, not necessarily in the form $\langle \xi_n \rangle$, the non-asymptotic estimates for the mean $\mathbb{E}[\tau_A]$, variance $\mathrm{Var}(\tau_A)$, and other statistics in terms of properties of A seem novel. Of course, as noted earlier, there are better results when A is of the form $\langle \xi_n \rangle$ and $d = 2$. Also, the LLN shape limit theorem given in a class of monoids seems to be one of the first generalizations of the shape theorem beyond the Euclidean setting. In particular, its application to inhomogeneous independent exponential weight Euclidean LPP models in $d \geq 3$ seems new.

In terms of proofs, the bounds follow from analyzing the backward equation of the increasing process X_t in terms of a time-reversed generator Δ. In Lemma 4.1, a useful identity is derived by applying the backward equation to indicator functions of sets $\mathbb{1}[A \subseteq X_t]$. We show an important comparison inequality Proposition 4.3 which allows one to bound functions of sets $g(A)$ by $f(A)$ when $(\Delta g)(A) \leq (\Delta f)(A)$ and $g(\emptyset) \leq f(\emptyset)$.

Accordingly, we obtain our estimates by choosing appropriate functions g and f. For example, for the variance upper bound in Theorem 3.1, we will consider functions $g(A) = \mathrm{Var}(\tau_A)$ and $f(A) = \mathbb{E}[\tau_A]$. To bound $\mathbb{E}[\tau_A]$, we use a bound on the moment generating function $\mathbb{E}[e^{u\tau_A}]$ (see Proposition 3.14). We obtain this bound by considering types of 'path functions', which are discrete surrogates of the exponential function (cf. [19] and [42] on bounding 'greedy lattice animals' for related objects). The structure of the poset Λ also plays a role.

Our bounding techniques seem quite different from previous methods. They in a sense allow the 'worst' case of $A \in L(\Lambda)$, and so are agnostic to the size of $\mathcal{M}(A)$ which could be as small as 1 or as large as $|A|$. However, our techniques require that X_t is a Markov process. That is, the weights $\{G_\alpha\}_{\alpha \in \Lambda}$ are exponentially distributed, although we allow their rates $\{\lambda_\alpha\}$ to be inhomogeneous and the sets $A \in L(\Lambda)$ to be arbitrary.

For the monoid LLN shape limit result, as in the known Euclidean context, we make use of the superadditivity of $\mathbb{E}[\tau_{A^n}]/n$. However, to show the limit is finite in the general setting, we analyze an effective mean upper bound $\left(\sqrt{\ell(A^n)} + \sqrt{\kappa(A^n) + \eta(A^n)}\right)^2$ via estimates on $\kappa(A^n)/\ell(A^n)$ and a natural 'steadiness' assumption $\ell(A^n) \leq C\ell_*(A^n)$, where $\ell_*(A^n)$ is a 'minimal length' that we show is a subadditive sequence.

In part, our results extend to non-exponential independent weights $\{G_\alpha\}$. When the weights are all stochastically less or all stochastically more than an exponential, the bounds for the mean and moment-generating function extend in a natural way. Similarly, some of the results with respect to the shape limit theorem may be extended as well (see Section 3.6).

1.3 Plan of the Article

We begin with a formal definition of the process X_t in Section 2, followed by statements of the results in Section 3. After introducing preliminary results for the time-reversed generator of X_t in Section 4, we prove our results in Section 5.

Contents

2 Crystal Growth Markov Process

We define now the class of partial ordered sets Λ considered in the article. This class includes the Euclidean model where $\Lambda = \mathbb{N}_0^d$. For any elements $\alpha, \beta \in \Lambda$ with $\alpha < \beta$, we say β is an **upper neighbor** of α or, reciprocally, α is a **lower neighbor** of β if there does *not* exist $x \in \Lambda$ with $\alpha < x < \beta$. We may write this relationship as $\alpha \to \beta$.

Definition 2.1. We say Λ is **locally finite** if

1. for any $\alpha \in \Lambda$, $\{x \in \Lambda \,:\, x \leq \alpha\}$ is a finite set,
2. Λ has finitely many minimal elements,
3. for any $\alpha \in \Lambda$, the element α has finitely many upper neighbors.

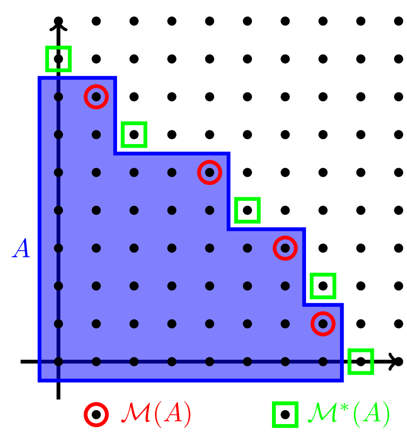

Figure 3: The maximal elements, $\mathcal{M}(A)$, and the growth elements, $\mathcal{M}^*(A)$, for a lower set $A \subset \Lambda = \mathbb{N}_0^2$.

Throughout the article, unless otherwise stated, we will take Λ to be a locally finite poset. For example, $\Lambda = \mathbb{N}_0^d$ is locally finite since $0 \in \mathbb{N}_0^d$ is the only minimal element and every element has d upper neighbors. Also, a rooted tree where the degree of a node is finite but increases, say with its depth, is locally finite. On the other hand, a tree, where every vertex has an infinite number of vertices below it, is *not* locally finite.

As before, let $L(\Lambda)$ be the set of finite lower sets in Λ. Let $\mathcal{M}(A)$ be the set of maximal elements in A and let $\mathcal{M}^*(A)$ be the set of minimal elements in $\Lambda \setminus A$. Note that every element of $\mathcal{M}^*(A)$ will be a minimal element of Λ or an upper neighbor of an element in A. Note also, when $\Lambda = \mathbb{N}_0^2$, that $\mathcal{M}^*(A) = \mathcal{M}(A) + 1$ (as in Figure 3); such a relation does not hold for general Λ.

Since A is finite, $\mathcal{M}(A)$ is finite. Further, since Λ is locally finite, $\mathcal{M}^*(A)$ is finite. Observe that $\mathcal{M}^*(\emptyset)$ consists of the minimal elements of Λ. Also, $|\mathcal{M}(\emptyset)| = 0$, but for non-empty A, we have $|\mathcal{M}(A)| \geq 1$ (where $|\cdot|$ denotes the cardinality). Also, if $A \in L(\Lambda)$ and $A \neq \Lambda$, we have $|\mathcal{M}^*(A)| \geq 1$. If Λ is finite then $\mathcal{M}^*(\Lambda) = \emptyset$.

We define a Markov process X_t on $L(\Lambda)$, with respect to rates $\{\lambda_\alpha\}_{\alpha \in \Lambda}$, with $\lambda_\alpha \in (0, \infty)$, so that for any $A \in L(\Lambda)$ and $\alpha \in \mathcal{M}^*(A)$, the process X_t transitions from A to $A \cup \alpha$ with rate λ_α. Here, we may think of $\mathcal{M}^*(X_t)$ as the set of 'available cells' for X_t to grow into. Throughout, we assume $X_0 = \emptyset$, though one may consider other initial conditions $X_0 \in L(\Lambda)$.

For non-empty $A \subseteq \Lambda$, let $\lambda_+(A) := \max_{x \in A} \lambda_x$ and $\lambda_-(A) := \min_{x \in A} \lambda_x$ be the upper and lower bounds of $\{\lambda_\alpha\}$ in A. When $A = \emptyset$, our convention is that $\lambda_-(A) = \lambda_+(A) = 0$. To avoid pathologies, it will be convenient to assume throughout that the rates satisfy

$$0 < \lambda_-(\Lambda) \leq \lambda_+(\Lambda) < \infty.$$

Hence, for non-empty $A \subseteq \Lambda$, we have $0 < \lambda_-(A) \leq \lambda_+(A) < \infty$. When the set A is clear from context, we may write $\lambda_+ = \lambda_+(A)$ and $\lambda_- = \lambda_-(A)$.

The process X_t is a continuous time, countable state Markov process with infinitesimal generator

$$(\mathcal{L}f)(A) := \sum_{\alpha \in \mathcal{M}^*(A)} \lambda_\alpha (f(A \cup \alpha) - f(A)) \tag{3}$$

for functions $f : L(\Lambda) \to \mathbb{R}$. Given $\lambda_+(\Lambda) < \infty$, the process is non-explosive and well-defined for all $t \geq 0$.

The time-reversed Markov process X_{-t} is a 'crystal etching' process with 'backward' generator $\mathcal{L}^* =: -\Delta$. For real functions f whose domain includes $A \in L(\Lambda)$ and $A \setminus \alpha$ for all $\alpha \in \mathcal{M}(A)$,

$$(\Delta f)(A) = \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \left(f(A) - f(A \setminus \alpha) \right) \tag{4}$$

with convention $(\Delta f)(\emptyset) = 0$.

We will denote by $\mathbb{P}$ and $\mathbb{E}$ the probability measure and expectation when $X_0 = \emptyset$.

Recall the stopping time τ_A defined in eq. (1) for X_t and $A \in L(\Lambda)$.

Lemma 2.2. *For any $A \in L(\Lambda)$ and $t \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(\tau_A \leq t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(A \subseteq X_t) = \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha [\mathbb{P}(\tau_{A \setminus \alpha} \leq t) - \mathbb{P}(\tau_A \leq t)] = -\Delta \mathbb{P}(\tau_A \leq t),$$

where $A \mapsto \mathbb{P}(\tau_A \leq t)$ is treated as a function on $L(\Lambda)$.

Proof. For $A = \emptyset$, as $X_0 = \emptyset$, we have $\mathbb{P}(\tau_A \leq t) \equiv 1$ and the statement holds. Otherwise, we fix non-empty $A \in L(\Lambda)$ and use $\mathbb{1}[\cdot]$ as the indicator of a condition. We define $s_A : L(\Lambda) \to \mathbb{R}$ as $s_A(B) := \mathbb{1}[A \subseteq B]$. Then, we can write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(A \subseteq X_t) &= \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}[s_A(X_t)] = \mathbb{E}[(\mathcal{L}s_A)(X_t)] \\
&= \mathbb{E}\left[\sum_{\alpha \in \mathcal{M}^*(X_t)} \lambda_\alpha \left(s_A(X_t \cup \alpha) - s_A(X_t) \right) \right] = \mathbb{E}\left[\sum_{\alpha \in \mathcal{M}^*(X_t)} \lambda_\alpha \mathbb{1}[A \subseteq X_t \cup \alpha, A \not\subseteq X_t] \right].
\end{aligned}$$

Notice that $\{A \subseteq X_t \cup \alpha, A \not\subseteq X_t\} = \{A \setminus \alpha \subseteq X_t, A \not\subseteq X_t\}$. Also, $A \subseteq X_t \cup \alpha$ and $A \not\subseteq X_t$ implies $\alpha \in \mathcal{M}(A)$. Conversely, when $\alpha \in \mathcal{M}(A)$, if $A \subseteq X_t \cup \alpha$ and $A \not\subseteq X_t$ then $\alpha \in \mathcal{M}^*(X_t)$. Thus,

$$\sum_{\alpha \in \mathcal{M}^*(X_t)} \lambda_\alpha \mathbb{1}[A \subseteq X_t \cup \alpha, A \not\subseteq X_t] = \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \mathbb{1}[A \setminus \alpha \subseteq X_t, A \not\subseteq X_t] = \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \left(s_{A \setminus \alpha}(X_t) - s_A(X_t) \right).$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(A \subseteq X_t) = \mathbb{E}\left[\sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \left(s_{A \setminus \alpha}(X_t) - s_A(X_t) \right) \right] = \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \left(\mathbb{P}(A \setminus \alpha \subseteq X_t) - \mathbb{P}(A \subseteq X_t) \right). \qquad \square$$

We now demonstrate the integrability of $f(\tau_A)$ when f has at most exponential growth, using the tail probabilities of τ_A.

Corollary 2.3. *For any $A \in L(\Lambda)$ and all $t \geq 0$, we have the bound*

$$\mathbb{P}(\tau_A > t) \leq (\lambda_+(A)t + 1)^{|A|} e^{-t\lambda_-(A)}, \tag{5}$$

where $|A|$ is the cardinality of A. Further, let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be such that there exist $C, t_0 > 0$ and $\mu < \lambda_-(A)$ with $|f(t)| \leq Ce^{\mu t}$ for all $t \geq t_0$. Then, $\mathbb{E}[|f(\tau_A)|] < \infty$.

Proof. We will prove the result by induction on $|A| \geq 0$. For $A = \emptyset$, $\mathbb{P}(\tau_A > t) = 0$ for all $t \geq 0$, and so the inequality holds. When $|A| \geq 1$, we consider the derivative of $e^{t\lambda_-(A)} \mathbb{P}(\tau_A > t) = e^{t\lambda_-(A)}(1 - \mathbb{P}(\tau_A \leq t))$. Using Lemma 2.2 and the fact that $1 \leq |\mathcal{M}(A)| \leq |A|$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} e^{t\lambda_-(A)} \mathbb{P}(\tau_A > t) &= \lambda_- e^{t\lambda_-} \mathbb{P}(\tau_A > t) + e^{t\lambda_-} \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{P}(\tau_A > t) \\
&= \lambda_- e^{t\lambda_-} \mathbb{P}(\tau_A > t) + e^{t\lambda_-} \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha [\mathbb{P}(\tau_{A \setminus \alpha} > t) - \mathbb{P}(\tau_A > t)] \\
&\leq \lambda_- e^{t\lambda_-} \mathbb{P}(\tau_A > t) - \lambda_- e^{t\lambda_-} \mathbb{P}(\tau_A > t) + e^{t\lambda_-} \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \mathbb{P}(\tau_{A \setminus \alpha} > t) \\
&\leq \lambda_+(A) e^{t\lambda_-(A)} \sum_{\alpha \in \mathcal{M}(A)} (\lambda_+(A \setminus \alpha)t + 1)^{|A|-1} e^{-t\lambda_-(A \setminus \alpha)} \\
&\leq \lambda_+(A) e^{t\lambda_-(A)} |A| \cdot (\lambda_+(A)t + 1)^{|A|-1} e^{-t\lambda_-(A)} = \lambda_+ |A| \cdot (\lambda_+ t + 1)^{|A|-1} = \lambda_+^{|A|} |A| \left(t + \frac{1}{\lambda_+} \right)^{|A|-1}.
\end{aligned}$$

Integrating both sides from 0, we obtain

$$e^{t\lambda_-}\mathbb{P}\left(\tau_A > t\right) - 1 \le \lambda_+^{|A|}\left[\left(t + \frac{1}{\lambda_+}\right)^{|A|} - \frac{1}{\lambda_+^{|A|}}\right] = (\lambda_+ t + 1)^{|A|} - 1$$

proving eq. (5).

Next, let $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ fulfill the given conditions. If $A = \emptyset$ then $\mathbb{E}\left[|f(\tau_A)|\right] = \mathbb{E}\left[|f(0)|\right] < \infty$. Otherwise, set $K = \int_{t=0}^{t_0} |f(t)| \, \mathrm{d}\mathbb{P}\left(\tau_A \le t\right) < \infty$ and note that $\mathbb{P}\left(\tau_A = 0\right) = 0$ as $A \neq \emptyset$ and $X_0 = \emptyset$. Then, from Lemma 2.2 and eq. (5), we obtain

$$\begin{aligned}
\mathbb{E}\left[|f(\tau_A)|\right] &= |f(0)|\mathbb{P}\left(\tau_A = 0\right) - \int_{t=0}^{\infty} |f(t)| \, \mathrm{d}\mathbb{P}\left(\tau_A > t\right) \\
&= K - \int_{t_0}^{\infty} |f(t)| \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\left(\tau_A > t\right) \, \mathrm{d}t \\
&= K - \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \int_{t_0}^{\infty} |f(t)| \left[\mathbb{P}\left(\tau_{A\setminus\alpha} > t\right) - \mathbb{P}\left(\tau_A > t\right)\right] \, \mathrm{d}t \\
&\le K + \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \int_{t_0}^{\infty} |f(t)| \cdot \mathbb{P}\left(\tau_A > t\right) \, \mathrm{d}t \\
&\le K + \lambda_+ |A| \int_{t_0}^{\infty} Ce^{t\mu} \cdot (\lambda_+ t + 1)^{|A|} e^{-t\lambda_-} \, \mathrm{d}t \\
&= K + \lambda_+^{|A|+1} C|A| \int_{t_0}^{\infty} (t + 1/\lambda_+)^{|A|} e^{t(\mu - \lambda_-)} \, \mathrm{d}t < \infty. \qquad \square
\end{aligned}$$

2.1 Well-definedness of Functions on $L(\Lambda)$

Because of the inhomogeneity of $\{\lambda_\alpha\}_{\alpha\in\Lambda}$, we remark that many functions $A \mapsto \mathbb{E}\left[f(\tau_A)\right]$ of interest (e.g. those where $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ grows exponentially) may only be defined on a subset of Λ. For instance, for $f(t) = e^{\mu t}$, the expectation $\mathbb{E}\left[f(\tau_A)\right]$ is well defined when $\mu < \lambda_-(A)$, but if $\lambda_-(A) \to 0$ as A grows then $\mathbb{E}\left[f(\tau_A)\right]$ will cease to be defined for sufficiently large A. On the other hand, if $\mu < \lambda_-(B)$ then we can say $\mathbb{E}\left[f(\tau_A)\right]$ is well defined for all $A \in \langle B \rangle$.

So, sometimes our results are stated for restricted-domain functions $g : \langle D_g \rangle \to \mathbb{R}$ where $D_g \subseteq \Lambda$ and $g(A)$ is defined for $A \in \langle D_g \rangle$. Of course, the unrestricted case $g : L(\Lambda) \to \mathbb{R}$, when g is well-defined, can be recovered using $D_g = \Lambda$. These notions are used in Lemma 4.1 and its applications as well as in the proofs of Proposition 3.14 and Proposition 3.16.

2.2 Relationship to Last Passage Percolation

We now observe the connection between the crystal growth model and last passage percolation on the locally finite poset Λ. Such a correspondence is well known when $\Lambda = \mathbb{N}_0^d$.

To this end, we define a **path** in Λ to be a (possibly empty) sequence $\pi = (\pi_1, \ldots, \pi_n)$ where $\pi_1, \ldots, \pi_n \in \Lambda$, π_1 is minimal, and for all $1 \le i < n$, $\pi_i < \pi_{i+1}$ and there exists no element $\beta \in \Lambda$ such that $\pi_i < \beta < \pi_{i+1}$ (i.e. π_{i+1} is an upper neighbor of π_i). Alternatively, for any $\alpha \in \Lambda$, a path to α is simply a chain maximal among those bounded by α. The **length** of the path $\pi = (\pi_1, \ldots, \pi_n)$ will be denoted $\ell(\pi) := n$. For each $A \in L(\Lambda)$, we take $\Pi(A)$ to be the set of paths π with $\pi_i \in A$ for all i. Similarly, we take $\Pi_m(A) \subseteq \Pi(A)$ to be the set of **maximal paths** in A. When $A = \emptyset$, we have $\Pi(A) = \Pi_m(A) = \{\emptyset\}$ where $\emptyset$ is the empty path. Also, by convention, empty summations vanish.

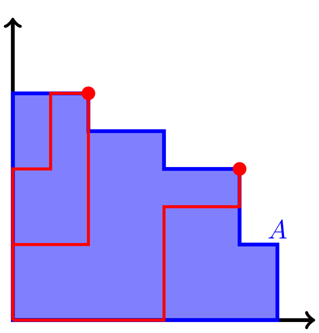

Figure 4: Three maximal paths in the lower set $A \subseteq \Lambda = \mathbb{N}_0^2$

Proposition 2.4. *Suppose* $\{G_\alpha\}_{\alpha\in\Lambda}$ *is a collection of independent exponential random variables with rates* $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ *and for any* $A \in L(\Lambda)$*, define*

$$\chi_A := \max_{\pi\in\Pi_m(A)} \sum_{i=1}^{\ell(\pi)} G_{\pi_i}. \tag{6}$$

Then, the stochastic process $Y_t = \bigcup\{A \in L(\Lambda) : \chi_A \le t\}$ *is a Markov process with generator* $\mathcal{L}$ *from eq.* (3)*. In particular,* Y_t *and* χ_A *have the same distributions as* X_t *and* τ_A*.*

We give two proofs of the proposition in Section 5.5, for the convenience and interest of the reader.

3 Results

In the following, we begin with bounds on the variance $\mathrm{Var}(\tau_A)$, higher moments $\mathbb{E}[\tau_A^n]$, and central moments $\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]$ written in terms of $\mathbb{E}[\tau_A]$. Bounds on the mean $\mathbb{E}[\tau_A]$ and moment generating function $\mathbb{E}[e^{u\tau_A}]$ are supplied later. Then, we state a LLN shape limit in a general monoid setting. Finally, in Section 3.6, we state some extensions to non-exponential weights $\{G_\alpha\}_{\alpha\in\Lambda}$.

3.1 Upper Bounds on $\mathrm{Var}(\tau_A)$ and Higher Moments in Terms of $\mathbb{E}[\tau_A]$

First, we state a sublinear upper bound for the variance of τ_A in terms of its mean. Estimates of the mean $\mathbb{E}[\tau_A]$ are provided in Section 3.3.

Theorem 3.1. *For any* $A \in L(\Lambda)$*, we have* $\mathbb{E}\left[\tau_A^2\right] < \infty$ *and* $\lambda_-(A) \cdot \mathrm{Var}(\tau_A) \le \mathbb{E}[\tau_A]$*.*

Because of Theorem 3.1, we have $\mathrm{Var}(\tau_A) \le C\mathbb{E}[\tau_A]^p$ for $p = 1$. However, it is known that this inequality holds for $p < 1$ in certain cases. Namely, as remarked in the introduction, in the homogeneous two dimensional Euclidean model, with $\Lambda = \mathbb{N}_0^2$ and $\lambda_\alpha \equiv \lambda$, the model is exactly solvable. One obtains for sets $A = \langle n(e_1 + e_2)\rangle$ that $\mathrm{Var}(\tau_A) \sim n^p \sim \mathbb{E}[\tau_A]^p$ with $p = 2/3$ (see [3]). Since Theorem 3.1 implies $\tau_A/\mathbb{E}[\tau_A] \to 1$ in probability and L^2, by Fatou's lemma, we have $\liminf \mathbb{E}[\tau_A^p]/\mathbb{E}[\tau_A]^p \ge 1$. So, for large n, one may verify the inequality $\mathrm{Var}(\tau_A) \le K\mathbb{E}[\tau_A^p]$.

We now assert that if $\mathrm{Var}(\tau_A) \le K\mathbb{E}[\tau_A]^p$ for some $p < 1$ then one can obtain similar bounds on the higher moments of τ_A. Related bounds are obtained for the higher central moments, giving a measure of concentration.

Proposition 3.2. *Suppose that there exist real numbers* $K > 0$ *and* $0 < p \le 1$ *such that, for all* $A \in L(\Lambda)$*, we have* $\mathrm{Var}(\tau_A) \le K\mathbb{E}[\tau_A^p]$*. For each integer* $n \ge 0$ *and every* $A \in L(\Lambda)$*, we have*

$$\mathbb{E}[\tau_A^n] - \mathbb{E}[\tau_A]^n \le K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right]. \tag{7}$$

Corollary 3.3. *Suppose there exists* $K > 0$ *and* $0 < p \le 1$ *such that* $\mathrm{Var}(\tau_A) \le K\mathbb{E}[\tau_A^p]$ *for all* $A \in L(\Lambda)$*. Then, for every* n *and every* $A \in L(\Lambda)$*, the central moment is bounded by*

$$\left|\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]\right| \le K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^p(\tau_A + \mathbb{E}[\tau_A])^{n-2}\right].$$

Also, for any sequence $A_1, A_2, \ldots \in L(\Lambda)$ *with* $\mu_j := \mathbb{E}\left[\tau_{A_j}\right] \to \infty$ *as* $j \to \infty$*, we have* $\mathbb{E}\left[(\tau_{A_j} - \mu_j)^n\right] = O(\mu_j^{p+n-2})$ *as* $j \to \infty$*.*

Note that for $p = 1$, the assumption $\mathrm{Var}(\tau_A) \le K\mathbb{E}[\tau_A^p]$ with $K = 1/\lambda_-(A)$ follows from Theorem 3.1 and so is unnecessary.

3.2 Lower Bound on $\mathrm{Var}(\tau_A)$

We can also obtain lower bounds for the variance as long as one can limit the growth rate of the lower sets. In particular, if we know that the number of maximal elements of A does not grow too quickly then we can ensure that the variance of τ_A does not grow too slowly.

Proposition 3.4. *Suppose there exists an increasing function* $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *such that for all* $A \in L(\Lambda)$*, we have* $\sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \le f(|A|)$*. Then, for all* $A \in L(\Lambda)$*,*

$$\mathrm{Var}(\tau_A) \ge \int_1^{|A|+1} \frac{\mathrm{d}x}{f(x)^2}.$$

Corollary 3.5. *There exists* $C > 0$ *such that* $\mathrm{Var}(\tau_A) \ge C$ *for all non-empty* $A \in L(\Lambda)$*.*

Proof. Because Λ is locally finite, every element $x \in \Lambda$ has finitely many upper neighbors. Hence, for any $n \in \mathbb{N}_0$, there are finitely many sets $A \in L(\Lambda)$ with $|A| \le n$. So we can define a positive increasing function

$$f(n) = \max_{|A| \le n} \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha < \infty.$$

We take $C = \int_1^2 \frac{\mathrm{d}x}{f(x)^2} > 0$. Then, by Proposition 3.4, when A is non-empty, so that $|A| \ge 1$, we have

$$\mathrm{Var}\,(\tau_A) \ge \int_1^{|A|+1} \frac{\mathrm{d}x}{f(x)^2} \ge C > 0. \qquad \square$$

Corollary 3.6. *When $\Lambda = \mathbb{N}_0^2$ and $\lambda_\alpha = \lambda$ for all $\alpha \in \Lambda$ then $\mathrm{Var}\,(\tau_A) \ge \frac{1}{2\lambda^2} \log(|A|+1)$ for all $A \in L(\Lambda)$.*

Proof. First, we show that $|\mathcal{M}(A)| \le \sqrt{2|A|}$ for all $A \in L(\Lambda)$. Suppose $A \in L(\Lambda) = L(\mathbb{N}_0^2)$ and $\{(x_1, y_1), \ldots, (x_m, y_m)\} = \mathcal{M}(A)$ are its maximal elements ordered such that $x_1, \ldots, x_m$ is an ascending sequence. For any $i < j$, we observe $x_i < x_j$, but by maximality (x_i, y_i) cannot be less than (x_j, y_j) and so $y_i > y_j$. Because $\{y_j\}$ are integers, we have $y_{j-1} \ge 1 + y_j$. Since $y_m \ge 0$, by induction, we have $y_j \ge m - j$. Note that for every $0 \le b \le y_i$, the element $(x_i, b) \in A$ since A is a lower set. Thus, taking $k = m - i + 1$,

$$|A| \ge \left| \bigcup_{i=1}^{m} \bigcup_{b=0}^{y_i} \{(x_i, b)\} \right| = \sum_{i=1}^{m} (y_i + 1) \ge \sum_{i=1}^{m} (m - i + 1) = \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \ge \frac{1}{2} m^2 = \frac{1}{2} |\mathcal{M}(A)|^2$$

implying that $|\mathcal{M}(A)|^2 \le 2|A|$ proving the result. Thus, the result follows by Proposition 3.4. $\square$

Corollary 3.7. *Suppose there exists $b > 0$ such that $|\mathcal{M}(A)| \le b$ for all $A \in L(\Lambda)$. Then, $\mathrm{Var}\,(\tau_A) \ge \frac{|A|}{\lambda_+(\Lambda)^2 b^2}$, and so $\mathrm{Var}\,(\tau_A)$ diverges linearly in $|A|$ as $|A| \to \infty$.*

Proof. We define $f(|A|) := \lambda_+(\Lambda) b \ge \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha$. Then, by Proposition 3.4,

$$\mathrm{Var}\,(\tau_A) \ge \int_1^{|A|+1} \frac{\mathrm{d}x}{\lambda_+(\Lambda)^2 b^2} = \frac{|A|}{\lambda_+(\Lambda)^2 b^2}. \qquad \square$$

Example 3.8. A toy poset Λ satisfying the assumptions of Corollary 3.7 is $\Lambda = \mathbb{N}_0 \times S$ where S is a finite set and for any $(n_1, s_1), (n_2, s_2) \in \mathbb{N}_0 \times S$, $(n_1, s_1) < (n_2, s_2)$ if and only if $n_1 < n_2$. This ordering forces the growth process to fill each layer (e.g. $\{n\} \times S$) sequentially so one always has $|\mathcal{M}(A)| \le |S|$.

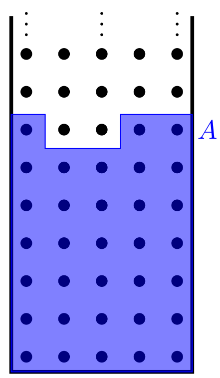

Figure 5: The lower set A in the poset $\mathbb{N}_0 \times \{1, \ldots, 5\}$ where the ordering is determined by $\mathbb{N}_0$ (up is greater) and (n, a) is incomparable to (n, b) whenever $a \ne b$

3.3 Bounds on $\mathbb{E}\,[\tau_A]$

For any $A \in L(\Lambda)$, we define the **length**, $\ell(A)$, of A as the maximum length of a path in A:

$$\ell(A) := \max_{\pi \in \Pi(A)} \ell(\pi) = \max_{\pi \in \Pi_m(A)} \ell(\pi).$$

Here, we take $\ell(\emptyset) = 0$. Naturally, one expects that the stopping time τ_A would be bounded below in terms of $\ell(A)$, since every path in A must be filled in order for A to be occupied.

Proposition 3.9. *For any $B \in L(\Lambda)$, we have $\lambda_+(B)\mathbb{E}[\tau_B] \geq \ell(B)$.*

Although simple, this bound is sharp when considering arbitrary Λ. In particular, one can take $\Lambda = \mathbb{N}_0$ and $\tau_{[0,n-1]}$ is the sum of n exponential random variables with rate λ. Hence, $\mathbb{E}\left[\tau_{[0,n-1]}\right] = \lambda n = \lambda\ell([0, n-1])$. One can also consider any model where the poset $\mathbb{N}_0$ arises as a lower set of Λ, such as when $\Lambda = \mathbb{N}_0^d$.

If we know something about the size of the set A (i.e. the number of paths in A), then we can also upper bound the mean $\mathbb{E}[\tau_A]$. To express this bound, define the **width** $\kappa(A)$ for any $A \in L(\Lambda)$ as the logarithm of the number of maximal paths in A and the **rate spread** $\eta(A)$ for any non-empty $A \subseteq \Lambda$ as the logarithm of the ratio of the maximum and minimum rates λ_α in A:

$$\kappa(A) := \log|\Pi_m(A)| \quad \text{and} \quad \eta(A) := \log\left(\frac{\lambda_+(A)}{\lambda_-(A)}\right). \tag{8}$$

We specify that $\kappa(\emptyset) = \eta(\emptyset) = 0$. Note the width will typically grow as $|A| \to \infty$, while since $0 < \lambda_-(\Lambda) \leq \lambda_+(\Lambda) < \infty$, we have $\eta(A) \leq \eta(\Lambda) < \infty$.

Theorem 3.10. *For any $A \in L(\Lambda)$,*

$$\lambda_-(A)\mathbb{E}[\tau_A] \leq \left(\sqrt{\ell(A)} + \sqrt{\kappa(A) + \eta(A)}\right)^2.$$

One can think of $\kappa(A)/\ell(A)$ as describing a 'shape' or 'aspect ratio' of A independent of its size. Naturally, one may want to bound $\kappa(A)$ in terms of $\ell(A)$ to ensure at most linear growth of the mean $\mathbb{E}[\tau_A]$ in terms of $\ell(A)$. In general, this may not be possible. Consider a tree poset where the number of branches for each element increases as one moves up the tree.

Lemma 3.11. *Let $A \in L(\Lambda)$ and suppose that there exists $d \in \mathbb{N}_0$ such that A has at most d minimal elements and for every $x \in A$, there are at most d upper neighbors of x in A. Then, $\kappa(A) \leq \log(d)\,\ell(A)$.*

If we restrict Theorem 3.10 to the case of the d-dimensional lattice where $\Lambda = \mathbb{N}_0^d$ then we can explicitly calculate $\kappa(A)$ in terms of $\ell(A)$ using Stirling's approximation. In particular, the bounds in Theorem 3.12 give information on the shape function $g(\alpha) = \lim_{n\to\infty} \tau_{\langle n\alpha\rangle}/n$, known to exist in the homogeneous rate $\lambda_x \equiv \lambda$ setting (cf. [43]), but non-explicit in $d \geq 3$. We comment, the limit below extends the known limit in $d \geq 3$ for homogeneous weights to an inhomogeneous setting.

Theorem 3.12. *Suppose $\Lambda = \mathbb{N}_0^d$ for $d \geq 1$. For any $\alpha = (\alpha_1, \ldots, \alpha_d) \neq (0, \ldots, 0)$, define $p_i := \frac{\alpha_i}{\ell(\alpha)}$ for each i. Then,*

$$\lim_{n\to\infty} \frac{\kappa(\langle n\alpha\rangle)}{\ell(\langle n\alpha\rangle)} = -\sum_{i=1}^{d} p_i \log(p_i), \tag{9}$$

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[\tau_{\langle n\alpha\rangle}\right]}{n\ell(\alpha)} \leq \frac{1}{\lambda_-(\Lambda)}\left(1 + \sqrt{-\sum_i p_i \log(p_i)}\right)^2. \tag{10}$$

Moreover, suppose in addition that $\lambda_x \geq \lambda_y$ for $x, y \in \Lambda$ with $x \leq y$. Then, by the later Theorem 3.18, the limit of $\frac{\mathbb{E}[\tau_{\langle n\alpha\rangle}]}{n\ell(\alpha)}$ exists. That is, one can replace '$\limsup$' with '$\lim$' in eq. (10). Also, the limit of $\tau_{\langle n\alpha\rangle}/n$ converges in L^2.

3.4 Bounds on the Moment Generating Function of τ_A

We introduce types of 'path functions' that will allow us to estimate the moment generating function of τ_A, tight up to first order expansion as $u \to 0^+$. Part of the rationale, as will be seen in the proofs, is that they behave nicely under the operator Δ.

Definition 3.13. For any $f : \mathbb{N}_0 \to \mathbb{R}$, we define the **greater path function** $\Gamma_\geq f : L(\Lambda) \to \mathbb{R}$ as

$$(\Gamma_\geq f)(A) := \sum_{\pi\in\Pi(A)} f(\ell(\pi))$$

where $\Pi(A)$ is the set of paths in A.

We can apply $\Gamma_\geq$ to the geometric function $f(n) = r^n$ for a real $r > 1$.

Proposition 3.14. *For any $B \in L(\Lambda)$ and real $0 < u < 1$,*

$$\sum_{\substack{\pi\in\Pi(B)\\ \pi\neq\emptyset}} \frac{u}{(1-u)^{\ell(\pi)}} = u\left(\Gamma_\geq\left\{\frac{1}{(1-u)^n}\right\}(B) - 1\right) \geq \mathbb{E}\left[e^{\lambda_-(B)\,u\tau_B}\right] - 1.$$

We now construct a lower bound for the moment generating functions. To do this, though, we must use information about how quickly Λ branches.

Definition 3.15. We define a **branching allocation** ψ to be a collection of non-negative real numbers

$$\{\psi_{\alpha\to\beta} \in \mathbb{R}_{\geq 0} \,:\, \alpha,\beta \in \Lambda, \alpha \text{ is a lower neighbor of } \beta\} \cup \{\psi_\mu \,:\, \mu \in \Lambda \text{ is minimal}\}$$

such that for each α, $\sum_\beta \psi_{\alpha\to\beta} \leq 1$ and $\sum_{\mu \text{ is minimal}} \psi_\mu \leq 1$. Given a branching allocation ψ, we say the **weight** of a path $\pi \in \Pi(\Lambda)$ with respect to a branching allocation ψ is

$$\omega_\psi(\pi) := \psi_{\pi_1} \prod_{i=2}^{\ell(\pi)} \psi_{\pi_{i-1}\to\pi_i}$$

and $\omega_\emptyset = 1$. Then, for any $f : \mathbb{N}_0 \to \mathbb{R}$, we define the **lesser path function** $\Gamma_{\leq}^{\psi} f : L(\Lambda) \to \mathbb{R}$ as

$$\left(\Gamma_{\leq}^{\psi} f\right)(A) := \sum_{\pi\in\Pi(A)} f(\ell(\pi))\omega_\psi(\pi).$$

when clear from context, we may omit ψ and simply write $\Gamma_{\leq} f$.

As with the greater path function, we can consider the lesser path functions for $f(n) = r^n$.

Proposition 3.16. *Let ψ be a branching allocation. Then, for non-empty $B \in L(\Lambda)$ and $0 < u < \lambda_-(B)/\lambda_+(B)$,*

$$u\left(\Gamma_{\leq}^{\psi}\left\{\frac{1}{(1-u)^n}\right\}(B) - 1\right) \leq \mathbb{E}\left[e^{\lambda_+(B)\,u\tau_B}\right] - 1.$$

We remark that the upper bound Proposition 3.14 is used to make the upper bounds of $\mathbb{E}[\tau_A]$ in Theorem 3.10. On the other hand, the companion lower bound Proposition 3.16 can be used to give an alternate proof of Proposition 3.9 (see Remark 5.5).

3.5 Partial Orders on Monoids

When $\Lambda = \mathbb{N}_0^d$, we notice that every element of Λ 'looks like' every other element of Λ. In particular, for any $\alpha \in \Lambda$, we have the transformation $x \mapsto x+\alpha$ (inherited from the group structure of $\mathbb{Z}$) which carries $0 \in \Lambda$ and its upper ray $[0,\infty)_\Lambda = \{x \in \Lambda \,:\, x \geq 0\}$ to α and its upper ray $[\alpha,\infty)_\Lambda$. This transformation structure gives natural sequences of elements and hence lower sets on which to consider their stopping times. In particular, for any $A \in L(\mathbb{N}_0^d)$, we may define $nA := \{(nx_1,\ldots,nx_d) \,:\, (x_1,\ldots,x_d) \in \mathbb{N}_0^d\}$ and consider the behavior of $\mathbb{E}[\tau_{nA}]/n$ as $n \to \infty$. The key ingredient here is the order preserving transformation provided by translation.

To generalize, let $\overline{\Lambda}$ be a group with a partial ordering $\leq$. We say that $\leq$ is **compatible** with $\overline{\Lambda}$ if for all $a_1,a_2,b_1,b_2 \in \overline{\Lambda}$, $a_1 \leq a_2$ and $b_1 \leq b_2$ implies $a_1b_1 \leq a_2b_2$. Then, we define $\Lambda := \{x \in \overline{\Lambda} \,:\, x \geq 1_{\overline{\Lambda}}\}$. where $1_{\overline{\Lambda}} = 1_\Lambda$ is the identity of $\overline{\Lambda}$ and Λ. So, being a subset of a group, Λ naturally has the structure of a monoid (i.e. a set with an associative binary operation and an identity for that operation) with a partial order.

For the remainder of the section, we assume that $\overline{\Lambda}$ is a group with a compatible partial ordering and $\Lambda \subseteq \overline{\Lambda}$ is the non-negative cone so that $\Lambda := \{x \geq 1_\Lambda \,:\, x \in \overline{\Lambda}\}$. In the later results, we will assume that Λ is locally finite. We comment that the second condition of Definition 2.1 automatically holds as 1_Λ is the unique minimal element in Λ.

The length $\ell(A)$ of $A \in L(\Lambda)$ may appear superficially similar to the diameter (as are defined in metric spaces). However, as seen in Lemma 5.7, the mapping$A \mapsto \ell(A)$ is superadditive, contradicting this analogy. We define a related notion of 'length', which however is subadditive (cf. eq. (20)): for every $A \in L(\Lambda)$, let

$$\ell_*(A) := \max_{x\in A} \min\{\ell(\pi) \,:\, \pi \in \Pi_m(\langle x\rangle)\}.$$

In words, $\min\{\ell(\pi) \,:\, \pi \in \Pi_m(\langle x\rangle)\}$ is the smallest number of steps to get to x, and $\ell_*(A)$ is the largest such number over $x \in A$. Note $\ell_*(\emptyset) = 0$. Since $\ell(A) = \max_{x\in A} \max\{\ell(\pi) \,:\, \pi \in \Pi_m(\langle x\rangle)\}$, clearly $\ell_*(A) \leq \ell(A)$.

Definition 3.17. For any two lower sets $A, B \in L(\Lambda)$, we define the set

$$AB := \{x \in \Lambda \,:\, \exists a \in A, b \in B \text{ s.t. } x \leq ab\}.$$

Later in Lemma 5.6, we show $AB \in L(\Lambda)$ and $A^n \in L(\Lambda)$. Then, for any $n \in \mathbb{N}_0$, we can define, recursively, $A^n := AA^{n-1}$.

We will now state the existence of a shape function to which τ_{A^n}/n converges as $n \to \infty$. A sufficient condition we impose is that $\ell(A)$ does not have faster than linear growth. For any monoid Λ, we say Λ is **steady** if there exists $C \geq 1$ such that for all $x \in \Lambda$, we have

$$\max_{\pi \in \Pi_m(\langle x \rangle)} \ell(\pi) \leq C \min_{\pi \in \Pi_m(\langle x \rangle)} \ell(\pi).$$

So, steadiness implies that for all $A \in L(\Lambda)$, $\ell(A) \leq C\ell_*(A)$. Because of the subadditivity of $\ell_*(A)$ (shown later in Lemma 5.7), steadiness ensures that $\ell(A^n)$ has at most linear growth in n. See Remark 3.22 for an example of 'non-steadiness'.

Theorem 3.18. *Suppose the monoid Λ is locally finite and steady. Further, assume that $\lambda_x \geq \lambda_y$ for any $x, y \in \Lambda$ with $x \leq y$. Then, for any non-empty $A \in L(\Lambda)$, the following limit converges and fulfills the inequality:*

$$g(A) := \lim_{n \to \infty} \frac{\mathbb{E}\left[\tau_{A^n}\right]}{n} \leq \frac{1}{\lambda_-(\Lambda)} \left(\sqrt{\lim_{n \to \infty} \frac{1}{n} \kappa(A^n)} + \sqrt{\lim_{n \to \infty} \frac{1}{n} \ell(A^n)} \right)^2. \tag{11}$$

Additionally, $\frac{1}{n}\tau_{A^n} \to g(A)$ in L^2 as $n \to \infty$.

Note that trivially, as $\tau_\emptyset = 0$, we have $\tau_{\emptyset^n}/n \equiv 0$. Also, if Λ is finite then $\tau_{A^n} \leq \tau_\Lambda < \infty$ almost surely and so $\tau_{A^n}/n \to 0$. In addition, recall the 'width' $\kappa(A)$ defined in eq. (8) is the logarithm of the number of maximal paths in A. If S is the finite set of upper neighbors of 1_Λ, then one can show that every element of Λ has at most $|S|$ upper neighbors (see the beginning of the proof of Theorem 3.18). Also, as Λ has a unique minimal element, namely 1_Λ, we may apply Lemma 3.11 to bound $\kappa(A)$ in terms of $\ell(A)$, giving a further bound to eq. (11).

Additionally, the steadiness condition allows several examples. For instance, referring to Example 1.1, in the Euclidean case, when $\Lambda = \mathbb{N}_0^d$, or when Λ are the words constructed from d generators, we have $\ell(\alpha) = \ell^*(\alpha)$ since, for any element $\alpha \in \Lambda$, every path from the minimal element to α has the same length.

We say the group $\overline{\Lambda}$ is **finitely presented** if there exist a finite set of generators S and a finite set of relations R among them such that $\overline{\Lambda} = \langle S \,|\, R \rangle$ (cf. [36]). The group $\langle S \,|\, R \rangle$ is the collection of equivalence classes of words consisting of elements of S and S^{-1} where the equivalence relation is determined by reduction by the relations R. With respect to the compatible partial ordering $\leq$, we will assume the corresponding partially ordered monoid $\Lambda = \{x \in \overline{\Lambda} : x \geq 1_{\overline{\Lambda}}\}$ is finitely generated. So, by Lemma 5.8, if Λ is steady then it is locally finite.

Example 3.19. Both the Euclidean and free generator settings in Example 1.1 correspond to finitely presented $\overline{\Lambda}$ and finitely generated Λ. Another example on the lattice may be described by its group presentation: $S = \{a, b, c\}$ and $R = \{bc = a^3\}$. Then, let $\overline{\Lambda} = \langle a, b, c \,|\, bc = a^3 \rangle$ be the abelian group generated by S. Further, let $\Lambda = \{x \in \overline{\Lambda} : x \geq 0 = 1_{\overline{\Lambda}}\} \subseteq \overline{\Lambda}$ be the associated abelian monoid generated by $\{a, b, c\}$ subject to $bc = a^3$. One can see that $\overline{\Lambda} \cong \mathbb{Z}^2$ and Λ will have the structure of a poset: the cone in Figure 1 with the binary operation written additively and

$$a = e_1 + e_2, \quad b = 2e_1 + e_2, \quad \text{and} \quad c = e_1 + 2e_2.$$

Here, it may be seen that $\ell(x) > \ell_*(x)$. Indeed, this is the case say for $x = b + c = 3a$ where $\ell(x) = 3$ and $\ell_*(x) = 2$.

We consider a generalization of the above example in the following statement.

Corollary 3.20. *Suppose S is a finite set of generators and R is a finite set of relations on S. Assume $\overline{\Lambda} = \langle S \,|R \rangle$ is abelian and Λ is generated by S. Then, Λ is steady and locally finite so that for any $A \in L(\Lambda)$, eq. (11) holds.*

We note in passing that $\overline{\Lambda}$ may be a finitely presented group, while Λ is not finitely generated. Indeed, if $\overline{\Lambda} = \mathbb{Z}^2$ and $\Lambda = \{(x, y) \in \mathbb{Z}^2 : x \geq 0 \text{ and } 0 \leq y < \pi x\}$ then there are infinitely many lattice points arbitrarily close to the line $y = \pi x$ (with irrational slope) none of which can be written as sums of non-negative multiples of other elements in the cone. Thus, each of these must be in any generating set of Λ. However, specifying that Λ is locally finite precludes such cases.

If one places some restrictions on the relations of a non-abelian group then we can also achieve steadiness.

Corollary 3.21. *Suppose the (possibly non-abelian) monoid Λ is finitely generated by S and suppose $\overline{\Lambda} = \langle S \,|\, R \rangle$ where R is finite (possibly empty). Further, assume that any relation $(a_1 \dots a_n = 1_{\overline{\Lambda}}) \in R$ is such that $|\{i \in [1, n] : a_i \in S\}| = |\{i \in [1, n] : a_i \in S^{-1}\}|$. Then, Λ is steady with $C = 1$ and locally finite so that for any $A \in L(\Lambda)$, eq. (11) holds.*

One may fit the non-abelian free group setting in Example 1.1 (without relations) into the assumptions of Corollary 3.21. Another simple example is $\overline{\Lambda} = \langle a, b, c \,|\, abc = cab = bca \rangle$ (the group which allows cyclic permutations of a, b, and c) and $\Lambda \subseteq \overline{\Lambda}$ to be the monoid generated by $\{a, b, c\}$. The relations $abc = cab = bca$ may be written as $abcb^{-1}a^{-1}c^{-1} = abca^{-1}c^{-1}b^{-1} = 1$, fulfilling the conditions of Corollary 3.21.

Remark 3.22 (On Non-steadiness)**.** If the relations on a noncommutative group are not so well behaved then one can display groups where the minimum and maximum length paths to an element $x \in \Lambda$ differ quite substantially. Take $\overline{\Lambda} = \langle a, b, c \,|\, ab = bac^2, ac = ca, bc = cb \rangle$ so that $x_n := (ab)^n = b^n a^n c^{n(n+1)}$. Then, the minimum length path to x_n (i.e. $\ell_*(\langle x \rangle)$) is at most $2n$, while the maximum length path (i.e. $\ell(\langle x \rangle)$) is at least $n^2 + 3n$ and $\frac{n^2+3n}{2n} \to \infty$ as $n \to \infty$. Hence, the associated poset Λ is not steady.

3.6 Extension to Stochastically Monotone Weights

We comment here on immediate extensions of some of the results to non-exponential independent weights $\{G_\alpha\}_{\alpha\in\Lambda}$. We begin with the bounding of the moments and moment generating function.

Remark 3.23. Suppose $\{G_\alpha\}_{\alpha\in\Lambda}$ and $\{H_\alpha\}_{\alpha\in\Lambda}$ are collections of independent random variables such that $G_\alpha \preceq H_\alpha$ for every $\alpha \in \Lambda$ (where $\preceq$ indicates stochastic ordering). For every $A \in L(\Lambda)$, let $\tau_A^{(G)}$ and $\tau_A^{(H)}$ be the stopping times defined by eq. (6) associated with each collection. Then, because maximum and summation are monotonic, we have $\tau_A^{(G)} \preceq \tau_A^{(H)}$ for every A. Thus, for every $k \geq 1$, we have $\mathbb{E}\left[\left(\tau_A^{(G)}\right)^k\right] \leq \mathbb{E}\left[\left(\tau_A^{(H)}\right)^k\right]$. Similarly, for every $u > 0$, we have $\mathbb{E}\left[e^{u\tau_A^{(G)}}\right] \leq \mathbb{E}\left[e^{u\tau_A^{(H)}}\right]$.

In particular, if $\{G_\alpha\}_{\alpha\in\Lambda}$ is a collection of independent random variables and $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ is a collection of positive reals such that for every α, G_α is stochastically less than an exponential distribution with rate λ_α, then the upper bounds of the mean and exponential moments in Theorem 3.10, Theorem 3.12 and Proposition 3.14 (using $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ as the rates) apply to the stopping times $\tau_A^{(G)}$ as well. Correspondingly, if for all α, G_α is stochastically greater than an exponential distribution with rate λ_α then the lower bound for the mean and exponential moment Proposition 3.9 and Proposition 3.16 apply.

We now discuss extension of part of the LLN shape limit theorem Theorem 3.18.

Remark 3.24. Suppose $\{G_\alpha\}$ is a collection of independent random variables, but not necessarily exponentially distributed. Instead, we assume that there exists $\lambda > 0$ such that for all $\alpha \in \Lambda$, the variable G_α is stochastically less than an exponential distribution with rate λ. Further, assume that $G_x \preceq G_y$ for any $x, y \in \Lambda$ with $x \leq y$.

Because the proof of Lemma 5.7 only relies on the stochastic ordering of the weights $\{G_\alpha\}_{\alpha\in\Lambda}$ (as in Remark 3.23), it is still true that $A \mapsto \mathbb{E}\left[\tau_A^{(G)}\right]$ is superadditive. By Remark 3.23, as $\{G_\alpha\}_{\alpha\in\Lambda}$ are all stochastically less than an exponential distribution, the upper bound on the mean from Theorem 3.10 applies to $\mathbb{E}\left[\tau_A^{(G)}\right]$ as well. Hence, the limit in eq. (11) will converge and fulfill the given inequality.

Finally, we comment that these stochastic ordering arguments, however, cannot be easily applied to the bounds of variances, such as that of $\operatorname{Var}(\tau_A)$ in Theorem 3.1 for instance. In particular, the L^2 convergence stated in Theorem 3.18 may not be applicable to $\tau_A^{(G)}$.

4 Properties of the Backward Operator Δ

Recall the operator Δ given in eq. (4). We derive several key relations with respect to well-defined functions (see Section 2.1) useful in the sequel. The first result computes the action of Δ on a function $A \mapsto \mathbb{E}[f(\tau_A)]$ in terms of the derivative of f.

Lemma 4.1. *Let $f \in C^1(\mathbb{R}_{\geq 0})$ be a continuously differentiable function such that there exist $D_f \in L(\Lambda)$, $C, t_0 > 0$, and $\mu < \lambda_-(D_f)$ with $|f'(t)| \leq Ce^{t\mu}$ for all $t \geq t_0$. Then, $A \mapsto \mathbb{E}[f(\tau_A)]$ and $A \mapsto \mathbb{E}[f'(\tau_A)]$ are well-defined real functions on $\langle D_f\rangle$ and for all non-empty $A \in \langle D_f\rangle$, we have*

$$\Delta\mathbb{E}[f(\tau_A)] = \mathbb{E}\left[f'(\tau_A)\right].$$

Proof. First, from Corollary 2.3, we know $\mathbb{E}[f'(\tau_A)]$ exists for all $A \subseteq D_f$. Similarly, notice that if $K = |f(0)| + \int_0^{t_0} |f'(t)|\,\mathrm{d}t$ then for any $t \geq t_0$,

$$\begin{aligned}|f(t)| &\leq |f(0)| + \int_0^t |f'(s)|\,\mathrm{d}s \leq K + C\int_{t_0}^t e^{\mu s}\,\mathrm{d}s \\ &\leq K + \frac{C}{\mu}e^{\mu(t-t_0)} \leq \left(K + \frac{C}{\mu}\right)e^{\mu(t-t_0)}\end{aligned}$$

implying that $\mathbb{E}[f(\tau_A)]$ also exists for all $A \subseteq D_f$. We will write $p_A(t) := \mathbb{P}(\tau_A \le t)$. Then, using Lemma 2.2, we can write

$$\begin{aligned}
\mathbb{E}\left[f'(\tau_A)\right] &= f'(0)p_A(0) + \int_0^\infty f'(t)\, \mathrm{d}p_A(t) = \int_0^\infty f'(t)\frac{\mathrm{d}}{\mathrm{d}t}p_A(t)\, \mathrm{d}t \\
&= \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \int_0^\infty f'(t)\left[p_{A\setminus\alpha}(t) - p_A(t)\right]\, \mathrm{d}t \\
&= \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[f(t)[p_{A\setminus\alpha}(t) - p_A(t)]\Big|_{t=0}^\infty - \int_0^\infty f(t)\, \mathrm{d}p_{A\setminus\alpha}(t) + \int_0^\infty f(t)\, \mathrm{d}p_A(t)\right] \\
&= \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[\left(f(0)p_A(0) + \int_0^\infty f(t)\, \mathrm{d}p_A(t)\right) - \left(f(0)p_{A\setminus\alpha}(0) + \int_0^\infty f(t)\, \mathrm{d}p_{A\setminus\alpha}(t)\right)\right] \\
&= \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[\mathbb{E}\left[f(\tau_A)\right] - \mathbb{E}\left[f(\tau_{A\setminus\alpha})\right]\right] = \Delta\mathbb{E}\left[f(\tau_A)\right]. \qquad \square
\end{aligned}$$

Next, we derive a product rule for the backward operator Δ. In order to state this relation, for any two functions $f, g : \langle D\rangle \to \mathbb{R}$ where $D \in L(\Lambda)$ or $D = \Lambda$, define the **quadratic covariance** of f and g as

$$[f,g](A) := \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[f(A) - f(A\setminus\alpha)\right]\cdot\left[g(A) - g(A\setminus\alpha)\right]$$

for all non-empty $A \subseteq D$ and $[f,g](\emptyset) = 0$.

Lemma 4.2. *For any $f, g : \langle D\rangle \to \mathbb{R}$ and any $A \subseteq D \in L(\Lambda)$, we have*

$$\Delta(fg)(A) = f(A)\cdot(\Delta g)(A) + (\Delta f)(A)\cdot g(A) - [f,g](A).$$

Proof. When $A = \emptyset$, we have $\Delta(fg)(A) = (\Delta f)(A) = (\Delta g)(A) = [f,g](A) = 0$. Otherwise, we can write

$$\begin{aligned}
\Delta(fg)(A) + [f,g](A) = &\sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \Big[f(A)g(A) - f(A\setminus\alpha)g(A\setminus\alpha) \\
&+ f(A)g(A) - f(A)g(A\setminus\alpha) - f(A\setminus\alpha)g(A) + f(A\setminus\alpha)g(A\setminus\alpha)\Big] \\
= &\sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[f(A)g(A) - f(A)g(A\setminus\alpha) + f(A)g(A) - f(A\setminus\alpha)g(A)\right] \\
= &\ f(A)\cdot(\Delta g)(A) + (\Delta f)(A)\cdot g(A). \qquad \square
\end{aligned}$$

We will now state the main vehicle for our results, a difference inequality on well-defined functions. We give an analytic argument, although a 'martingale' style proof can also be envisioned.

Proposition 4.3. *Consider functions $f, g : \langle D\rangle \to \mathbb{R}$ with $D \in L(\Lambda)$ and $\varphi : \langle D\rangle \times \mathbb{R} \to \mathbb{R}$ such that for any $A \subseteq D$ and for all $x, y \in \mathbb{R}$ with $x \ne y$, we have $|\varphi(A,x) - \varphi(A,y)| < |x-y|\sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha$. Suppose $f(\emptyset) \ge g(\emptyset)$ and for all $A \subseteq D$ that*

$$(\Delta f)(A) \ge \varphi(A, f(A)) \quad \textit{and} \quad (\Delta g)(A) \le \varphi(A, g(A)).$$

Then, for all $A \subseteq D$, we have $f(A) \ge g(A)$. In particular, if $f(\emptyset) \ge g(\emptyset)$ and for all $A \subseteq D$, we have $(\Delta f)(A) \ge (\Delta g)(A)$, then $f(A) \ge g(A)$ for all $A \subseteq D$.

Proof. We will prove by induction on $|A|$. If $|A| = 0$ then $A = \emptyset$ and $f(A) = f(\emptyset) \ge g(\emptyset) = g(A)$. Otherwise, we consider when $|A| \ge 1$. By the induction hypothesis, we know $f(A\setminus\alpha) \ge g(A\setminus\alpha)$ for all $\alpha \in \mathcal{M}(A)$. Then, take $s := \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha$ and notice that

$$(\Delta f)(A) = \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha[f(A) - f(A\setminus\alpha)] = sf(A) - \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha f(A\setminus\alpha).$$

If $f(A) = g(A)$ then $f(A) \ge g(A)$. Otherwise, $f(A) \ne g(A)$ and we write the difference as

$$\begin{aligned}
sf(A) - sg(A) &= (\Delta f)(A) - (\Delta g)(A) + \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[f(A\setminus\alpha) - g(A\setminus\alpha)\right] \\
&\ge (\Delta f)(A) - (\Delta g)(A) \ge \varphi(A, f(A)) - \varphi(A, g(A)) > -s|f(A) - g(A)|.
\end{aligned}$$

Rearranging, we obtain

$$0 < s(f(A) - g(A)) + s|f(A) - g(A)| = s\left(\operatorname{sgn}(f(A) - g(A)) + 1\right)|f(A) - g(A)|$$

which implies $\operatorname{sgn}(f(A) - g(A)) + 1 > 0$ and $f(A) \neq g(A)$. Then, $\operatorname{sgn}(f(A) - g(A)) > -1$ and therefore $\operatorname{sgn}(f(A) - g(A)) = 1$ and $f(A) > g(A)$. For the last statement, we may take $\varphi(A, x) \equiv \varphi(A) = (\Delta g)(A)$ (or $(\Delta f)(A)$), and note that since $x \neq y$, $|\varphi(A, x) - \varphi(A, y)| \equiv 0 < |x - y| \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha$ is satisfied. ☐

Corollary 4.4. *Suppose $h : \langle D\rangle \to \mathbb{R}$ with $D \in L(\Lambda)$ and $|u| < \lambda_-(D)$. If $(\Delta h)(A) \geq uh(A)$ for all $A \subseteq D$ then $h(A) \geq h(\emptyset)\mathbb{E}\left[e^{u\tau_A}\right]$ for all $A \subseteq D$. Alternatively, if $(\Delta h)(A) \leq uh(A)$ for all $A \subseteq D$ then $h(A) \leq h(\emptyset)\mathbb{E}\left[e^{u\tau_A}\right]$ for all $A \subseteq D$.*

Proof. For the first case, we apply Proposition 4.3 with $f(A) = h(A)$, $g(A) = h(\emptyset)\mathbb{E}\left[e^{u\tau_A}\right]$, and $\varphi(A, x) = ux$. When $A = \emptyset$, we have $f(\emptyset) = h(\emptyset) = g(\emptyset)$, and the claim holds. Otherwise, A has at least one maximal element, and

$$|\varphi(A, x) - \varphi(A, y)| = |u| \cdot |x - y| < \lambda_-(D)|x - y| \leq |x - y| \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha.$$

From Lemma 4.1, we have $(\Delta g)(A) = u\mathbb{E}\left[h(\emptyset)e^{u\tau_A}\right] = ug(A) = \varphi(A, g(A))$. From the hypothesis, we have $(\Delta f)(A) = (\Delta h)(A) \geq uh(A) = uf(A) = \varphi(A, f(A))$. Hence, the conditions of Proposition 4.3 have been fulfilled and so $h(A) = f(A) \geq g(A) = h(\emptyset)\mathbb{E}\left[e^{u\tau_A}\right]$ for $A \subseteq D$.

Similarly, for the second case, we apply Lemma 4.1 with $f(A) = h(\emptyset)\mathbb{E}\left[e^{u\tau_A}\right]$, $g(A) = h(A)$, and $\varphi(A, x) = ux$. Then, we get $h(A) = g(A) \leq f(A) = h(\emptyset)\mathbb{E}\left[e^{u\tau_A}\right]$ for $A \subseteq D$. ☐

5 Proofs

We now prove the results, mostly in succession, as stated in Section 3. We note the exponential bounds in Section 3.4 are proven before those for the means, as they are used in the arguments for the mean bounds in Section 3.3.

5.1 Proofs of Variance and Moment Bounds

We now supply the arguments for the bounds on the variance $\operatorname{Var}(\tau_A)$ and moments given in Section 3.1. These rely on the difference inequalities from Section 4. The variance bounds are given first as they are shorter.

Proof of Theorem 3.1. By Lemma 4.1, since $t \mapsto t^2$ is sub-exponential, the function $A \mapsto \mathbb{E}\left[\tau_A^2\right]$ is well-defined on all of $L(\Lambda)$ with $D = \Lambda$. When $A = \emptyset$, we know $\operatorname{Var}(\tau_\emptyset) = 0 = \mathbb{E}\left[\tau_\emptyset\right]$, and the desired statement holds.

Otherwise, for non-empty A, we apply Δ to $\operatorname{Var}(\tau_A)$. Then, we use Lemma 4.1 and Lemma 4.2, noting $\Delta E[\tau_A] = 1$, to obtain

$$\Delta \operatorname{Var}(\tau_A) = \Delta\mathbb{E}\left[\tau_A^2\right] - \Delta\left(\mathbb{E}\left[\tau_A\right] \cdot \mathbb{E}\left[\tau_A\right]\right) = 2\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_A\right] + \left[\mathbb{E}\left[\tau_A\right], \mathbb{E}\left[\tau_A\right]\right] = \left[\mathbb{E}\left[\tau_A\right], \mathbb{E}\left[\tau_A\right]\right]. \tag{12}$$

Now, since $\mathbb{E}\left[\tau_A\right] \geq \mathbb{E}\left[\tau_B\right]$ whenever $B \subseteq A$, we have $\mathbb{E}\left[\tau_A - \tau_{A\setminus\alpha}\right] \geq 0$. Hence, the quadratic variation is bounded by

$$\lambda_-(A)\left[\mathbb{E}\left[\tau_A\right], \mathbb{E}\left[\tau_A\right]\right] = \lambda_- \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \mathbb{E}\left[\tau_A - \tau_{A\setminus\alpha}\right]^2 \leq \left(\sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha \mathbb{E}\left[\tau_A - \tau_{A\setminus\alpha}\right]\right)^2 = \left(\Delta\mathbb{E}\left[\tau_A\right]\right)^2 = 1 = \Delta\mathbb{E}\left[\tau_A\right]$$

implying that $\Delta\operatorname{Var}(\tau_A) \leq \frac{1}{\lambda_-(A)}\Delta\mathbb{E}\left[\tau_A\right]$. So, by Proposition 4.3, $\operatorname{Var}(\tau_A) \leq \frac{1}{\lambda_-(A)}\mathbb{E}\left[\tau_A\right]$. ☐

Proof of Proposition 3.4. When $A = \emptyset$, the claim holds trivially. Otherwise, consider a non-empty set A. For $n \geq 1$, define $V_n := \min\{\operatorname{Var}(\tau_B) : B \in L(\Lambda), |B| = n\} \in \mathbb{R}$. Also, define $s_A := \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha$. From Theorem 3.1 and Jensen's inequality, we have

$$\begin{aligned}\Delta\operatorname{Var}(\tau_A) = \left[\mathbb{E}\left[\tau_A\right], \mathbb{E}\left[\tau_A\right]\right] &= s_A \sum_{\alpha \in \mathcal{M}(A)} \frac{\lambda_\alpha}{s_A}\mathbb{E}\left[\tau_A - \tau_{A\setminus\alpha}\right]^2 \\ &\geq s_A\left(\sum_{\alpha \in \mathcal{M}(A)} \frac{\lambda_\alpha}{s_A}\mathbb{E}\left[\tau_A - \tau_{A\setminus\alpha}\right]\right)^2 = s_A\left(\frac{\Delta\mathbb{E}\left[\tau_A\right]}{s_A}\right)^2 = \frac{1}{s_A}.\end{aligned}$$

Now, for any n, let $A \in L(\Lambda)$ with $|A| = n$. From the definition of Δ and noting that $|A \setminus \alpha| = |A| - 1 = n - 1$, we obtain

$$\begin{aligned}\mathrm{Var}\left(\tau_A\right) &= \frac{1}{s_A}\sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathrm{Var}\left(\tau_A\right) - \mathrm{Var}\left(\tau_{A\setminus\alpha}\right)\right) + \frac{1}{s_A}\sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha \mathrm{Var}\left(\tau_{A\setminus\alpha}\right)\\ &= \frac{1}{s_A}\Delta\mathrm{Var}\left(\tau_A\right) + \frac{1}{s_A}\sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\mathrm{Var}\left(\tau_{A\setminus\alpha}\right)\\ &\geq \frac{1}{s_A^2} + \frac{1}{s_A}\sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha V_{|A|-1} \geq \frac{1}{f(|A|)^2} + V_{|A|-1} = \frac{1}{f(n)^2} + V_{n-1}.\end{aligned}$$

This holds for all A with $|A| = n$. Here, $V_n \geq \frac{1}{f(n)^2} + V_{n-1}$. Thus, since $f(k)$ increases as k increases, we have as desired,

$$V_n \geq \sum_{k=1}^{n}\frac{1}{f(k)^2} \geq \int_1^{n+1}\frac{\mathrm{d}x}{f(x)^2}. \qquad \square$$

Proof of Proposition 3.2. As $t \mapsto t^n$ is sub-exponential, we may define $q_n : L(\Lambda) \to \mathbb{R}$ by $q_n(A) := \mathbb{E}\left[\tau_A^n\right] - \mathbb{E}\left[\tau_A\right]^n$ for every $A \in L(\Lambda)$ (with $D = \Lambda$). When $A = \emptyset$, the claim holds trivially.

Otherwise, for non-empty A, applying Δ to $\mathbb{E}\left[\tau_A\right]^n$ yields

$$\Delta\mathbb{E}\left[\tau_A\right]^n = \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathbb{E}\left[\tau_A\right]^n - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]^n\right) = \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\sum_{k=0}^{n-1}\mathbb{E}\left[\tau_A\right]^k\mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-k-1}.$$

Then, using Lemma 4.1,

$$\begin{aligned}n\mathbb{E}\left[\tau_A\right]^{n-1}\cdot 1 = n\mathbb{E}\left[\tau_A\right]^{n-1}\cdot\Delta\mathbb{E}\left[\tau_A\right] &= n\mathbb{E}\left[\tau_A\right]^{n-1}\sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\\ &= \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\sum_{k=0}^{n-1}\mathbb{E}\left[\tau_A\right]^{n-1}.\end{aligned}$$

Hence, the backward operator Δ applied to q_n gives

$$\begin{aligned}\Delta q_n &= \Delta\mathbb{E}\left[\tau_A^n\right] - \Delta\mathbb{E}\left[\tau_A\right]^n = n\left(\mathbb{E}\left[\tau_A^{n-1}\right] - \mathbb{E}\left[\tau_A\right]^{n-1}\right) + n\mathbb{E}\left[\tau_A\right]^{n-1} - \Delta\mathbb{E}\left[\tau_A\right]^n\\ &= nq_{n-1}(A) + \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\sum_{k=0}^{n-1}\left(\mathbb{E}\left[\tau_A\right]^{n-1} - \mathbb{E}\left[\tau_A\right]^k\mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-k-1}\right).\end{aligned} \tag{13}$$

Now, we consider just the sum over k,

$$\begin{aligned}\sum_{k=0}^{n-1}\left(\mathbb{E}\left[\tau_A\right]^{n-1} - \mathbb{E}\left[\tau_A\right]^k\mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-k-1}\right) &= \sum_{k=0}^{n-2}\mathbb{E}\left[\tau_A\right]^k\left(\mathbb{E}\left[\tau_A\right]^{n-k-1} - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-k-1}\right)\\ &= \left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\sum_{k=0}^{n-2}\mathbb{E}\left[\tau_A\right]^k\sum_{j=0}^{n-k-2}\mathbb{E}\left[\tau_A\right]^j\mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-j-k-2}\\ &= \left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\sum_{s=0}^{n-2}\mathbb{E}\left[\tau_A\right]^s\mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-s-2}\sum_{j=0}^{s}1\\ &\leq \left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\sum_{s=0}^{n-2}(s+1)\mathbb{E}\left[\tau_A\right]^{n-2}\\ &= \left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)\frac{n(n-1)}{2}\mathbb{E}\left[\tau_A\right]^{n-2}\end{aligned}$$

where $s = k + j$ and $\mathbb{E}\left[\tau_{A\setminus\alpha}\right] \leq \mathbb{E}\left[\tau_A\right]$. Substituting this back into eq. (13) and recalling eq. (12) gives

$$\begin{aligned}\Delta q_n &\leq nq_{n-1}(A) + \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left(\mathbb{E}\left[\tau_A\right] - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]\right)^2\cdot\frac{n(n-1)}{2}\mathbb{E}\left[\tau_A\right]^{n-2}\\ &= nq_{n-1}(A) + \frac{n(n-1)}{2}\mathbb{E}\left[\tau_A\right]^{n-2}\Delta\mathrm{Var}\left(\tau_A\right).\end{aligned} \tag{14}$$

Next, we notice that $\mathrm{Var}\left(\tau_{A\setminus\alpha}\right) \geq 0$ and $\mathbb{E}\left[\tau_A\right]^{n-2}$ is non-decreasing in A. Then,

$$\begin{aligned}\left[\mathrm{Var}\left(\tau_A\right), \mathbb{E}\left[\tau_A\right]^{n-2}\right] &= \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left(\mathrm{Var}\left(\tau_A\right) - \mathrm{Var}\left(\tau_{A\setminus\alpha}\right)\right)\left(\mathbb{E}\left[\tau_A\right]^{n-2} - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-2}\right) \\ &\leq \mathrm{Var}\left(\tau_A\right) \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left(\mathbb{E}\left[\tau_A\right]^{n-2} - \mathbb{E}\left[\tau_{A\setminus\alpha}\right]^{n-2}\right) = \mathrm{Var}\left(\tau_A\right)\Delta\mathbb{E}\left[\tau_A\right]^{n-2}.\end{aligned}$$

Applying Lemma 4.2,

$$\begin{aligned}\Delta\left(\mathrm{Var}\left(\tau_A\right)\mathbb{E}\left[\tau_A\right]^{n-2}\right) &= \mathbb{E}\left[\tau_A\right]^{n-2}\Delta\mathrm{Var}\left(\tau_A\right) + \mathrm{Var}\left(\tau_A\right)\Delta\mathbb{E}\left[\tau_A\right]^{n-2} - \left[\mathrm{Var}\left(\tau_A\right), \mathbb{E}\left[\tau_A\right]^{n-2}\right] \\ &\geq \mathbb{E}\left[\tau_A\right]^{n-2}\Delta\mathrm{Var}\left(\tau_A\right) + \mathrm{Var}\left(\tau_A\right)\Delta\mathbb{E}\left[\tau_A\right]^{n-2} - \mathrm{Var}\left(\tau_A\right)\Delta\mathbb{E}\left[\tau_A\right]^{n-2} = \mathbb{E}\left[\tau_A\right]^{n-2}\Delta\mathrm{Var}\left(\tau_A\right).\end{aligned}$$

So, eq. (14) becomes

$$\Delta q_n \leq nq_{n-1}(A) + \frac{n(n-1)}{2}\mathbb{E}\left[\tau_A\right]^{n-2}\Delta\mathrm{Var}\left(\tau_A\right) \leq nq_{n-1}(A) + \frac{n(n-1)}{2}\Delta\left(\mathrm{Var}\left(\tau_A\right)\mathbb{E}\left[\tau_A\right]^{n-2}\right).$$

Notice that for $n = 0$ and $n = 1$, we have $q_0(A) = q_1(A) = 0$, and so the inequality eq. (7) holds. Next, we show by induction on $n \geq 2$ that $q_n(A) \leq K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right]$. For $n = 2$, we have by assumption that

$$q_2(A) = \mathbb{E}\left[\tau_A^2\right] - \mathbb{E}\left[\tau_A\right]^2 = \mathrm{Var}\left(\tau_A\right) \leq K\mathbb{E}\left[\tau_A^p\right] = K\frac{2(2-1)^2}{2}\mathbb{E}\left[\tau_A^{p+2-2}\right].$$

Then, when $n \geq 3$,

$$\begin{aligned}\Delta q_n(A) - \frac{n(n-1)}{2}\Delta\left(\mathrm{Var}\left(\tau_A\right)\mathbb{E}\left[\tau_A\right]^{n-2}\right) &\leq nq_{n-1}(A) \\ &\leq Kn\frac{(n-1)(n-2)^2}{2}\mathbb{E}\left[\tau_A^{p+n-3}\right] \\ &\leq Kn\frac{(n-1)(n-2)(n-2+p)}{2}\mathbb{E}\left[\tau_A^{p+n-3}\right] \\ &= K\frac{n(n-1)(n-2)}{2}\Delta\mathbb{E}\left[\tau_A^{p+n-2}\right],\end{aligned}$$

using Lemma 4.1, while

$$q_n(\emptyset) - \frac{n(n-1)}{2}\mathrm{Var}\left(\tau_\emptyset\right)\mathbb{E}\left[\tau_\emptyset\right]^{n-2} = 0 \leq 0 = K\frac{n(n-1)(n-2)}{2}\mathbb{E}\left[\tau_\emptyset^{p+n-2}\right].$$

Then, from Proposition 4.3, using $\mathbb{E}\left[\tau_A^p\right] \leq \mathbb{E}\left[\tau_A\right]^p$ as $0 < p \leq 1$ and $\mathbb{E}\left[\tau_A\right]^p\mathbb{E}\left[\tau_A\right]^{n-2} \leq \mathbb{E}\left[\tau_A^{p+n-2}\right]$ as $p + n - 2 \geq 1 + p$, we get

$$\begin{aligned}q_n(A) &\leq \frac{n(n-1)}{2}\mathrm{Var}\left(\tau_A\right)\mathbb{E}\left[\tau_A\right]^{n-2} + K\frac{n(n-1)(n-2)}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right] \\ &\leq K\frac{n(n-1)}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right] + K\frac{n(n-1)(n-2)}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right] = K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right]\end{aligned}$$

completing the induction and proving the desired result. □

Proof of Corollary 3.3. As in the proof of Proposition 3.2, consider $q_n(A) := \mathbb{E}\left[\tau_A^n\right] - \mathbb{E}\left[\tau_A\right]^n$. We know $q_n(A) \geq 0$ and, from

Proposition 3.2, that $q_n(A) \le K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^{p+n-2}\right]$. Then, expanding the central moment,

$$\begin{aligned}
|\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]| &= \left|\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathbb{E}\left[\tau_A^k\right]\mathbb{E}[\tau_A]^{n-k}\right| \\
&= \left|\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\mathbb{E}[\tau_A]^k\,\mathbb{E}[\tau_A]^{n-k} + \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}q_k(A)\mathbb{E}[\tau_A]^{n-k}\right| \\
&= \left|(\mathbb{E}[\tau_A] - \mathbb{E}[\tau_A])^n + \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}q_k(A)\mathbb{E}[\tau_A]^{n-k}\right| \\
&\le \sum_{k=0}^{n}\binom{n}{k}K\frac{k(k-1)^2}{2}\mathbb{E}\left[\tau_A^{p+k-2}\right]\mathbb{E}[\tau_A]^{n-k} \\
&= \frac{K}{2}\sum_{k=2}^{n}n(n-1)\binom{n-2}{k-2}(k-1)\mathbb{E}\left[\tau_A^{p+k-2}\right]\mathbb{E}[\tau_A]^{n-k} \\
&\le K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^p\sum_{k=0}^{n-2}\binom{n-2}{k}\tau_A^k\mathbb{E}[\tau_A]^{n-2-k}\right] = K\frac{n(n-1)^2}{2}\mathbb{E}\left[\tau_A^p\left(\tau_A + \mathbb{E}[\tau_A]\right)^{n-2}\right].
\end{aligned}$$

To address the last claim, we observe that $\mathbb{E}\left[\tau_A^p\right] \le \mathbb{E}[\tau_A]^p$ as $0 < p \le 1$, and

$$\begin{aligned}
|\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]| &\le \mathbb{E}\left[\tau_A^p(\tau_A + \mathbb{E}[\tau_A])^{n-2}\right] \le 2^{n-3}\big(\mathbb{E}\left[\tau_A^{p+n-2}\right] + \mathbb{E}\left[\tau_A^p\right]\mathbb{E}[\tau_A]^{n-2}\big) \\
&\le 2^{n-3}2^{p+n-3}\big(\mathbb{E}\left[|\tau_A - \mathbb{E}[\tau_A]|^{p+n-2}\right] + \mathbb{E}[\tau_A]^{p+n-2}\big) + \mathbb{E}[\tau_A]^{p+n-2}\big).
\end{aligned} \tag{15}$$

Suppose n is even. Note that $\mathbb{E}\left[|\tau_A - \mathbb{E}[\tau_A]|^{p+n-2}\right] \le \mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]^{\frac{p+n-2}{n}} \le \epsilon\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]$ for every $0 < \epsilon < 1$ when $\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]$ diverges as $A = A_j$ grows. Hence, by choosing $\epsilon 2^{p+2n-6} \le 1/2$ and rearranging eq. (15), we have $|\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]| \le C\mu_j^{p+n-2}$ and the claim holds. If $|\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n|]$ does not diverge, the claim trivially holds.

For odd n, by Schwarz inequality,

$$|\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^n]| \le \mathbb{E}\left[(\tau_A - \mathbb{E}[\tau_A])^{n-1}(\mathbb{E}[|\tau_A - \mathbb{E}[\tau_A]|]\cdot 1)\right] \le \sqrt{\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^{n+1}]\,\mathbb{E}[(\tau_A - \mathbb{E}[\tau_A])^{n-1}]}. \tag{16}$$

By applying the previous argument to the even power terms, we bound eq. (16) by $C\sqrt{\mu_j^{p+n+1-2}\mu_j^{p+n-1+2}} = O(\mu_j^{p+n-2})$ as desired. ☐

5.2 Proofs of Moment Generating Function Bounds

Recall the definition of the 'greater path function' $\Gamma_\ge f : L(\Lambda) \to \mathbb{R}$. We also define the **backward difference operator** $\delta f : \mathbb{N}_0 \to \mathbb{R}$ of f as

$$(\delta f)(n) := f(n) - f(n-1) \text{ for } n \ge 1 \quad\text{and}\quad (\delta f)(0) = f(0).$$

Also, we define the function $\epsilon : \mathbb{N}_0 \to \mathbb{R}$ by $\epsilon(0) = 1$ and $\epsilon(n) = 0$ for $n > 0$. Note that $\delta 1 = \epsilon$ (where 1 is the constant function).

Lemma 5.1. *For any $A \in L(\Lambda)$, we have $\Pi_m(A) = \bigcup_{\alpha\in\mathcal{M}(A)}(\Pi(A) \setminus \Pi(A \setminus \alpha))$.*

Proof. Let $\pi \in \Pi_m(A)$. Since π is maximal in A we have $\beta := \pi_{\ell(\pi)} \in \mathcal{M}(A)$. Since $\beta \notin A \setminus \beta$, we have $\pi \notin \Pi(A \setminus \beta)$. Hence, $\pi \in \Pi(A) \setminus \Pi(A \setminus \beta)$.

Conversely, let $\pi \in \bigcup_{\alpha\in\mathcal{M}(A)}(\Pi(A)\setminus\Pi(A\setminus\alpha))$ then $\pi \in \Pi(A)\setminus\Pi(A\setminus\beta)$ for some $\beta \in \mathcal{M}(A)$. Because $\pi \subseteq \Pi(A)$ and $\pi \nsubseteq \Pi(A\setminus\beta)$, we must have $\beta \in \pi$. Since β is maximal in A, we know $\pi_{\ell(\pi)} = \beta$ and may conclude π is maximal in A. ☐

Proposition 5.2. *For any non-decreasing $f : \mathbb{N}_0 \to \mathbb{R}$ with $f(0) \ge 0$ and any $A \in L(\Lambda)$, we have*

$$\Delta\left(\Gamma_\ge f\right)(A) \ge \lambda_-(A)\Gamma_\ge(\delta f)(A).$$

Proof. The display holds for $A = \emptyset$ by our conventions. Otherwise, for non-empty A, applying Δ to $\Gamma_\ge f$ and recalling Lemma 5.1, for all $A \in L(\Lambda)$, we have

$$\begin{aligned}
\left(\Delta\Gamma_\ge f\right)(A) &= \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\left[\sum_{\pi\in\Pi(A)}f(\ell(\pi)) - \sum_{\pi\in\Pi(A\setminus\alpha)}f(\ell(\pi))\right] \\
&= \sum_{\alpha\in\mathcal{M}(A)}\lambda_\alpha\sum_{\pi\in\Pi(A)\setminus\Pi(A\setminus\alpha)}f(\ell(\pi)) = \sum_{\pi\in\Pi_m(A)}\lambda_{\pi_{\ell(\pi)}}f(\ell(\pi)).
\end{aligned} \tag{17}$$

Note that $\delta f \geq 0$ since f is non-decreasing and $f(0) \geq 0$. Because every path in A is contained in at least one maximal path and every subpath of a maximal path may be labelled by its length $k = \ell(\xi)$ for $\xi \subseteq \pi$, we may write

$$\begin{aligned}
\lambda_-(A)\left(\Gamma_{\geq}\delta f\right)(A) &= \lambda_- \sum_{\xi\in\Pi(A)} \delta f(\ell(\xi)) \leq \lambda_- \sum_{\pi\in\Pi_m(A)} \sum_{\substack{\xi\in\Pi(A)\\ \xi\subseteq\pi}} \delta f(\ell(\xi)) = \lambda_- \sum_{\pi\in\Pi_m(A)} \sum_{k=0}^{\ell(\pi)} \delta f(k) \\
&= \lambda_- \sum_{\pi\in\Pi_m(A)} \left(f(0) + \sum_{k=1}^{\ell(\pi)} f(k) - f(k-1) \right) = \sum_{\pi\in\Pi_m(A)} \lambda_-(A) f(\ell(\pi)) \\
&\leq \left(\Delta\Gamma_{\geq} f\right)(A).
\end{aligned}$$

□

Proof of Proposition 3.14. One may verify that the claim holds when $B = \emptyset$. Otherwise, we fix a non-empty $B \in L(\Lambda)$ and will consider functions defined on $\langle B\rangle$. All applications of the operator Δ will be with respect to $A \in \langle B\rangle$. Note that $(\Gamma_{\geq}\epsilon)(A) = \sum_{\pi\in\Pi(A)} \epsilon(\ell(\pi)) = \epsilon(\ell(\emptyset)) = \epsilon(0) = 1$. By Proposition 5.2, we have $(\Delta\Gamma_{\geq}1)(A) \geq \lambda_-(A)(\Gamma_{\geq}\epsilon)(A) = \lambda_-(A)$.

For any $r > 1$, consider $f : \mathbb{N}_0 \to \mathbb{R}$ given by $f(n) = \frac{r^{n+1}-1}{r-1}$. Then, $\delta f(0) = \frac{r^{0+1}-1}{r-1} = 1 = r^0$ and, for $n \geq 1$, we have

$$\delta f(n) = \frac{r^{n+1}-1}{r-1} - \frac{r^n - 1}{r-1} = \frac{r^n(r-1)}{r-1} = r^n.$$

Denote by r^n the function $n \mapsto r^n$. Using the linearity of Δ and $\Gamma_{\geq}$ as well as Proposition 5.2, for any non-empty $A \subseteq B$, we have

$$\begin{aligned}
\lambda_-(B)\Gamma_{\geq}\{r^n\}(A) &\leq \lambda_-(A)\Gamma_{\geq}\{r^n\}(A) = \lambda_-(A)\Gamma_{\geq}\{\delta f\}(A) \\
&\leq \Delta\Gamma_{\geq}\{f\}(A) = \frac{r}{r-1}\Delta\Gamma_{\geq}\{r^n\}(A) - \frac{1}{r-1}\Delta\Gamma_{\geq}\{1\}(A) \\
&\leq \frac{r}{r-1}\Delta\Gamma_{\geq}\{r^n\}(A) - \lambda_-(A)\frac{1}{r-1} \leq \frac{r}{r-1}\Delta\Gamma_{\geq}\{r^n\}(A) - \lambda_-(B)\frac{1}{r-1}.
\end{aligned}$$

Hence, $\Delta\Gamma_{\geq}\{r^n\}(A) \geq \lambda_-(B)\left(1 - \frac{1}{r}\right)\Gamma_{\geq}\{r^n\}(A) + \lambda_-(B)\frac{1}{r}$.

Consider the function $A \mapsto \frac{1}{r-1}\mathbb{E}\left[re^{\lambda_-(B)(1-1/r)\tau_A} - 1\right]$ which is well-defined for $A \in \langle B\rangle$ (see Section 2.1). Let $u := 1-1/r < 1$. Then, by Lemma 4.1, we have

$$\Delta\mathbb{E}\left[\frac{re^{\lambda_-(B)(1-1/r)\tau_A} - 1}{r-1}\right] = \lambda_-(B)\left(1 - \frac{1}{r}\right)\mathbb{E}\left[\frac{re^{\lambda_-(B)(1-1/r)\tau_A}}{r-1}\right] = \lambda_-(B)\left(1 - \frac{1}{r}\right)\mathbb{E}\left[\frac{re^{\lambda_-(B)u\tau_A} - 1}{r-1}\right] + \frac{\lambda_-(B)}{r} \tag{18}$$

with $\frac{1}{r-1}\mathbb{E}\left[re^{\lambda_-(B)\,(1-1/r)\tau_\emptyset} - 1\right] = 1$ and $\Gamma_{\geq}\{r^n\}(\emptyset) = \sum_{\pi\in\Pi(\emptyset)} r^{\ell(\pi)} = r^{\ell(\emptyset)} = 1$.

Therefore, by Proposition 4.3 with $\varphi(A, x) = \lambda_-(B)\left(1 - \frac{1}{r}\right)x + \lambda_-(B)\frac{1}{r}$, we have $\Gamma_{\geq}\{r^n\}(A) \geq \frac{1}{r-1}\mathbb{E}\left[re^{\lambda_-(B)(1-1/r)\tau_A} - 1\right]$. Taking $A = B$, we can write

$$u\Gamma_{\geq}\left\{\frac{1}{(1-u)^n}\right\}(B) - u = \frac{r-1}{r}\Gamma_{\geq}\{r^n\}(B) - 1 + \frac{1}{r} \geq \mathbb{E}\left[e^{\lambda_-(B)\,(1-1/r)\tau_B}\right] - 1 = \mathbb{E}\left[e^{\lambda_-(B)\,u\tau_B}\right] - 1.$$

□

We will use a similar approach to construct a lower bound for the moment generating functions. Recall the definitions of a 'branching allocation' ψ, its 'weight' ω_ψ along a path $\pi \in \Pi(\Lambda)$, and 'lesser path functions' $\Gamma^{\psi}_{\leq} f : L(\Lambda) \to \mathbb{R}$ in Definition 3.15. As with the greater path function, there is another difference inequality with respect to the lesser path function. To derive it, we first consider how a branching allocation distributes weight among paths.

Lemma 5.3. *For any branching allocation ψ and any $\xi \in \Pi(A)$, we have*

$$\sum_{\substack{\pi\in\Pi_m(A)\\ \xi\subseteq\pi}} \omega_\psi(\pi) \leq \omega_\psi(\xi).$$

Proof. We will induct on $\max_\pi(\ell(\pi) - \ell(\xi))$. When $\max_\pi(\ell(\pi) - \ell(\xi)) = 0$, since $\xi = \pi$, we have ξ is maximal in A and

$$\sum_{\substack{\pi\in\Pi_m(A)\\ \xi\subseteq\pi}} \omega_\psi(\pi) = \sum_{\pi=\xi} \omega_\psi(\pi) = \omega_\psi(\xi).$$

Otherwise, when $\max_\pi(\ell(\pi) - \ell(\xi)) > 0$, we have that ξ is not maximal in A. When ξ is empty, we have $\max_\pi(\ell(\pi) - \ell(\xi)) = \max_\pi \ell(\pi) > \max_\pi(\ell(\pi) - 1) = \max_\pi(\ell(\pi) - \ell((\mu)))$, where (μ) is a singleton path contained in π. Then, using the induction hypothesis,

$$\sum_{\substack{\pi\in\Pi_m(A)\\ \xi\subseteq\pi}} \omega_\psi(\pi) = \sum_{\pi\in\Pi_m(A)} \omega_\psi(\pi) = \sum_{\substack{\mu\in A\\ \mu \text{ is minimal}}} \sum_{\substack{\pi\in\Pi_m(A)\\ (\mu)\subseteq\pi}} \omega_\psi(\pi) \leq \sum_{\substack{\mu\in A\\ \mu \text{ is minimal}}} \omega_\psi((\mu)) = \sum_{\substack{\mu\in A\\ \mu \text{ is minimal}}} \psi_\mu \leq 1.$$

When ξ is non-empty and $\alpha = \xi_{\ell(\xi)}$. Let $\{\xi^{(\beta)}\}_{\alpha\to\beta}$ be the paths in A obtained by extending ξ by an upper neighbor β of α. By definition of ω_ψ, we have $\omega_\psi(\xi^{(\beta)}) = \psi_{\alpha\to\beta}\cdot\omega_\psi(\xi)$. Then, because every maximal path contains one of $\{\xi^{(\beta)}\}$ for $\alpha\to\beta$, using the induction hypothesis $\max_\pi(\ell(\pi) - \ell(\xi)) > \max_\pi(\ell(\pi) - \ell(\xi^{(\beta)}))$ where $\xi^{(\beta)} \subseteq \pi$, we have

$$\sum_{\substack{\pi\in\Pi_m(A)\\ \xi\subseteq\pi}} \omega_\psi(\pi) \leq \sum_{\alpha\to\beta} \sum_{\substack{\pi\in\Pi_m(A)\\ \xi^{(\beta)}\subseteq\pi}} \omega_\psi(\pi) \leq \sum_{\alpha\to\beta} \omega(\xi^{(\beta)}) = \omega_\psi(\xi) \sum_{\alpha\to\beta} \psi_{\alpha\to\beta} \leq \omega_\psi(\xi). \qquad \square$$

Proposition 5.4. *Let ψ be a branching allocation. For any non-decreasing $f : \mathbb{N}_0 \to \mathbb{R}$ with $f(0) \geq 0$ and $A \in L(\Lambda)$, we have*

$$\Delta\left(\Gamma_\leq^\psi f\right)(A) \leq \lambda_+(A)\Gamma_\leq^\psi(\delta f)(A).$$

Proof. The display holds when $A = \emptyset$ by our conventions. Otherwise, for non-empty $A \in L(\Lambda)$, applying the operator Δ to $\Gamma_\leq^\psi f$ and using Lemma 5.1, we get

$$\left(\Delta\Gamma_\leq^\psi f\right)(A) = \sum_{\alpha\in\mathcal{M}(A)} \lambda_\alpha \left[\sum_{\pi\in\Pi(A)} f(\ell(\pi))\omega_\psi(\pi) - \sum_{\pi\in\Pi(A\setminus\alpha)} f(\ell(\pi))\omega_\psi(\pi)\right] = \sum_{\pi\in\Pi_m(A)} \lambda_{\pi_{\ell(\pi)}} f(\ell(\pi))\omega_\psi(\pi).$$

Recalling the scheme of proof of Proposition 5.2, using Lemma 5.3, we can write

$$\begin{aligned}
\lambda_+(A)\left(\Gamma_\leq^\psi(\delta f)\right)(A) &= \lambda_+(A) \sum_{\xi\in\Pi(A)} \delta f(\ell(\xi))\omega_\psi(\xi) \\
&\geq \lambda_+(A) \sum_{\xi\in\Pi(A)} \sum_{\substack{\pi\in\Pi_m(A)\\ \xi\subseteq\pi}} \delta f(\ell(\xi))\omega_\psi(\pi) = \lambda_+(A) \sum_{\pi\in\Pi_m(A)} \omega_\psi(\pi) \sum_{k=0}^{\ell(\pi)} \delta f(k) \\
&\geq \sum_{\pi\in\Pi_m(A)} \lambda_{\pi_{\ell(\pi)}} f(\ell(\pi))\omega_\psi(\pi) = \left(\Delta\Gamma_\leq^\psi f\right)(A). \qquad \square
\end{aligned}$$

Proof of Proposition 3.16. Consider non-empty $B \in L(\Lambda)$ and $A \in \langle B\rangle$. Note that $(\Gamma_\leq^\psi\epsilon)(A) = \sum_{\pi\in\Pi(A)} \epsilon(\ell(\pi))\omega_\psi(\pi) = \omega_\psi(\emptyset) = 1$. Then, by Proposition 5.4, $\Delta(\Gamma_\leq^\psi 1)(A) \leq \lambda_+(A)(\Gamma_\leq^\psi\epsilon)(A) = \lambda_+(A)$. As in the proof of Proposition 3.14, let $r > 1$ and $f : \mathbb{N}_0 \to \mathbb{R}$ with $f(n) := \frac{r^{n+1}-1}{r-1}$ so that $\delta f(n) = r^n$. Then, using the linearity of Δ and $\Gamma_\leq$ as well as Proposition 5.4, we have

$$\begin{aligned}
\lambda_+(B)\Gamma_\leq^\psi\{r^n\}(A) &\geq \lambda_+(A)\Gamma_\leq^\psi\{r^n\}(A) = \lambda_+(A)\Gamma_\leq^\psi(\delta f) \\
&\geq \Delta\Gamma_\leq^\psi\{f\}(A) = \frac{r}{r-1}\Delta\Gamma_\leq^\psi\{r^n\}(A) - \frac{1}{r-1}\Delta\Gamma_\leq^\psi\{1\}(A) \\
&\geq \frac{r}{r-1}\Delta\Gamma_\leq^\psi\{r^n\}(A) - \lambda_+(A)\frac{1}{r-1} \geq \frac{r}{r-1}\Delta\Gamma_\leq^\psi\{r^n\}(A) - \lambda_+(B)\frac{1}{r-1}.
\end{aligned}$$

Hence, $\Delta\Gamma_\leq^\psi\{r^n\}(A) \leq \lambda_+(B)\left(1 - \frac{1}{r}\right)\Gamma_\leq^\psi\{r^n\}(A) + \lambda_+(B)\frac{1}{r}$.

Recall eq. (18) (using $\lambda_+(B)$ instead of $\lambda_-(B)$). Then, $\Gamma_\leq^\psi\{r^n\}(A) \leq \frac{1}{r-1}\mathbb{E}\left[re^{\lambda_+(B)(1-1/r)\tau_A} - 1\right]$, the right-hand side function being well-defined for $u = 1 - 1/r < \lambda_-(B)/\lambda_+(B)$. Taking now $A = B$ and rewriting, yields the desired result. $\square$

5.3 Proofs of Mean Bounds

We now turn to the estimation of the mean $\mathbb{E}[\tau_A]$ based on the bounds in Section 5.2.

Proof of Proposition 3.9. The desired statement holds for $B = \emptyset$. Otherwise, consider non-empty $B \in L(\Lambda)$ and a non-empty $A \in \langle B \rangle$. Either all maximum length paths in A end at a single element $\tilde{\alpha} \in \mathcal{M}(A)$ or not. Suppose such a $\tilde{\alpha}$ exists. Then, for all $\alpha \in \mathcal{M}(A) \setminus \tilde{\alpha}$, we have $\tilde{\alpha} \in A \setminus \alpha$ so $\ell(A \setminus \alpha) = \ell(A)$. On the other hand, $\ell(A \setminus \tilde{\alpha}) = \ell(A) - 1$ since all maximum paths in $A \setminus \tilde{\alpha}$ have length at most $\ell(A) - 1$ and there exists a maximal path in A which, after truncating $\tilde{\alpha}$, is a maximal path in $A \setminus \tilde{\alpha}$ with length $\ell(A) - 1$. Applying Δ to $\ell(A)$, we have

$$\Delta \ell(A) = \sum_{\alpha \in \mathcal{M}(A)} \lambda_\alpha [\ell(A) - \ell(A \setminus \alpha)] = \lambda_{\tilde{\alpha}} [\ell(A) - \ell(A \setminus \tilde{\alpha})] = \lambda_{\tilde{\alpha}} \le \lambda_+(B).$$

Alternatively, suppose no such $\tilde{\alpha}$ exists. Then, there exist at least two elements in $\mathcal{M}(A)$ that are the terminuses of maximum length paths. Thus, for any $\alpha \in \mathcal{M}(A)$, the set $A \setminus \alpha$ will still contain a path of length $\ell(A)$ implying that $\ell(A \setminus \alpha) = \ell(A)$. Hence, $\Delta \ell(A) = 0$.

Therefore, in all cases when A is non-empty, noting $\Delta \mathbb{E}[\tau_A] = 1$ (see Lemma 4.1), we have $\Delta \ell(A) \le \lambda_+(B) \le \Delta (\lambda_+(B) \mathbb{E}[\tau_A])$. The inequality $0 = \Delta \ell(\emptyset) \le \Delta (\lambda_+(B) \mathbb{E}[\tau_\emptyset]) = 0$ also holds. Since $\ell(\emptyset) = 0 = \lambda_+(B) \mathbb{E}[\tau_\emptyset]$, we conclude $\ell(A) \le \lambda_+(B) \mathbb{E}[\tau_A]$ by Proposition 4.3. The result follows by taking $A = B$. □

Remark 5.5. We give an alternate proof of Proposition 3.9 for the reader's interest. Consider the inequality in Proposition 3.16. Dividing by u, the right-hand side converges to $\lambda_+(B) \mathbb{E}[\tau_B]$ as $u \to 0^+$. For the left-hand side, consider a branching allocation ψ such that $\sum_\beta \psi_{\alpha \to \beta} = 1$ and $\sum_{\mu \text{ is minimal}} \psi_\mu = 1$. Indeed, one may take $\psi_{\alpha \to \beta} = 1/d_\alpha$ where d_α is the number of upper neighbors of α and $\psi_\mu = 1/d_{\min}$ where $d_{\min}$ is the number of minimal elements. Then, we may write

$$\lim_{u \to 0^+} \Gamma^\psi_\le \left\{ \frac{1}{(1-u)^n} \right\} - 1 = \lim_{u \to 0^+} \left(\sum_{\pi \in \Pi(B)} \frac{1}{(1-u)^{\ell(\pi)}} \omega_\psi(\pi) \right) - 1 = -1 + \sum_{\pi \in \Pi(B)} \omega_\psi(\pi) = -1 + \sum_{k=0}^{\ell(B)} \sum_{\substack{\pi \in \Pi(B) \\ \ell(\pi) = k}} \omega_\psi(\pi). \tag{19}$$

Now, as $\omega_\psi(\pi) = \psi_{\pi_1} \prod_{i=2}^{\ell(\pi)} \psi_{\pi_{i-1} \to \pi_i}$, summing successively over the possible points π_i, we have that $\sum_{\substack{\pi \in \Pi(B) \\ \ell(\pi) = k}} \omega_\psi(\pi) = 1$, and so eq. (19) equals $-1 + \ell(B) + 1 = \ell(B)$ as desired.

Proof of Theorem 3.10. The claim follows for $A = \emptyset$ by our conventions. Otherwise, consider a non-empty set A. For any $0 < u < 1$, as the mapping $t \mapsto e^{\lambda_-(A)\, ut}$ is convex, by Jensen's inequality, $\mathbb{E}\left[e^{\lambda_-(A)\, u \tau_A} \right] \ge e^{u \lambda_-(A) \mathbb{E}[\tau_A]}$. Consider $r := \frac{1}{1-u}$. Then, using Proposition 3.14 and Proposition 5.2 and recalling eq. (17), we have

$$\begin{aligned}
e^{u\lambda_-(A)\mathbb{E}[\tau_A]} - 1 &\le u \left(\Gamma_\ge \left\{ \frac{1}{(1-u)^n} \right\}(A) - 1 \right) = \left(1 - \frac{1}{r} \right) \left(\Gamma_\ge \{ r^n \}(A) - 1 \right) \\
&\le \frac{1}{\lambda_-(A)} \left(1 - \frac{1}{r} \right) \Delta \Gamma_\ge \left\{ \frac{r^{n+1} - 1}{r - 1} \right\}(A) - 1 + \frac{1}{r} = -1 + \frac{1}{r} + \frac{r-1}{r} \frac{1}{\lambda_-(A)} \sum_{\pi \in \Pi_m(A)} \lambda_{\pi_{\ell(\pi)}} \frac{r^{\ell(\pi)+1} - 1}{r - 1} \\
&\le -1 + \frac{1}{r} + \frac{1}{\lambda_-(A)} \sum_{\pi \in \Pi_m(A)} \lambda_+(A) \left[r^{\ell(A)} - \frac{1}{r} \right] = -1 + \frac{1}{r} \left(1 - \frac{\lambda_+(A)}{\lambda_-(A)} |\Pi_m(A)| \right) + \frac{\lambda_+(A)}{\lambda_-(A)} |\Pi_m(A)| r^{\ell(A)} \\
&\le -1 + \frac{\lambda_+(A)}{\lambda_-(A)} |\Pi_m(A)| r^{\ell(A)}.
\end{aligned}$$

Hence, as $r > 1$, we have $e^{u\lambda_-(A)\mathbb{E}[\tau_A]} \le \frac{\lambda_+(A)}{\lambda_-(A)} |\Pi_m(A)| r^{\ell(A)} = e^{\eta(A) + \kappa(A)} r^{\ell(A)}$. After taking the logarithm, we obtain

$$\lambda_-(A) \mathbb{E}[\tau_A] \le \frac{\kappa(A) + \eta(A) + \ell(A) \log(r)}{u} = \frac{1}{u} \left(\kappa(A) + \eta(A) + \ell(A) \log \left(1 + \frac{u}{1-u} \right) \right) \le \frac{1}{u} \left(\kappa(A) + \eta(A) + \ell(A) \frac{u}{1-u} \right).$$

We may optimize our choice of u to minimize this value. Differentiate $\frac{\kappa(A)+\eta(A)}{u} + \frac{\ell(A)}{1-u}$ and consider

$$0 = -\frac{\kappa(A) + \eta(A)}{u_0^2} + \frac{\ell(A)}{(1-u_0)^2} \quad \text{and} \quad \sqrt{\frac{\kappa(A) + \eta(A)}{\ell(A)}} = \frac{u_0}{1-u_0} = \frac{1}{1-u_0} - 1$$

implying $u_0 = \frac{\sqrt{\kappa(A)+\eta(A)}}{\sqrt{\kappa(A)+\eta(A)} + \sqrt{\ell(A)}}$. Hence, the desired result follows from

$$\begin{aligned}
\lambda_-(A) \mathbb{E}[\tau_A] &\le (\kappa(A) + \eta(A)) \frac{\sqrt{\kappa(A) + \eta(A)} + \sqrt{\ell(A)}}{\sqrt{\kappa(A) + \eta(A)}} + \ell(A) \frac{\sqrt{\kappa(A) + \eta(A)} + \sqrt{(\ell(A))}}{\sqrt{\ell(A)}} \\
&\le \sqrt{\kappa(A) + \eta(A)} \left(\sqrt{\kappa(A) + \eta(A)} + \sqrt{\ell(A)} \right) + \sqrt{\ell(A)} \left(\sqrt{\kappa(A) + \eta(A)} + \sqrt{\ell(A)} \right) \\
&= \left(\sqrt{\kappa(A) + \eta(A)} + \sqrt{\ell(A)} \right)^2.
\end{aligned}$$

□

Proof of Lemma 3.11. We consider a branching allocation ψ specified by

$$\psi_\mu = \begin{cases} \frac{1}{d} & \text{when } \mu = \mu_i \\ 0 & \text{otherwise} \end{cases} \quad \text{for minimal elements } \mu \in \Lambda \quad \text{and}$$

$$\psi_{\alpha\to\beta} = \begin{cases} \frac{1}{d} & \text{when } \alpha, \beta \in A \\ 0 & \text{otherwise.} \end{cases} \quad \text{for any } \alpha, \beta \in \Lambda \text{ where } \beta \text{ is an upper neighbor of } \alpha$$

Then, we have $\sum_\mu \psi_\mu \le 1$ since A has at most d minimal elements. Also, for each $\alpha \in \Lambda$, we have $\sum_\beta \psi_{\alpha\to\beta} \le 1$ because $\alpha \in A$ has at most d upper neighbors in A. Now, by Lemma 5.3, we write

$$|\Pi_m(A)| \cdot d^{-\ell(A)} = \sum_{\pi\in\Pi_m(A)} \left(\frac{1}{d}\right)^{\ell(A)} \le \sum_{\pi\in\Pi_m(A)} \left(\frac{1}{d}\right)^{\ell(\pi)} = \sum_{\pi\in\Pi_m(A)} \omega_\psi(\pi) \le \omega_\psi(\emptyset) = 1.$$

Hence, $|\Pi_m(A)| \le d^{\ell(A)}$, giving $\kappa(A) = \log|\Pi_m(A)| \le \log(d)\,\ell(A)$. □

Proof of Theorem 3.12. First, as every maximal path in $A = \langle n\alpha\rangle$ must contain $n\alpha_i$ steps in the i-th dimension, we have $\ell(\langle n\alpha\rangle) = n\ell(\alpha)$. Thus, every maximal path may be enumerated by counting the labelings of the $n\ell(\alpha)$ steps across the different dimensions. Hence, there are $|\Pi_m(\langle n\alpha\rangle)| = \binom{n\ell(\alpha)}{n\alpha_1,\ldots,n\alpha_d}$ different maximal paths.

Using Stirling's approximation, as $n \to \infty$,

$$\binom{n\ell\alpha}{n\alpha_1,\ldots,n\alpha_d} = \frac{(n\ell(\alpha))!}{(n\alpha_1)!\cdots(n\alpha_d)!} \sim \frac{\sqrt{2\pi n\ell(\alpha)}\left(\frac{n\ell(\alpha)}{e}\right)^{n\ell(\alpha)}}{\prod_{i=1}^d \sqrt{2\pi n\alpha_i}\left(\frac{n\alpha_i}{e}\right)^{n\alpha_i}} = \sqrt{2\pi n\ell(\alpha)}^{1-d}\left(\prod_{i=1}^d p_i^{-1/2}\right)\left(\prod_{i=1}^d p_i^{p_i}\right)^{-n\ell(\alpha)}$$

where $\sim$ means that the ratio of the two quantities approaches one. Taking the logarithm, we obtain

$$\log(|\Pi_m(\langle n\alpha\rangle)|) - \frac{1-d}{2}\log(2\pi n\ell(\alpha)) + \sum_{i=1}^d \left[\frac{1}{2}\log(p_i) + n\ell(\alpha)p_i\log(p_i)\right] \to 0 \quad \text{and}$$

$$\frac{\kappa(\langle n\alpha\rangle)}{\ell(\langle n\alpha\rangle)} = \frac{\log(|\Pi_m(\langle n\alpha\rangle)|)}{n\ell(\alpha)} \to -\sum_{i=1}^d p_i\log(p_i),$$

proving eq. (9).

Since $0 < \lambda_-(\Lambda) \le \lambda_+(\Lambda) < \infty$, we have $\eta(\langle n\alpha\rangle) \le \eta(\Lambda) < \infty$, and so $\eta(\langle n\alpha\rangle)/\ell(\langle n\alpha\rangle) \to 0$. By applying Theorem 3.10, we may derive eq. (10):

$$\begin{aligned}\limsup_{n\to\infty} \frac{\lambda_-(\Lambda)\mathbb{E}\left[\tau_{\langle n\alpha\rangle}\right]}{n\ell(\alpha)} &\le \lim_{n\to\infty} \frac{\left(\sqrt{\kappa(\langle n\alpha\rangle)+\eta(\Lambda)} + \sqrt{\ell(\langle n\alpha\rangle)}\right)^2}{n\ell(\alpha)} \\ &= \lim_{n\to\infty}\left(\sqrt{\frac{\kappa(\langle n\alpha\rangle)+\eta(\Lambda)}{\ell(\langle n\alpha\rangle)}} + 1\right)^2 = \left(1 + \sqrt{-\sum_i p_i\log(p_i)}\right)^2.\end{aligned}$$

□

5.4 Proofs for the Shape Theorem for Partially Ordered Monoids

We would like to establish the existence of a shape function. In particular, we would like τ_{A^n}/n to converge in some appropriate sense. Recall the definition of AB from Definition 3.17. First, we prove that AB and, hence, A^n lie in $L(\Lambda)$.

Lemma 5.6. *For any $A, B \in L(\Lambda)$ and $n \in \mathbb{N}_0$, we have $AB, A^n \in L(\Lambda)$.*

Proof. If the result holds for AB then, by induction, the result follows for A^n. Since AB is a lower set by construction, it suffices to show that AB is finite.

To this end, let $\gamma \in \mathcal{M}(AB)$. Then, there exist $a \in A$ and $b \in B$ such that $\gamma \le ab$. Since A and B are finite, there exist $\alpha \in \mathcal{M}(A)$ and $\beta \in \mathcal{M}(B)$ with $a \le \alpha$ and $b \le \beta$. By the assumed compatibility, $\gamma \le ab \le \alpha\beta \in AB$ which, by maximality of γ, gives $\gamma = \alpha\beta$. Thus, $\mathcal{M}(AB) \subseteq \{\alpha\beta : \alpha \in \mathcal{M}(A), \beta \in \mathcal{M}(B)\}$. Hence,

$$AB = \bigcup_{\gamma\in\mathcal{M}(AB)} \langle\gamma\rangle \subseteq \bigcup_{\alpha\in\mathcal{M}(A)} \bigcup_{\beta\in\mathcal{M}(B)} \langle\alpha\beta\rangle,$$

where $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are finite, since A and B are finite, and $\langle\alpha\beta\rangle$ is finite by local finiteness of Λ. Therefore, AB is finite and $AB \in L(\Lambda)$. □

Via Theorem 3.1, we may obtain L^2 convergence of τ_{A^n}/n as long as $\mathbb{E}[\tau_{A^n}]/n$ is convergent. We will use the superadditivity of $\mathbb{E}[\tau_A]$, $\ell(A)$ and $\kappa(A)$, as well as the subadditivity of $\ell_*(A)$ to establish the desired convergence of $\mathbb{E}[\tau_{A^n}]/n$.

Lemma 5.7. *Let $A, B \in L(\Lambda)$. Then,*

$$\kappa(A) + \kappa(B) \leq \kappa(AB), \quad \ell(A) + \ell(B) \leq \ell(AB), \quad \text{and} \quad \ell_*(AB) \leq \ell_*(A) + \ell_*(B).$$

Additionally, suppose for all $x, y \in \Lambda$ with $x \leq y$ that $\lambda_x \geq \lambda_y$. Then, for any $A, B \in L(\Lambda)$, we have $\mathbb{E}[\tau_A] + \mathbb{E}[\tau_B] \leq \mathbb{E}[\tau_{AB}]$.

Proof. If either A or B are empty, the claims hold. Otherwise, let both A and B be non-empty sets. For any $\pi = (\pi_1, \ldots, \pi_k) \in \Pi(A)$ and $\xi = (\xi_1, \ldots, \xi_n) \in \Pi(B)$, define $\pi \circ \xi \in \Pi(AB)$ as

$$\pi \circ \xi := (\xi_1, \ldots, \xi_n, \pi_1\xi_n, \ldots, \pi_k\xi_n).$$

If $\xi \in \Pi_m(B)$, notice that any extension of ξ is not in $\Pi(B)$, and so $(\xi_1, \ldots, \xi_n, \pi_1\xi_n) \notin \Pi(B)$. Thus, $\max\{i \in [1, k+n] : (\pi \circ \xi)_i \in B\} = n = \ell(\xi)$. So, one can recover the location of the division between ξ and π from $\pi \circ \xi$. Further, from the group structure of $\overline{\Lambda}$ by multiplying by ξ_n^{-1}, we conclude $(\pi, \xi) \mapsto \pi \circ \xi$ is injective for $(\pi, \xi) \in \Pi(A) \times \Pi_m(B)$.

We will define $\Pi_m(A) \circ \Pi_m(B) := \{\pi \circ \xi : \pi \in \Pi_m(A), \xi \in \Pi_m(B)\}$. Due to the injectivity, $|\Pi_m(A) \circ \Pi_m(B)| = |\Pi_m(A) \times \Pi_m(B)| = |\Pi_m(A)| \cdot |\Pi_m(B)|$.

First, we have

$$\ell(A) + \ell(B) = \max_{\pi \in \Pi_m(A)} \ell(\pi) + \max_{\xi \in \Pi_m(B)} \ell(\xi) = \max_{\substack{\pi \in \Pi_m(A) \\ \xi \in \Pi_m(B)}} (\ell(\pi) + \ell(\xi)) = \max_{\zeta \in \Pi_m(A) \circ \Pi_m(B)} \ell(\zeta) \leq \max_{\zeta \in \Pi(AB)} \ell(\zeta) = \ell(AB),$$

implying ℓ is superadditive.

For any $x \in \Lambda$, we define $|x| := \min_{\pi \in \Pi_m(\langle x \rangle)} \ell(\pi)$ so that $\ell_*(A) = \max_{x \in A} |x|$. For $x, y \in \Lambda$, we have

$$|x| + |y| = \min_{\pi \in \Pi_m(\langle x \rangle)} \ell(\pi) + \min_{\xi \in \Pi_m(\langle y \rangle)} \ell(\xi) = \min_{\substack{\pi \in \Pi_m(\langle x \rangle) \\ \xi \in \Pi_m(\langle y \rangle)}} (\ell(\pi) + \ell(\xi)) = \min_{\zeta \in \Pi_m(\langle x \rangle) \circ \Pi_m(\langle y \rangle)} \ell(\zeta) \geq \min_{\zeta \in \Pi(\langle xy \rangle)} \ell(\zeta) = |xy|.$$

Then, for any $A, B \in L(\Lambda)$, using the fact that $|x| \leq |y|$ whenever $x \leq y$,

$$\begin{aligned} \ell_*(AB) &= \max\{|z| : z \in AB\} = \max\{|xy| : x \in A, y \in B\} \\ &\leq \max\{|x| + |y| : x \in A, y \in B\} = \max\{|x| : x \in A\} + \max\{|y| : y \in B\} \\ &\leq \ell_*(A) + \ell_*(B), \end{aligned} \tag{20}$$

demonstrating that ℓ_* is subadditive.

Second, we show that no path in $\Pi_m(A) \circ \Pi_m(B)$ is a subpath of another. Suppose $\pi^{(1)}, \pi^{(2)} \in \Pi_m(A)$ and $\xi^{(1)}, \xi^{(2)} \in \Pi_m(B)$ with $\zeta^{(1)} := \pi^{(1)} \circ \xi^{(1)}$ and $\zeta^{(2)} := \pi^{(2)} \circ \xi^{(2)}$ where $\zeta^{(1)} \subseteq \zeta^{(2)}$. Then,

$$\xi^{(1)}_{\ell(\xi^{(1)})} = \max\left\{\zeta^{(1)}_i \in B : i \in [1, \ell(\zeta^{(1)})]\right\} \leq \max\left\{\zeta^{(2)}_i \in B : i \in [1, \ell(\zeta^{(2)})]\right\} = \xi^{(2)}_{\ell(\xi^{(2)})}$$

which, by the maximality of $\xi^{(1)}$ and $\xi^{(2)}$, implies $\xi^{(1)}_{\ell(\xi^{(1)})} = \xi^{(2)}_{\ell(\xi^{(2)})}$ and so $\xi^{(1)} = \xi^{(2)}$. Then, using $n := \ell(\xi^{(1)}) = \ell(\xi^{(2)})$, we have $\zeta^{(1)}_n = \xi^{(1)}_n = \xi^{(2)}_n = \zeta^{(2)}_n$. Hence,

$$\pi^{(1)} = \left(\zeta^{(1)}_{n+1}(\zeta^{(1)}_n)^{-1}, \ldots, \zeta^{(1)}_{\ell(\zeta^{(1)})}(\zeta^{(1)}_n)^{-1}\right) \subseteq \left(\zeta^{(2)}_{n+1}(\zeta^{(2)}_n)^{-1}, \ldots, \zeta^{(2)}_{\ell(\zeta^{(2)})}(\zeta^{(2)}_n)^{-1}\right) = \pi^{(2)},$$

implying $\pi^{(1)} = \pi^{(2)}$ by maximality of $\pi^{(1)}$ in A.

Therefore, none of the elements of $\Pi_m(A) \circ \Pi_m(B)$ are strict subpaths of each other. Thus, we can extend each to a unique maximal path in AB. Then, $|\Pi_m(A) \circ \Pi_m(B)| \leq |\Pi_m(AB)|$, and $\log|\Pi_m(A)| + \log|\Pi_m(B)| \leq \log|\Pi_m(AB)|$. Hence, κ is superadditive.

Third, we assume $\lambda_x \geq \lambda_y$ for all $x, y \in \Lambda$ with $x \leq y$. For every $\pi = (\pi_1, \ldots, \pi_k) \in \Pi_m(A)$ and $\xi = (\xi_1, \ldots, \xi_n) \in \Pi_m(B)$, consider $\zeta := \pi \circ \xi$. Then, since the identity is a minimum, $1_\Lambda \leq \xi_n$ and by compatibility $\pi_i \leq \pi_i\xi_n = \zeta_{i+n}$, we have $\lambda_{\pi_i} \geq \lambda_{\zeta_{i+n}}$.

Now, suppose $\{G_\alpha\}_{\alpha \in \Lambda}$ are defined as in Proposition 2.4, and $\{\tilde{G}_\alpha\}_{\alpha \in \Lambda}$ is an independent copy of the process (on a common probability space, using the same symbol $\mathbb{E}$ for the expectation). Then, $\lambda_{\pi_i} \geq \lambda_{\zeta_{i+n}}$ implies $\tilde{G}_{\pi_i} \preceq G_{\zeta_{i+n}}$ where we recall $\preceq$ represents stochastic ordering. Also, as $\zeta_i = \xi_i$ for all $i \in [1, n]$, we have $G_{\xi_i} \preceq G_{\zeta_i}$. From the independence of $\{G_\alpha\}_{\alpha \in \Lambda}$ and $\{\tilde{G}_\alpha\}_{\alpha \in \Lambda}$ as well as the monotonicity of the maximum and summation, we obtain

$$\max_{\pi \in \Pi_m(A)} \max_{\xi \in \Pi_m(B)} \left[\sum_{i=1}^{\ell(\pi)} \tilde{G}_{\pi_i} + \sum_{i=1}^{\ell(\xi)} G_{\xi_i}\right] \preceq \max_{\pi \in \Pi_m(A)} \max_{\xi \in \Pi_m(B)} \sum_{i=1}^{\ell(\zeta)} G_{\zeta_i} \leq \tau_{AB}.$$

Using Proposition 2.4, this implies $\tilde{\tau}_A + \tau_B \preceq \tau_{AB}$ where $\tilde{\tau}_A$ is the stopping time of A for the process associated with $\{\tilde{G}_\alpha\}_{\alpha \in \Lambda}$. Hence, $\mathbb{E}[\tau_A] + \mathbb{E}[\tau_B] = \mathbb{E}[\tilde{\tau}_A + \tau_B] \leq \mathbb{E}[\tau_{AB}]$. □

Proof of Theorem 3.18. Let S be the finite set of upper neighbors of 1_Λ. Consider $x, y \in \Lambda$ where y is an upper neighbor of x. Regarding $x, y \in \overline{\Lambda}$, we have $yx^{-1} > 1_\Lambda$, and so $yx^{-1} \in \Lambda$ by definition of Λ. For any z with $1_\Lambda < z \leq yx^{-1}$, we have $x < zx \leq y$, implying $zx = y$ as y is an upper neighbor of x. Thus, yx^{-1} is an upper neighbor of 1_Λ and $yx^{-1} \in S$. Therefore, x has at most $|S|$ upper neighbors. Because this holds for all $x \in \Lambda$ and 1_Λ is the unique minimal element of Λ, by Lemma 3.11 with $d = |S| \geq 1$, for all $A \in L(\Lambda)$, we have $\kappa(A) \leq \ell(A) \log |S|$.

From Lemma 5.7, we know $\{\ell(A^n)\}_{n=1}^\infty$, $\{\kappa(A^n)\}_{n=1}^\infty$, and $\{\mathbb{E}[\tau_{A^n}]\}_{n=1}^\infty$ are superadditive sequences. Also, by subadditivity of ℓ_* and steadiness of Λ, we have $\ell(A^n) \leq C\ell_*(A^n) \leq nC\ell_*(A)$. Then, the sequence $\{\ell(A^n)/n\}_{n=1}^\infty$ is bounded by $C\ell_*(A)$. and the sequence $\{\kappa(A^n)/n\}_{n=1}^\infty$ is bounded by $C\ell_*(A) \log |S|$. Applying Theorem 3.10, for $n \geq 1$, we have

$$\frac{\lambda_-(\Lambda)\mathbb{E}[\tau_{A^n}]}{n} \leq \frac{\left(\sqrt{\kappa(A^n) + \eta(A^n)} + \sqrt{\ell(A^n)}\right)^2}{n} \leq C\ell_*(A)\left(1 + \sqrt{\log |S| + \eta(\Lambda)}\right)^2.$$

Since, $\eta(\Lambda)/n \to 0$, by superadditivity (Fekete's lemma), the following limits exist:

$$\ell_\infty := \lim_{n\to\infty} \frac{\ell(A^n)}{n} \leq C\ell_*(A), \qquad \kappa_\infty := \lim_{n\to\infty} \frac{\kappa(A^n)}{n} \leq C\ell_*(A) \log |S|,$$
$$\text{and } \lim_{n\to\infty} \frac{\lambda_-(\Lambda)\mathbb{E}[\tau_{A^n}]}{n} \leq \left(\sqrt{\kappa_\infty} + \sqrt{\ell_\infty}\right)^2.$$

Lastly, define $g(A) := \lim_{n\to\infty} \mathbb{E}[\tau_{A^n}]/n$ so that, by Theorem 3.1,

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{\tau_{A^n}}{n} - g(A)\right)^2\right] = \lim_{n\to\infty} \frac{\operatorname{Var}(\tau_{A^n})}{n^2} \leq \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\mathbb{E}[\tau_{A^n}]}{n} = \lim_{n\to\infty} \frac{g(A)}{n} = 0,$$

proving $\tau_{A^n}/n \to g(A)$ in L^2. □

Lemma 5.8. *If Λ is finitely generated and steady then Λ is locally finite.*

Proof. First, 1_Λ is the unique minimal element of Λ fulfilling the second condition of Definition 2.1. Second, take S to be the upper neighbors of 1_Λ. Because Λ is finitely generated, S is finite. By the first part of the proof of Theorem 3.18, there are at most $|S|$ upper neighbors of any $\alpha \in \Lambda$, fulfilling the third condition of Definition 2.1.

Third, for any $\alpha \in \Lambda$, we can write $\alpha = s_1 \cdots s_n$ for $s_1, \ldots, s_n \in S$. Thus, $(1_\Lambda, s_1, s_1s_2, \ldots, \alpha)$ is a maximal path in $\langle\alpha\rangle$ so $\min_{\pi\in\Pi_m(\langle\alpha\rangle)} \ell(\pi) \leq n+1$. By steadiness, $\max_{\pi\in\Pi_m(\langle\alpha\rangle)} \ell(\pi) \leq C(n+1)$. Then, for every $x \in \langle\alpha\rangle$, we can write $x = \tilde{s}_1 \cdots \tilde{s}_m$ with $\tilde{s}_1, \ldots, \tilde{s}_m \in S$. We define the path $\tilde{\xi}_x = (1_\Lambda, \tilde{s}_1, \tilde{s}_1\tilde{s}_2, \ldots, x) \in \Pi(\langle\alpha\rangle)$. Then, we can extend $\tilde{\xi}_x$ to a maximal path $\xi_x \in \Pi_m(\langle\alpha\rangle)$. Thus, every element of $\langle\alpha\rangle$ can be mapped into $\Pi_m(\langle\alpha\rangle)$ and for every $\pi \in \Pi_m(\langle\alpha\rangle)$, there are $\ell(\pi) \leq C(n+1)$ elements which map to it. Hence, $|\langle\alpha\rangle| \leq C(n+1) \cdot |\Pi_m(\langle\alpha\rangle)|$. Because the maximum length of a path in $\Pi_m(\langle\alpha\rangle)$ is $C(n+1)$ and there are $|S|$ different generators, we also know that $|\Pi_m(\langle\alpha\rangle)| \leq |S|^{C(n+1)}$. Finally,

$$|\langle\alpha\rangle| \leq C(n+1) \cdot |\Pi_m(\langle\alpha\rangle)| \leq C(n+1) \cdot |S|^{C(n+1)} < \infty$$

fulfilling the first condition of Definition 2.1. □

Proof of Corollary 3.20. First, we will construct a group homomorphism $\varphi : \overline{\Lambda} \to \mathbb{R}$ such that $\varphi(x) > 0$ for all $x \in \Lambda \setminus \{1_{\overline{\Lambda}}\}$. Because of the commutativity of $\overline{\Lambda}$, we will write the group operation additively.

Let S and R be a finite set of generators and relations, respectively, of $\overline{\Lambda}$. Without loss of generality, we may choose R to not contain the trivial relation $0 = 0$ and S so that it does not contain the identity and is a generating set for Λ as a monoid. We enumerate the generators as $S = \{s_1, \ldots, s_d\}$. Similarly, we enumerate the relations $R = \{r^{(1)}, \ldots, r^{(n)}\}$ such that for each i, $r_1^{(i)} s_1 + \ldots + r_d^{(i)} s_d = 0$ where $r_1^{(i)}, \ldots, r_d^{(i)} \in \mathbb{Z}$.

Fix $j_0 \in [1, d]$. Now, we construct a matrix M from the relations excluding the column corresponding to j_0 so $M := (r_j^{(i)})_{i, j\neq j_0} \in \mathbb{Z}^{n\times(d-1)}$. Then, we define the vector $b = (-r_{j_0}^{(i)})_i \in \mathbb{Z}^n$. By the Farkas' lemma for rational matrices (see Corollary 3.5 of [45]), either $Mc = b$ has a solution $c = (c_j)_{j\neq j_0} \in \mathbb{Q}^{d-1}$ with $c_j \geq 0$ or $M^T w \leq 0$ has a solution $w \in \mathbb{Q}^n$ with $b^T w > 0$, but not both. Assume for sake of contradiction that there exists a solution $w = (w_i) \in \mathbb{Q}^n$ to the latter so $M^T w \leq 0$ and $b^T w > 0$. Without loss of generality, by scaling w appropriately, we may assume $w \in \mathbb{Z}^n$. Then, $M^T w \leq 0$ means that $w_1 r_j^{(1)} + \ldots + w_n r_j^{(n)} \leq 0$ for all $j \neq j_0$. Additionally, $b^T w > 0$ means that $-w_1 r_{j_0}^{(1)} - \ldots - w_n r_{j_0}^{(n)} > 0$ and, because it is an integer, it must be greater than or equal to one. Now, using the ordering on Λ,

$$0 = \sum_{i=1}^n w_i \cdot 0 = \sum_{i=1}^n w_i \sum_{j=1}^d r_j^{(i)} s_j = \left(w_1 r_{j_0}^{(1)} + \ldots + w_n r_{j_0}^{(n)}\right) s_{j_0} + \sum_{j\neq j_0} \left(w_1 r_j^{(1)} + \ldots + w_n r_j^{(n)}\right) s_j \leq_\Lambda -s_{j_0} <_\Lambda 0,$$

which is a contradiction. Therefore, we conclude there exists a solution $c = (c_j)_{j\neq j_0} \in \mathbb{Q}^{d-1}$ with $Mc = b$ and $c_j \geq 0$. Define $\varphi_{j_0} : \Lambda \to \mathbb{Q}$ as

$$\varphi_{j_0}\left(\sum_{j=1}^{d} x_j s_j\right) = x_{j_0} + \sum_{j\neq j_0} x_j c_j$$

for any $x_1, \ldots, x_d \in \mathbb{Z}$. Because $Mc = b$, we know $\varphi_{j_0}\left(r_1^{(i)} s_1 + \ldots + r_d^{(i)} s_d\right) = 0$ for all i. So, φ_{j_0} is well-defined on $\overline{\Lambda}$. For any $x = x_1 s_1 + \ldots + x_d s_d \in \Lambda$ as $x_1, \ldots x_d \geq 0$, we have $\varphi_{j_0}(x) \geq 0$. Also, $\varphi_{j_0}(s_{j_0}) = 1 > 0$.

Now, we define $\varphi : \Lambda \to \mathbb{R}$ as $\varphi(x) = \varphi_1(x) + \ldots + \varphi_d(x)$ so that $\varphi(s) > 0$ for all $s \in S$. Define $m_- := \min_{s\in S} \varphi(s) > 0$ and $m_+ := \max_{s\in S} \varphi(s) > 0$. For any $x \in \Lambda$, let $\pi \in \Pi_m(\langle x\rangle)$. Then, for each $i \in [1, \ell(\pi)]$, $\pi_i - \pi_{i-1}$ is minimal in $\Lambda \setminus \{0\}$ and so $\pi_i - \pi_{i-1} \in S$. Thus,

$$\varphi(x) = \sum_{i=1}^{\ell(\pi)} \varphi(\pi_i - \pi_{i-1}) \geq \sum_{i=1}^{\ell(\pi)} m_- = m_- \ell(\pi)$$

and, similarly, $\varphi(x) \leq m_+ \ell(\pi)$. Hence, $\frac{1}{m_+}\varphi(x) \leq \ell(\pi) \leq \frac{1}{m_-}\varphi(x)$. Therefore,

$$\max_{\pi\in\Pi_m(\langle x\rangle)} \ell(\pi) \leq \frac{1}{m_-}\varphi(x) \leq \frac{m_+}{m_-} \min_{\pi\in\Pi_m(\langle x\rangle)} \ell(\pi)$$

and so Λ is steady with $C = \frac{m_+}{m_-}$. By Lemma 5.8, we know Λ is also locally finite. We conclude, by Theorem 3.18, eq. (11) holds for all $A \in L(\Lambda)$. □

Remark 5.9. A concrete strictly positive homomorphism φ with respect to the cone in Example 3.19 given in Figure 1 can be found. One could choose $\varphi(x_1 a + x_2 b + x_3 c) = (2/3, 1, 1) \cdot (x_1, x_2, x_3) = (2/3)x_1 + x_2 + x_3$ which is strictly positive on the non-zero elements of $\Lambda = \langle a, b, c \,|\, 3a = b + c\rangle$. Since $\varphi(3a - b - c) = 0$, the function φ is well-defined.

Proof of Corollary 3.21. For each word $w = w_1 \ldots w_n \in \overline{\Lambda}$ composed of elements of $S \cup S^{-1}$, consider $f(w) = |\{i \in [1, n] : w_i \in S\}| - |\{i \in [1, n] : w_i \in S^{-1}\}|$, the number of positive generators minus the number of negative generators. Then, by assumption, $f(r) = 0$ for any relation $r \in R$. Hence, if two words w_1 and w_2 are equivalent under R, then $f(w_1) = f(w_2)$. Therefore, f is well-defined as a function on $\overline{\Lambda}$.

Suppose now $x \in \Lambda$. For every $\pi = (\pi_1, \ldots, \pi_n) \in \Pi_m(\langle x\rangle)$, we have $x = \pi_1 \cdot (\pi_1^{-1}\pi_2) \cdots (\pi_{n-1}^{-1}\pi_n)$ where $\pi_i^{-1}\pi_{i+1} \in S$ and $\pi_1 \in S$. Thus, $w = \pi_1 \cdot (\pi_1^{-1}\pi_2) \cdots (\pi_{n-1}^{-1}\pi_n)$ is a word and $f(x) = f(w) = n = \ell(\pi)$. Hence, $\ell(\pi)$ is the same for all $\pi \in \Pi_m(\langle x\rangle)$. We conclude

$$\max_{\pi\in\Pi_m(\langle x\rangle)} \ell(\pi) = \min_{\pi\in\Pi_m(\langle x\rangle)} \ell(\pi),$$

meaning Λ is steady with $C = 1$. By Lemma 5.8, we know Λ is also locally finite. Moreover, by Theorem 3.18, eq. (11) holds for all $A \in L(\Lambda)$. □

5.5 Proofs of Proposition 2.4

We now give two proofs of Proposition 2.4. The first is more 'standard', following from the construction of Markov jump processes, inspired by the proof of Proposition 1.1 of [46]. for the discrete time growth model on $\mathbb{N}_0^2$ with independent Geometric weights. The second is self-contained and of a different character. We will use $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}$ to denote the probability measure and expectation operator for the probability space of the process Y_t.

Proof of Proposition 2.4 via Markov Jump Process Construction. With respect to the process Y_t, let $\{T_i\}_{i\geq 0}$ and $\{Z_i\}_{i\geq 0}$ be the event times and the states visited at these event times. Here, $T_0 = 0$ and $Z_0 = \emptyset$. By the construction of Markov jump processes on countable state spaces, to show that Y_t is a Markov process with generator $\mathcal{L}$, it is enough to verify the following:

(a) $\{Z_i\}$ is a discrete time Markov chain with transition probability $\tilde{\mathbb{P}}\left(Z_{i+1} = A \cup \alpha \,\middle|\, Z_i = A\right) = \lambda_\alpha / \left(\sum_{\beta\in\mathcal{M}^*(A)} \lambda_\beta\right)$ for $\alpha \in \mathcal{M}^*(A)$.

(b) $\{T_{i+1} - T_i\}_{i\geq 0} \,|\, \{Z_i\}_{i\geq 0}$ are independent exponential random variables with parameters $\left\{\sum_{\beta\in\mathcal{M}^*(Z_i)} \lambda_\beta\right\}_{i\geq 0}$.

Let $\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_k$ be lower sets in Λ such that $A_j \setminus A_{j-1} = \alpha_j$. In particular, $A_j = \bigcup_{1\leq i\leq j} \alpha_i$ for $1 \leq j \leq k$. Items (a) and (b) will follow from verifying

$$\begin{aligned}&\tilde{\mathbb{P}}\left(Z_k = Z_{k-1} \cup \alpha_k, T_k - T_{k-1} > t_k, \ldots, Z_2 = Z_1 \cup \alpha_2, T_2 - T_1 > t_2, Z_1 = \alpha_1, T_1 > t_1\right)\\ &= \prod_{j=1}^{k} \frac{\lambda_{\alpha_j}}{\sum_{\beta\in\mathcal{M}^*(A_{j-1})} \lambda_\beta} \exp\left(-t_j \sum_{\beta\in\mathcal{M}^*(A_{j-1})} \lambda_\beta\right). \qquad (21)\end{aligned}$$

Indeed, by taking $t_j \equiv 0$, we see that item (a) would hold. Also, item (b) would hold since then, for any $L \geq k$,

$$\tilde{\mathbb{P}}\left(\bigcup_{j=1}^{k}\{T_j - T_{j-1} > t_j\} \,\middle|\, \{Z_j\}_{j=0}^{L}\right) = \tilde{\mathbb{P}}\left(\bigcup_{j=1}^{k}\{T_j - T_{j-1} > t_j\} \cup \bigcup_{j=k+1}^{L}\{T_j - T_{j-1} > 0\} \,\middle|\, \{Z_j\}_{j=0}^{L}\right)$$
$$= \prod_{j=1}^{k}\exp\left(-\sum_{\beta\in\mathcal{M}^*(Z_{j-1})}\lambda_\beta\right).$$

We now verify (a) and (b). Note for $\alpha \in \Lambda$ that $G_\alpha = \chi_{\langle\alpha\rangle} - \chi_{\langle\alpha\rangle\setminus\alpha}$ is the time between when α becomes first accessible and when it is achieved. Also, when $\alpha \in \mathcal{M}^*(\emptyset)$, we observe $\chi_{\langle\alpha\rangle\setminus\alpha} = \chi_\emptyset = 0$.

The left-hand side of eq. (21) equals

$$\begin{aligned}
&\tilde{\mathbb{P}}\Bigg(\chi_{\langle\alpha_k\rangle} - \chi_{A_{k-1}} = \min_{\beta\in\mathcal{M}^*(A_{k-1})}\{\chi_{\langle\beta\rangle} - \chi_{A_{k-1}})\} > t_k,\ \chi_{\langle\alpha_{k-1}\rangle} - \chi_{A_{k-2}} = \min_{\beta\in\mathcal{M}^*(A_{k-2})\setminus\mathcal{M}^*(A_{k-1})}\{\chi_{\langle\beta\rangle} - \chi_{A_{k-2}}\} > t_{k-1},\\
&\quad \ldots,\ \chi_{\langle\alpha_1\rangle} = \min_{\beta\in\mathcal{M}^*(A_0)\setminus\mathcal{M}^*(A_{k-1})}\chi_{\langle\beta\rangle} > t_1\Bigg) \qquad (22)\\
&= \tilde{\mathbb{P}}\Bigg(G_{\alpha_k} - (\chi_{A_{k-1}} - \chi_{\langle\alpha_k\rangle\setminus\alpha_k}) = \min_{\beta\in\mathcal{M}^*(A_{k-1})}\{G_\beta - (\chi_{A_{k-1}} - \chi_{\langle\beta\rangle\setminus\beta})\} > t_k,\\
&\quad G_{\alpha_{k-1}} - \left(\chi_{A_{k-2}} - \chi_{\langle\alpha_{k-1}\setminus\alpha_{k-1}\rangle}\right) = \min_{\beta\in\mathcal{M}^*(A_{k-2})\setminus\mathcal{M}^*(A_{k-1})}\{G_\beta - (\chi_{A_{k-2}} - \chi_{\langle\beta\rangle\setminus\beta}) > t_{k-1},\\
&\quad \ldots,\ G_{\alpha_1} = \min_{\beta\in\mathcal{M}^*(A_0)\setminus\mathcal{M}^*(A_{k-1})} G_\beta > t_1\Bigg).
\end{aligned}$$

By definition, for $\beta \in \mathcal{M}^*(A_j)$ and $1 \leq j \leq k-1$, the difference $\chi_{A_j} - \chi_{\langle\beta\rangle\setminus\beta}$ is expressible in terms of $\{G_\alpha : \alpha \in A_{k-1}\}$. Therefore, the event inside the probability in eq. (22) is expressible all in terms of $\{G_\alpha\}_{\alpha\in A_{k-1}\cup\mathcal{M}^*(A_{k-1})}$.

We now decompose the event. Observe

$$B_0 = \{G_{\alpha_k} - (\chi_{A_{k-1}} - \chi_{\langle\alpha_k\rangle\setminus\alpha_k}) = \min_{\beta\in\mathcal{M}^*(A_{k-1})}\{G_\beta - (\chi_{A_{k-1}} - \chi_{\langle\beta\rangle\setminus\beta})\} > t_k\}$$

is a subset of $B_1 = \cap_{\beta\in\mathcal{M}^*(A_{k-1})}\{G_\beta > \chi_{A_{k-1}} - \chi_{\langle\beta\rangle\setminus\beta}\}$. Let

$$\begin{aligned}
C = \Big\{&G_{\alpha_{k-1}} - (\chi_{A_{k-2}} - \chi_{\langle\alpha_{k-1}\rangle\setminus\alpha_{k-1}}) = \min_{\beta\in\mathcal{M}^*(A_{k-2})\setminus\mathcal{M}^*(A_{k-1})}\{G_\beta - (\chi_{A_{k-2}} - \chi_{\langle\beta\rangle\setminus\beta})\} > t_{k-1},\\
&\ldots, G_{\alpha_1} = \min_{\beta\in\mathcal{M}^*(A_0)\setminus\mathcal{M}^*(A_{k-1})} G_\beta > t_1\Big\}.
\end{aligned}$$

Observe that C involves only $\mathcal{F} = \sigma\{G_\alpha : \alpha \in A_{k-1}\}$, and B_0, B_1 involve only $\sigma\{G_\alpha : \alpha \in \mathcal{M}^*(A_{k-1})\}$. Since, $\mathcal{F}$ is independent of $\sigma\{G_\beta : \beta \in \mathcal{M}^*(A_{k-1})\}$, we have eq. (22) equals

$$\tilde{\mathbb{P}}(B_0 \cap C) = \tilde{\mathbb{E}}[B_0 \cap B_1 \cap C] = \tilde{\mathbb{E}}\left[1_C\tilde{\mathbb{P}}(B_0 \cap B_1 \,|\, \mathcal{F})\right] = \tilde{\mathbb{E}}\left[1_C\tilde{\mathbb{P}}(B_0 \,|\, B_1, \mathcal{F})\,\tilde{\mathbb{P}}(B_1 \,|\, \mathcal{F})\right].$$

Since $\{G_\alpha : \alpha \in \mathcal{M}^*(A_{k-1})\}$ are independent random variables, and also independent of $\mathcal{F}$, by properties of exponential distributions, we have that $\tilde{\mathbb{P}}(B_0 \,|\, B_1, \mathcal{F}) = \frac{\lambda_{\alpha_k}}{\sum_{\beta\in\mathcal{M}^*(A_{k-1})}\lambda_\beta}\exp\left(-t_k\sum_{\beta\in\mathcal{M}^*(A_{k-1})}\lambda_\beta\right)$. Hence, we have

$$\tilde{\mathbb{P}}(B_0 \cap B_1 \cap C) = \frac{\lambda_{\alpha_k}}{\sum_{\beta\in\mathcal{M}^*(A_{k-1})}\lambda_\beta}\exp\left(-t_k\sum_{\beta\in\mathcal{M}^*(A_{k-1})}\lambda_\beta\right)\tilde{\mathbb{P}}(C \cap B_1).$$

Now, the event $C \cap B_1$, expressed in terms of $\{G_\alpha : \alpha \in A_{k-1}\} \cup \{G_\beta : \beta \in \mathcal{M}^*(A_{k-1})\}$, states that the evolution up to time T_{k-1} fills in $\alpha_1, \ldots, \alpha_{k-1}$ in order and that $\beta \in \mathcal{M}^*(A_{k-1})$ (those states which are still accessible) are not filled by time T_{k-1}, and also that $T_j - T_{j-1} \geq t_j$ for $1 \leq j \leq k-1$. The event $C \cap B_1$ can be re-expressed as

$$C \cap B_1 = \{Z_{k-1} = Z_{k-2} \cup \alpha_{k-1}, T_{k-1} - T_{k-2} > t_{k-1}, \ldots, Z_1 = \alpha_1, T_1 > t_1\}.$$

Hence, we may iterate and verify the claim in eq. (21). ☐

For the reader's interest, we give an alternate self-contained proof of Proposition 2.4.

Alternate proof of Proposition 2.4. Fix some (not necessarily finite) lower set $D \subseteq \Lambda$. For any $t \geq 0$, let $\mathcal{F}_t := \sigma(Y_s : s \leq t)$ be the natural filtration for Y_t and $\sigma(Y_t \cap D)$ be the σ-algebra generated by $Y_t \cap D$. Also, for any $A \in L(\Lambda)$, let $\mathcal{G}_B := \sigma(\chi_A : A \subseteq B)$. We will now show that $Y_t \cap D$ is Markov with respect to the filtration $\mathcal{F}_t$. That is, for all $s \leq t$, $(\sigma(Y_t \cap D) \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(Y_s \cap D)$. In words, $\sigma(Y_t \cap D)$ is independent of $\mathcal{F}_s$ given $\sigma(Y_s \cap D)$. Note that $\sigma(Y_t \cap D) = \sigma(\chi_A \leq t : A \in L(\Lambda), A \subseteq D)$ and, similarly, $\mathcal{F}_t = \sigma(\chi_A \leq s : A \in L(\Lambda), s \leq t)$.

We prove by induction on $|A|$ that for all lower sets $A \subseteq D$ and all $s \geq 0$, $(s \vee \chi_A \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(Y_s \cap D)$. When $|A| = 0$, we have $A = \emptyset$ so $s \vee \chi_A = s \vee 0 = s$ which is independent of $\mathcal{F}_s$. For $|A| \geq 1$, either $|\mathcal{M}(A)| > 1$ or $|\mathcal{M}(A)| = 1$. If $|\mathcal{M}(A)| > 1$ then for all $\alpha \in A$, $|\langle\alpha\rangle| < |A|$ so, by the induction hypothesis, $(s \vee \chi_{\langle\alpha\rangle} \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(Y_s \cap D)$. Thus, $(s \vee \chi_A = \max_{\alpha\in A}(s \vee \chi_{\langle\alpha\rangle}) \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(Y_s \cap D)$.

Otherwise, $|\mathcal{M}(A)| = 1$ so $\mathcal{M}(A) = \{\alpha\}$ and $A = \langle\alpha\rangle$. Note $G_\alpha = \chi_{\langle\alpha\rangle} - \chi_{\langle\alpha\rangle\setminus\alpha}$. We condition on $\{\chi_A > s\} = \{G_\alpha > s - \chi_{A\setminus\alpha}\}$, Note that $\mathcal{F}_s$ is generated by events of the form $E := \{\chi_{B_1} \leq s_1, \ldots, \chi_{B_n} \leq s_n\}$ where $B_1, \ldots, B_n \in L(\Lambda)$ with $B_1 \subseteq \ldots \subseteq B_n$ and $s_1, \ldots, s_n \geq 0$ with $s_1 \leq \ldots \leq s_n = s$. Then, $\chi_{\langle\alpha\rangle} = \chi_A > s$ and $s \geq \chi_{B_n}$ implies $\alpha \notin B_n$ and $B_n \subseteq A \setminus \alpha$. So, if $B_n \not\subseteq A \setminus \alpha$ then $\tilde{\mathbb{P}}(E \,|\, \chi_A > s) = 0$ meaning $(s \vee \chi_A \perp\!\!\!\perp E) \,|\, (\chi_A > s)$. Otherwise, $B_n \subseteq A \setminus \alpha$ implying $E \in \mathcal{G}_{A\setminus\alpha}$ and so $G_\alpha \perp\!\!\!\perp E$. Also, $G_\alpha \perp\!\!\!\perp (s - \chi_{A\setminus\alpha})$ and so, by memorylessness of G_α, for all $t \geq s$,

$$\begin{aligned}\tilde{\mathbb{P}}\left(s \vee \chi_A = \chi_A > t \,\middle|\, \chi_A > s, E\right) &= \tilde{\mathbb{P}}\left(G_\alpha > t - \chi_{A\setminus\alpha} \,\middle|\, G_\alpha > s - \chi_{A\setminus\alpha}, E\right) \\ &= \tilde{\mathbb{P}}\left(G_\alpha > t - s\right) = \tilde{\mathbb{P}}\left(s \vee \chi_A = \chi_A > t \,\middle|\, \chi_A > s\right).\end{aligned}$$

Thus, $(s \vee \chi_A \perp\!\!\!\perp E) \,|\, (\chi_A > s)$ and, because this holds for all E, we get $(s \vee \chi_A \perp\!\!\!\perp \mathcal{F}_s) \,|\, (\chi_A > s)$. Conditioning on $\chi_A \leq s$, we also have $(s \vee \chi_A = s \perp\!\!\!\perp \mathcal{F}_s) \,|\, (\chi_A \leq s)$ so $(s \,\langle\chi\rangle_A \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(\{\chi_A > s\})$. Since $A \subseteq D$, we know $\sigma(\{\chi_A > s\}) \subseteq \sigma(Y_s \cap D) \subseteq \mathcal{F}_s$. So, by the weak union property of conditional independence, we can extend the conditioning on $\sigma(\{\chi_A > s\})$ to conditioning on all of $\sigma(Y_s \cap D)$. Therefore, we may conclude $(s \vee \chi_A \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(Y_s \cap D)$, completing the induction.

Observe that for all $t > s$, we have $\sigma(Y_t \cap D) = \sigma(\chi_A \leq t : A \in L(\Lambda), A \subseteq D) = \sigma(s \vee \chi_A \leq t : A \in L(\Lambda), A \subseteq D)$. Thus, $(\sigma(Y_t \cap D) \perp\!\!\!\perp \mathcal{F}_s) \,|\, \sigma(Y_s \cap D)$ meaning $Y_t \cap D$ is a Markov process with respect to $\mathcal{F}_t$. In particular, $Y_t = Y_t \cap \Lambda$ is Markov. Lastly, we note that for any $A \in L(\Lambda)$, let $\alpha \in \mathcal{M}^*(A)$ and take $D = \langle\alpha\rangle$. Then, since $Y_t \cap D$ is Markov and $\{Y_t = \langle\alpha\rangle \setminus \alpha\}, \{Y_t = A\} \in \mathcal{F}_t$, we can interchange conditioning on $Y_t = \langle\alpha\rangle \setminus \alpha$ or $Y_t = A$ with conditioning on $Y_t \cap D = \langle\alpha\rangle \setminus \alpha$:

$$\begin{aligned}\tilde{\mathbb{P}}\left(A \cup \alpha \subseteq Y_{t+h} \,\middle|\, Y_t = A\right) &= \tilde{\mathbb{P}}\left(\langle\alpha\rangle \subseteq Y_{t+h} \cap D \,\middle|\, Y_t = A\right) = \tilde{\mathbb{E}}\left[\tilde{\mathbb{P}}\left(\langle\alpha\rangle \subseteq Y_{t+h} \cap D \,\middle|\, \mathcal{F}_t\right) \,\middle|\, Y_t = A\right] \\ &= \tilde{\mathbb{P}}\left(\langle\alpha\rangle \subseteq Y_{t+h} \cap D \,\middle|\, Y_t \cap D = \langle\alpha\rangle \setminus \alpha\right) \\ &= \tilde{\mathbb{P}}\left(G_\alpha + \chi_{\langle\alpha\rangle\setminus\alpha} = \chi_\alpha \leq t + h \,\middle|\, \chi_{\langle\alpha\rangle\setminus\alpha} \leq t < \chi_\alpha\right) \\ &= 1 - \tilde{\mathbb{P}}\left(G_\alpha > h + (t - \chi_{\langle\alpha\rangle\setminus\alpha}) \,\middle|\, G_\alpha > t - \chi_{\langle\alpha\rangle\setminus\alpha} \geq 0\right) = 1 - e^{-\lambda_\alpha h}.\end{aligned}$$

Therefore, the generator for Y_t is $\mathcal{L}$ from eq. (3). ☐

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