Schauder estimate for quasilinear discrete PDEs of parabolic type

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Abstract

We investigate quasilinear discrete PDEs $\partial_t u = \Delta^N \varphi(u) + Kf(u)$ of reaction-diffusion type with nonlinear diffusion term defined on an n-dimensional unit torus discretized with mesh size $\frac{1}{N}$ for $N \in \mathbb{N}$, where Δ^N is the discrete Laplacian, φ is a strictly increasing C^5 function and f is a C^1 function. We establish L^∞ bounds and space-time Hölder estimates, both uniform in N, of the first and second spatial discrete derivatives of the solutions. In the equation, $K > 0$ is a large constant and we show how these estimates depend on K. The motivation for this work stems originally from the study of hydrodynamic scaling limits of interacting particle systems.

Our method is a two steps approach in terms of the Hölder estimate and Schauder estimate, which is known for continuous parabolic PDEs. We first show the discrete Hölder estimate uniform in N for the solutions of the associated linear discrete PDEs with continuous coefficients, based on the Nash estimate. We next establish the discrete Schauder estimate for linear discrete PDEs with uniform Hölder coefficients. The link between discrete and continuous settings is given by polylinear interpolations. Since this operation has a non-local nature, the method requires proper modifications.

We also discuss another method based on the study of the corresponding fundamental solutions.

Keywords. Schauder estimate, Hölder estimate, Nash estimate, quasilinear discrete PDE, Allen-Cahn equation, polylinear interpolation, fundamental solution.

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Contents

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1 Introduction

In this article, we investigate quasilinear discrete PDEs (partial differential equations) defined on an n-dimensional unit torus $\mathbb{T}^n \cong [0,1)^n$ with periodic boundary discretized with mesh size $\frac{1}{N}$ for $N \in \mathbb{N}$. The goal is to establish gradient estimates for the solutions, especially to show L^∞ bounds and also space-time Hölder estimates, both uniform in N, of the first and second spatial discrete derivatives of the solutions. As a model equation, we take a type of discrete reaction-diffusion equation with nonlinear diffusion term, which is a variant of Allen-Cahn equation; see (1.1) below.

Such a type of equation arises in the study of the hydrodynamic scaling limit of interacting particle systems on the n-dimensional periodic integer lattice of size N; see [15], [16] and [20], [21]. In particular, a motivation of the present article is the use of bounds of the discrete derivatives, in these studies, to prove the hydrodynamic limits. We expect that one can extend our results to a wide class of quasilinear discrete PDEs. As such the results and methods may also be of use in the context of numerical methods which approximate continuous quasilinear parabolic PDEs; see Section 1.6 for comments on previous work in this vein.

Our method is based on the two steps approach, which is originally exploited for continuous linear and quasilinear parabolic PDEs. The first step is to establish the discrete Hölder estimate uniform in N for the solutions of linear discrete PDEs, more precisely linear discrete diffusion equations of divergence form, with an external force term. The assumptions at this stage are minimal such that the diffusion coefficient is symmetric, nondegenerate, bounded, continuous in time and the external force term is bounded. Then, based on the so-called Nash estimate followed by the parabolic Harnack inequality already known in the discrete setting when there is no external force term, one can easily extend the uniform Hölder estimate in space and time variables to our equation with an external force term. In particular, such an approach applies to our quasilinear equations.

Once the Hölder estimate is in hand, the second step is to improve the regularity by establishing the discrete Schauder estimate for linear discrete PDEs with uniform Hölder coefficients. Our main task is to perform this step. The method follows that of Lieberman [41] for the continuous equations, though the present article is self-contained in that we do not rely on results in it.

More precisely, we compare the solution of our equation with that of the discrete heat equation, that is the discrete PDE with a constant diffusion coefficient, and derive a discrete energy inequality. At this point, we make use of the Hölder estimate established in the first step. Then, we derive an oscillation interior estimate for discrete heat equation, valid also for the gradient of the solution due to the property that the diffusion coefficient is constant.

We then note the fact that a certain Campanato-type integral estimate (cf. Section 3.4) implies the Hölder property; this is the tack taken for continuous PDE in [41] (which refers to [7]). We apply this approach for the gradient of the solution of our equation. To bound the integral estimate, we need a link between discrete and continuous settings, which will be given by polylinear interpolations. However, since the polylinear operation has a non-local nature, the method requires proper and substantial modifications. Indeed, the integral estimation method does not imply the Hölder property of the gradient of the solution in the short distance regime. For the short distance regime, we need to show the Hölder estimate separately. This proves the Schauder estimate for the first discrete derivatives, with help along the way of a discrete version of the interpolation inequality and an estimate on the time varying norm.

For the Schauder estimate for the second discrete derivatives, we introduce a nonlinear transform of the solution and consider its discrete gradient. Though the equation is not single discrete PDE but a system of linear discrete PDEs, one can apply a similar method as that for the first Schauder estimate to deduce the second Schauder estimate for the original equation.

Another method known to derive gradient estimates is based on the study of the corresponding fundamental solutions. We will also briefly discuss this approach.

In terms of novelty, our results appear to be one of the first to find Schauder estimates for a general class of semi-discrete (continuous in time but discrete in space) quasilinear parabolic PDE of both divergence and non-divergence forms. In particular, the work here provides uniform bounds for discrete first and second order gradients in terms of explicit dependence on the coefficient of the reaction term. Moreover, in terms of methodology, our use of Campanato-type integral estimates seems flexible and perhaps new with respect to obtaining gradient bounds of discrete PDE evolutions. As such we feel the methods of this work may be of potential use in other discrete PDE contexts.

1.1 Quasilinear discrete PDEs

Let $\mathbb{T}_N^n \equiv (\mathbb{Z}/N\mathbb{Z})^n = \{1, 2, \dots, N\}^n$ be the integer lattice of size N with periodic boundary. Let $u = u^N(t, \frac{x}{N}), x \in \mathbb{T}_N^n$, be a solution of the quasilinear discrete PDE

$$\partial_t u = \Delta^N \varphi(u) + K f(u), \quad \frac{x}{N} \in \frac{1}{N}\mathbb{T}_N^n, \tag{1.1}$$

where Δ^N is the discrete Laplacian on $\frac{1}{N}\mathbb{T}_N^n$ (see (2.3) for precise definition), φ is a strictly increasing function on $\mathbb{R}$ whose fifth derivative is bounded $\|\varphi^{(5)}\|_{L^\infty} < \infty$, and f is a function on $\mathbb{R}$ with bounded derivative $\|f'\|_{L^\infty} < \infty$ satisfying that $u \cdot f(u) < 0$ for $u \in \mathbb{R}$ with large enough $|u|$. These assumptions are explained later in this section.

In our problem, we are interested in how the estimates depend on $K > 0$, a large constant. The goal is to give both L^∞ and Hölder estimates on first and second discrete derivatives $\nabla_e^N u^N(t, \frac{x}{N})$ and $\nabla_{e_1}^N \nabla_{e_2}^N u^N(t, \frac{x}{N})$ of the solution uniformly in N; see (2.2) for the definition of discrete derivatives $\nabla^N = \{\nabla_e^N\}_{e \in \mathbb{Z}^n : |e|=1}$. Roughly, in a stationary situation, for example for a traveling wave type solution, the second derivative would behave as $\Delta^N \varphi(u^N) = O(K)$ so that one would expect $\nabla_e^N u^N = O(\sqrt{K})$ and $\nabla_{e_1}^N \nabla_{e_2}^N u^N = O(K)$, but actually our results are weaker than this, given the quasilinear nature of the problem.

As alluded to earlier, discrete PDEs such as (1.1) arise in the study of hydrodynamic limits of interacting particle systems. Indeed, the 'relative entropy' method [44], tries to compare the dependent variables $\{\eta_t^N(x) : x \in \mathbb{T}_N^n\}$ which count the number of particles at $x \in \mathbb{T}_N^n$, with independent variables with means $\{u^N(t, \frac{x}{N}) : x \in \mathbb{T}_N^n\}$ satisfying a discretization of the continuum hydrodynamic PDE. Specifically, estimates of the discrete derivatives of u^N are used to quantify relative entropy bounds and make successful this comparison. In the 'mean-curvature' problem in [15], a version of the equation (1.1) is found where K represents the rate of a 'reaction' mechanism. To achieve the continuum limit, $K = K(N)$ will diverge as $N \uparrow \infty$. How the discrete derivatives of u^N depend in terms of K is used in [15] to specify the form of $K = K(N)$ so that the relative entropy bounds are of the order needed.

Note that the equation (1.1) is a system of ODEs. Given an initial value $u^N(0, \frac{x}{N})$ which is uniformly bounded in N, x and K, by the comparison argument for the nonlinear Laplacian $\Delta^N \varphi(u)$ and noting the condition $u \cdot f(u) < 0$ for large $|u|$ on f, one can show that the solution u exists uniquely and globally in time. In addition, one can show that there exist $-\infty < u_- < u_+ < \infty$ such that

$$u_- \leq u^N(t, \tfrac{x}{N}) \leq u_+ \tag{1.2}$$

holds for all $t \geq 0$ and $x \in \mathbb{T}_N^n$; see Section 2.5. In particular, if $f(0) > 0$ and $u^N(0, \frac{x}{N})$ is uniformly positive, one can take $u_- > 0$ so that, in this case, we can discuss (1.1) for f and φ defined only on $[0, \infty)$. This is in fact the setting where (1.1) arises in the context of particle systems, where the positive solution u^N describes the macroscopic particle density.

Finally, we remark that conditions $\|\varphi^{(3)}\|_{L^\infty} < \infty$ and $\|f\|_{L^\infty} < \infty$ are used in estimating the first discrete gradient of u^N in Section 4.2, while condition $\|\varphi^{(5)}\|_{L^\infty} < \infty$ and $\|f'\|_{L^\infty} < \infty$ are used to estimate the second discrete gradient in Section 5.2. We note only in Corollary 5.10 do we use that φ is 5 times differentiable, while in the other results we use lesser regularity on φ.

1.2 Linear discrete PDEs

The following linear discrete PDE (1.6) of divergence form is associated to the nonlinear equation (1.1). To describe the equation, we introduce some notation. For $0 < c_- < c_+ < \infty$, let $\mathcal{A}(c_-, c_+)$ be the class of functions $a = \{a_{x,e}\}_{x \in \mathbb{T}_N^n, e \in \mathbb{Z}^n : |e|=1}$ satisfying the symmetry, uniform positivity (nondegeneracy) and boundedness conditions:

$$a_{x,e} = a_{x+e,-e}, \tag{1.3}$$

$$c_- \leq a_{x,e} \leq c_+. \tag{1.4}$$

For $a \in \mathcal{A}(c_-, c_+)$, we consider the linear discrete diffusion operator L_a^N with coefficient a defined by

$$\begin{aligned} L_a^N u(\tfrac{x}{N}) &= N^2 \sum_{e:|e|=1} a_{x,e} \{ u(\tfrac{x+e}{N}) - u(\tfrac{x}{N}) \} \\ &= -\tfrac{1}{2} \sum_{e:|e|=1} \nabla_e^{N,*} \big(a_{x,e} \nabla_e^N u \big)(\tfrac{x}{N}), \end{aligned} \tag{1.5}$$

where we have used (1.3) for the second line; see (2.4) for $\nabla_e^{N,*}$ and also (2.5) below. In particular, if $a \equiv 1$, then $L_a^N = \Delta^N$. Note that the operator L_a^N is symmetric: $(L_a^N)^* = L_a^N$ with respect to the natural L^2-inner product for functions on $\frac{1}{N}\mathbb{T}_N^n$; see (2.6) below. For given $a(t) \in \mathcal{A}(c_-, c_+)$ and a bounded function $g(t, \frac{x}{N})$, both continuous in $t \geq 0$, we consider the following linear discrete PDE for $u = u^N(t, \frac{x}{N})$:

$$\partial_t u = L_{a(t)}^N u + g(t, \tfrac{x}{N}), \quad \tfrac{x}{N} \in \tfrac{1}{N}\mathbb{T}_N^n. \tag{1.6}$$

Note that (1.6) has a unique solution globally in time.

1.3 Relation between the quasilinear equation (1.1) and the linear equation (1.6)

Once we a priori know the solution $u = u^N$ of (1.1), the nonlinear equation (1.1) is reduced to the linear equation (1.6) by taking $a(t) \equiv a^N(t)$ and $g(t, \frac{x}{N}) \equiv g^N(t, \frac{x}{N})$ as follows. The nonlinear Laplacian $\Delta^N \varphi(u)$ is regarded as the linear operator $L^N_{a(u^N(t))}u$, i.e.,

$$\Delta^N \varphi(u^N(t))(\tfrac{x}{N}) = L^N_{a(u^N(t))} u^N(t, \tfrac{x}{N}) \tag{1.7}$$

holds, if we define $a(u) = \{a_{x,e}(u)\}$ for $u = \{u(\frac{x}{N})\}$ as a discrete gradient of $\varphi(u)$ in direction e determined by

$$a_{x,e}(u) := \begin{cases} \dfrac{\varphi(u(\frac{x+e}{N})) - \varphi(u(\frac{x}{N}))}{u(\frac{x+e}{N}) - u(\frac{x}{N})}, & \text{if } \ u(\frac{x+e}{N}) \neq u(\frac{x}{N}), \\ \varphi'(u(\frac{x}{N})), & \text{if } \ u(\frac{x+e}{N}) = u(\frac{x}{N}). \end{cases} \tag{1.8}$$

Note that $a(u)$ satisfies the symmetry condition (1.3) and, by (1.2), $a(u^N(t)) \in \mathcal{A}(c_-, c_+)$ holds with

$$c_- = \min_{u \in [u_-, u_+]} \varphi'(u), \quad c_+ = \max_{u \in [u_-, u_+]} \varphi'(u). \tag{1.9}$$

Moreover, we take $g(t, \frac{x}{N}) = Kf(u^N(t, \frac{x}{N}))$. Note that g is bounded by (1.2).

Note also that, from (1.7) and (1.8) combined with the 'mean-value' lemma (cf. Lemma 2.4), the Hölder estimate of the solution of (1.1), shown by the Nash bound, implies a like estimate for the coefficient $a(u^N(t))$ when we view (1.1) as a linear equation (1.6).

1.4 Overview of the main results

Our main results are the L^∞ bounds and the space-time Hölder estimates, both uniform in N, for the first and second discrete derivatives, $\nabla^N_e u^N(t, \frac{x}{N})$ and $\nabla^N_{e_1}\nabla^N_{e_2} u^N(t, \frac{x}{N})$, of the solution of the equation (1.1). To illustrate part of our main estimates, let us introduce discrete C^k-norms for $u^N = \{u^N(\frac{x}{N})\}$ and for $k = 0, 1, 2, \dots$ by

$$\|u^N\|_{C^k_N} := \sum_{i=0}^{k} \sum_{e_1, \dots, e_i} \max_{x \in \mathbb{T}^n_N} \left|\nabla^N_{e_1} \cdots \nabla^N_{e_i} u^N(\tfrac{x}{N})\right| \tag{1.10}$$

where when $k = 0$, the norm reduces to $\|u^N\|_\infty = \|u^N\|_{L^\infty(\mathbb{T}^n_N)}$.

To derive the estimates for the first derivatives, we first study the linear equation (1.6) with bounded $g(t)$ and the coefficient $a(t)$, which satisfies the uniform α-Hölder estimate; see the assumption (A.2) in Section 4.1. This assumption is reasonable by the last comment in Section 1.3. We show the uniform estimate on the first derivative of the solution $u^N(t)$ of (1.6):

$$\|u^N(t)\|_{C^1_N} \le \frac{C}{\sqrt{t}}, \quad 0 < t \le T, \tag{1.11}$$

where C depends on $c_\pm$, Hölder constant and exponent $\alpha \in (0,1)$ of $a(t)$, $\|g\|_\infty$, $\|u^N(\cdot)\|_\infty = \sup_{t\in[0,T]} \|u^N(t)\|_\infty$, n and T; see (4.6) of Theorem 4.1. We note that the constant C is linearly growing in $\|g\|_\infty$ and $\|u^N(\cdot)\|_\infty$, in particular, in $\|u^N(0)\|_\infty$ due to the linearity of the equation (1.6). Dependence on $c_\pm$, n and T stem from the use of an initial Hölder inequality; see Theorem 2.2. A further Hölder estimate on $\nabla_e^N u^N(t, \frac{x}{N})$ in $(t, \frac{x}{N})$ is given in (4.5):

$$|\nabla_e^N u^N(t_1, \tfrac{x_1}{N}) - \nabla_e^N u^N(t_2, \tfrac{x_2}{N})| \le C \frac{|t_1 - t_2|^{\frac{\alpha}{2}} + |\frac{x_1}{N} - \frac{x_2}{N}|^\alpha}{(t_1 \wedge t_2)^{\frac{1+\alpha}{2}}}, \quad 0 < t_1, t_2 \le T. \tag{1.12}$$

By analyzing how C depends on $a(t)$ in (1.11), we deduce the following specific estimate for the solution of the quasilinear equation (1.1):

$$\|u^N(t)\|_{C_N^1} \le \frac{C(K^{\frac{1}{\sigma}} + 1)}{\sqrt{t}}, \quad 0 < t \le T, \tag{1.13}$$

where $\sigma \in (0,1)$ is the Hölder exponent in Nash estimate; see Corollary 4.3. The non-linearity of the above estimate in K is due to that of the equation (1.1). The constant C in (1.13) depends on the Hölder property of the coefficient $a(t)$, given in Corollary 2.3 applied to the equation (1.1), in terms of the Nash estimate.

The equation (1.1) can also be expressed in non-divergence form for the variable $\psi = \varphi(u)$:

$$\begin{aligned} \partial_t \psi &= \varphi'(u)\{\Delta^N \psi + K f(u)\} \\ &= \varphi'(\varphi^{-1}(\psi))\{\Delta^N \psi + K f(\varphi^{-1}(\psi))\}. \end{aligned} \tag{1.14}$$

To derive L^∞ and Hölder estimates on the second derivatives $\nabla_{e_1}^N \nabla_{e_2}^N u^N(t)$, it turns out to be efficient to use the equation (1.14). As above, we first consider a linear system of equations associated to the equation satisfied by $\xi = \nabla^N \psi \equiv \{\nabla_e^N \psi\}_e$ and derive the estimate for $\|\xi\|_{C_N^1}$ as well as the Hölder estimate for $\nabla_e^N \xi$; see the equation (5.4), Theorem 5.5 and Corollary 5.6.

These results are applied to the nonlinear equation (1.14) and we obtain the uniform estimate on the second derivatives of the solution $u^N(t)$ of (1.1):

$$\|u^N(t)\|_{C_N^2} \le \frac{C(K^{\frac{2}{\sigma}} + 1)}{t}, \quad 0 < t \le T; \tag{1.15}$$

see (5.59) in Corollary 5.8 combining with (1.13). The Hölder estimate on $\nabla_{e_1}^N \nabla_{e_2}^N u^N(t, \frac{x}{N})$ in $(t, \frac{x}{N})$ is also given in (5.60) of this corollary.

The local in time Schauder estimates (1.11), (1.13) and (1.15) improve the regularity, but these yield a diverging factor near $t = 0$, which is inherited from the Hölder estimate we find for $u^N(t)$ in Theorem 2.2 and therefore for $a(u^N(t))$ defined by (1.8). This is to be expected as the initial condition could be rough. To avoid this, we show that such a diverging factor does not appear in the L^∞ and Hölder estimates if we assume regularity for the initial value; see Section 2.4, Theorems 4.2 and 5.7.

Based on these observations, we obtain

$$\|u^N(t)\|_{C_N^1} \le C K_0^{\frac{1}{\sigma}}, \quad t \in [0,T], \tag{1.16}$$

$$\|u^N(t)\|_{C_N^2} \le C\big[\bar{K}_0^{\frac{2}{\sigma}}(1+\bar{C}_0^{24}) + \bar{K}_0^{\frac{2}{\sigma}-1}\bar{C}_0^{48}\big], \quad t \in [0,T], \tag{1.17}$$

in (4.21) and (5.93) in Corollaries 4.4 and 5.10 by improving (1.13) and (1.15) under the assumptions that $C_0 = \sup_N \|u^N(0)\|_{C_N^2} < \infty$ and $\bar{C}_0 = \sup_N \|u^N(0)\|_{C_N^4} < \infty$, respectively, where we set

$$K_0 = K + C_0 + 1 \quad \text{and} \quad \bar{K}_0 = K + \bar{C}_0 + 1. \tag{1.18}$$

These estimates clarify the dependence on the initial values. Such a clarification is useful, since, as we mentioned above, we may consider the situation when $\nabla^N u^N(0) = O(\sqrt{K})$ among others. An intermediate estimate on the second derivatives

$$\|u^N(t)\|_{C_N^2} \le \frac{C}{\sqrt{t}} K_0^{\frac{2}{\sigma}}, \quad t \in [0,T] \tag{1.19}$$

is given in Corollary 5.9 under the assumption $C_0 < \infty$, milder than that for (1.17). We also have an improvement of the Hölder estimates; see (4.22), (5.81) and (5.94).

The estimate on $\|u^N(t)\|_{C_N^2}$ can be obtained via the parametrix method applied for the fundamental solution associated with the operator $L_{a(u^N(t))} - \partial_t$; see Proposition 6.3. But the result obtained in this way is much weaker than (1.17) in terms of K so that this approach has a disadvantage. We note that, in the semilinear case with $\varphi(u) = cu$ with $c > 0$, the gradient estimates of the discrete heat kernel leads to the bound on $\|u^N(t)\|_{C_N^1}$, which is however reasonable in K; see [22], Proposition 4.3 and Remark 2.1 below.

1.5 Methods and contents of the article

In the continuous setting, the Schauder estimate for linear parabolic PDEs is well-known; see Lieberman [41] Theorems 4.8 and 6.48, cf. [24] for elliptic PDEs. The basic requirement to derive the Schauder estimate is the Hölder continuity of the coefficients.

In Section 2, as we mentioned above, we show the discrete Hölder estimate for the solution u^N of the equation (1.6) uniformly in N. We recall Nash continuity estimate for (1.6) with $g \equiv 0$ and then apply Duhamel's formula to treat the term g. This applies to the quasilinear equation (1.1). Such a step is fundamental to move to the next step to establish the discrete Schauder estimate. In this context, we also state the maximum principle. See [41] Theorem 6.29 in the continuous setting.

We comment to obtain regularity at $t = 0$, a 'time-reversal' method is used to extend the solution $u^N(t)$ to times $t < 0$. That is, we consider an evolution v^N of the initial condition $u^N(0)$ by the discrete heat equation where $a_{i,j} = \delta_{i,j}$. Then, we define $\hat{u}^N(t) = v^N(1-t)$ for $0 \le t \le 1$ and $\hat{u}^N(t) = u^N(t-1)$ for $1 \le t \le T+1$. One may see that $\hat{u}^N(t)$ satisfies also a discrete PDE with respect to an $\hat{a}(t)$ and $\hat{g}$. Applying the (interior time) Schauder estimates to $\hat{u}^N(t)$ for $t > 1$ gives non-divergent gradient bounds for $u^N(t)$ at $t = 0$. We comment that boundary regularity of continuous parabolic equations is achieved in the literature by various methods under different assumptions on $a(\cdot)$ and $g(\cdot)$; see for instance [18], [39], as well as [45] and references therein. In this respect, our development has been inspired by notions in [46].

Section 3 is a preparatory section. Some Hölder seminorms unweighted or weighted near $t = 0$ are introduced. To apply the continuous method, especially two basic lemmas explained later, we introduce a polylinear interpolation as a link between discrete and continuous settings by embedding discrete functions u^N into continuous ones $\tilde{u}^N$; see (3.11) below. However, the discrete setting has a minimum unit of size $\frac{1}{N}$ and the continuous PDE method does not cover this regime. We have to treat distances less than this minimum unit from different viewpoint.

In Section 4, recalling the last comment in Section 1.3, we work on the linear equation (1.6) with α-Hölder coefficient $a(t)$ and bounded $g(t)$ in general and derive both L^∞ and Hölder estimates for $\nabla^N u$. The results are applied to the quasilinear equation (1.1) and we obtain (1.11), (1.12) and (1.13).

With the α-Hölder assumption (A.2) on $a(t)$ in mind, we couple and compare the solution u^N of (1.6) with the solution $v = v^N$ of a simpler equation called the discrete heat equation

$$\partial_t v = \Delta_a^N v, \tag{1.20}$$

where $\Delta_a^N := L_a^N$ with a coefficient $a = \{a_e\}_{|e|=1}$, which is constant in space and time, satisfying $a_e = a_{-e}$ and $c_- \le a_e \le c_+$; see (2.8) and (3.2) below.

Differently from the continuous setting, the interior estimates of the polylinear interpolation $\tilde{v}^N$ of v^N on the parabolic ball $Q(r)$ of size $r > 0$ (see (3.1)), given the behavior of v^N on a wider discrete parabolic ball $Q_N(R)$, can be discussed only under the condition $r + \frac{\sqrt{n}}{N} \le R$. We call this gap the "non-local nature" of our discrete problem. One reason for this gap is to guarantee the definability of the polylinear interpolation $\tilde{v}^N$ of v^N on $Q(r)$. The other is to derive the interior estimate, severing the influence from the outside area. Thus, we need to consider a band area, that is, a gap region $Q_N(R) \setminus Q(r)$, which is unnecessary in the continuous setting.

The main idea is to apply two basic lemmas used in [41] in suitably modified forms. The first is that a Campanato-type integral estimate for oscillation of F (we take $F = \nabla^N \tilde{u}^N$) on $Q(r)$ implies Hölder estimates for F (cf. Lemma 3.6), and the other is an iteration lemma (cf. Lemmas 3.7 and 4.9).

To derive the integral estimate required for Lemma 3.6, as we mentioned, we couple u^N with the solution v^N of the discrete heat equation (1.20) with properly chosen a and derive the discrete energy inequality for $\nabla^N(u^N - v^N)$ based on the summation by parts formula; see Lemmas 4.6 and 4.7. Here, the Hölder property of $a(t)$ plays a role. The oscillation estimate for $\nabla^N \tilde{v}^N$ in a smaller domain by that in a wider domain is prepared in Proposition 3.4, especially (3.25). This is shown essentially by the maximum principle; cf. Lemmas 3.1 and 3.2. Note that the estimate on the oscillation in time follows from that in space and, for the small region of size of minimum unit such as $r \le \frac{c}{N}$, the polylinearity plays a role. Collecting these estimates, the modified iteration lemma (Lemma 4.9) implies the integral estimate for oscillation of $\nabla^N \tilde{u}^N$ on $Q(r)$ in the L^2-sense required for Lemma 3.6; see Proposition 4.10.

However, as formulated in the assumption in Lemma 3.6, the integral estimate can be shown only with $r_N = r + \frac{c}{N}$ instead of simply $r \geq 0$ due to the "non-local nature" of our discrete problem. As a result, we obtain the uniform Hölder estimate only for two points $X = (t_1, \frac{x_1}{N})$ and $Y = (t_2, \frac{x_2}{N})$ such that $|X - Y| \geq \frac{1}{MN}$, i.e., excluding the short distance regime, in terms of an additional small factor of weighted Hölder seminorm of $\nabla^N u^N$.

To fill the gap, we need a short distance regime estimate for $|X - Y| \leq \frac{1}{MN}$ for large enough M. This will be shown separately in Lemma 4.11. In addition, to complete the program, we need a discrete version of the interpolation inequality and an estimate on the time varying norm.

In Section 5, to derive the estimates on the second derivatives $\nabla^N \nabla^N u^N(t)$, we consider the equation (1.14) of non-divergence form. Setting $\bar{a}(t, \frac{x}{N}) := \varphi'(u^N(t, \frac{x}{N}))$ and $g(t, \frac{x}{N}) := K\bar{a}(t, \frac{x}{N}) f(u^N(t, \frac{x}{N}))$, the equation (1.14) is regarded as a linear discrete PDE with coefficients $\bar{a}(t)$ and $g(t)$. We actually study the system of equations obtained by applying the gradient operator ∇^N to (1.14); see (5.4). Then, the method of the discrete energy inequality works well for this system (see Lemma 5.1), and one can mostly repeat similar arguments given in Section 4 to obtain Theorem 5.5 and Corollary 5.6.

To derive the estimate such as (1.17) avoiding the singularity near $t = 0$, it turns out to be useful to work at the level of the equation (1.14) for ψ, especially to apply the maximum principle; see Theorem 5.7. In other words, the system of equations (5.4) has a gradient structure. When we apply this result for the discrete PDE (1.1), we require the following regularity: $\|\varphi^{(5)}\|_{L^\infty} < \infty$ and $\|f'\|_{L^\infty} < \infty$. In each statement in Corollaries 4.3, 4.4, 5.8, 5.9 and 5.10, we will make clear the regularity assumptions on φ and f by indicating the dependence of the constants C on the derivatives of φ and f.

It is well-known that the Schauder estimate may be shown also by gradient estimates on the associated fundamental solutions, in particular derived by the E.E. Levi's parametrix method; see Friedman [18], Il'in et al. [32], Ladyženskaja et al. [39] and Eidel'man [14]. We discuss the parametrix method in the discrete setting in Section 6. To complete this procedure, the Hölder estimate with singularity at $t = 0$ obtained in Theorem 2.2 is not enough–we will need to avoid it by using the Hölder estimate in Section 2.4.

1.6 Further comments

Finally, we give some further comments. Ladyženskaja et al. [39] took a different method to show the Hölder estimate. They first derive a local energy inequality with a cut-off function. This implies the parabolic Harnack inequality which then leads to the Hölder estimate. We replaced this route by use of the Nash estimate.

Approaches directly applicable to nonlinear PDEs are also known. Lieberman [41] discusses gradient bounds for quasilinear parabolic PDEs in Chapter XI. Evans [17] explains the method of (global) energy inequality. However, the nonlinearity in our equation impedes in the application of these methods.

Discrete PDEs on complex (random) graphs are well studied in the probability literature, and the parabolic Harnack inequality, the Gaussian bound called Aronson estimate on heat kernels, and Hölder estimates are known; see [5], [11], [12], [37], [4]. Semilinear equations with linear discrete Laplacian on general graphs are discussed by [28]. In our case, the graph is $\mathbb{T}_N^n$ and it is much simpler than those studied by these authors. See Section 2.6 for further comments. For $a(t) \in \mathcal{A}(c_-, c_+)$, the Hölder estimate of the fundamental solution p^N of $L_{a(t)}^N - \partial_t$ is available, but we do not have its pointwise gradient estimate in general; see [12].

Related to the stochastic homogenization, a weighted integral estimate on the gradient of p^N in case $a(t) \equiv a$ is obtained in [26], Theorem 3. Especially, their estimate is global in time with optimal decay in t. See [2], Proposition 2.1 for the gradient estimate of p^N in a random conductance model and [10], Theorem 1.3 for that on percolation clusters. See also [25], Lemmas 3.2 and 3.3 for estimates in quenched or annealed sense for the first and second derivatives of elliptic Green's function on $\mathbb{T}_N^n$.

In the context of numerical methods, we mention previous literature on related but different equations with more regularity assumptions. These include [1], [9] on convergence bounds for types of implicit discretizations of quasilinear parabolic equations with Lipschitz continuous a, making use of discrete maximal parabolic regularity ideas in [35], [40]. Also, other literature with respect to gradient bounds of discrete PDE equations with different structures includes [6], [33], [43] on discrete gradient and Schauder estimates for types of elliptic problems, and [30], [38] on Schauder estimates for fully nonlinear discrete equations subject to a concavity assumption. See also Chapter 9 of [34] especially, the semi-discrete approach) and Section 10.5 of [36] for general discussions about discretization and convergence of the solution of linear discrete PDEs with boundary conditions.

1.7 Notation

Here, we summarize notation used in this article. As we mentioned, we work on the space $\frac{1}{N}\mathbb{T}_N^n$ which is a discretization of the continuous torus $\mathbb{T}^n \simeq [0,1)^n$ of size 1 with mesh size $\frac{1}{N}$, where $\mathbb{T}_N^n = \{1, 2, \dots, N\}^n$ is the discrete torus of size N. The discrete derivatives ∇_e^N and $\nabla_e^{N,*}$ in the direction $e \in \mathbb{Z}^n : |e| = 1$ are defined in (2.2) acting on functions u on $\frac{1}{N}\mathbb{T}_N^n$, and sometimes for those on $\mathbb{T}^n$; see (3.31). We denote $\nabla^N = \{\nabla_e^N\}_{e\in\mathbb{Z}^n:|e|=1}$. The discrete Laplacian Δ^N is defined in (2.3). The linear discrete diffusion operator L_a^N of divergence form is given by (1.5). Its coefficient $a = \{a_{x,e}\}_{x\in\mathbb{T}_N^n, e\in\mathbb{Z}^n:|e|=1}$ is taken from the class $\mathcal{A}(c_-, c_+)$ satisfying the symmetry condition (1.3) and the nondegeneracy/boundedness condition (1.4) for $0 < c_- < c_+ < \infty$. The discrete C^k-norms $\|\cdot\|_{C_N^k}$ for functions on $\frac{1}{N}\mathbb{T}_N^n$ are defined in (1.10) for $k = 0, 1, 2, \dots$

We link the functions $u^N = \{u^N(\frac{x}{N})\}$ on $\frac{1}{N}\mathbb{T}_N^n$ to $\tilde{u}^N = \{\tilde{u}^N(z)\}$ on $\mathbb{T}^n$ via the polylinear interpolation

$$\tilde{u}^N(z) = \sum_{v\in\{0,1\}^n} \vartheta^N(v,z) u^N\big(\tfrac{[Nz]+v}{N}\big), \tag{1.21}$$

which satisfies $u^N(\frac{x}{N}) = \tilde{u}^N(\frac{x}{N})$ for every $\frac{x}{N} \in \frac{1}{N}\mathbb{T}_N^n$ in terms of nonnegative weights $\{\vartheta^N(v,z)\}_{v\in\{0,1\}^n}$ satisfying $\sum_{v\in\{0,1\}^n} \vartheta^N(v,z) = 1$; see (3.11) below for details.

In the space-time setting, for $T > 0$ fixed, we write $\Omega = [0,T] \times \mathbb{T}^n$ and $\Omega_N = [0,T] \times \frac{1}{N}\mathbb{T}_N^n$. For $X = (t,z) \in \Omega$, set $|X| = \max\{t^{\frac{1}{2}}, |z|\}$, called the parabolic norm, and $d(X) = t^{\frac{1}{2}}$, called the parabolic distance to the parabolic boundary $\mathcal{P}\Omega = \{t = 0\} \times \mathbb{T}^n$. For a fixed $X_0 = (t_0, z_0) \in \Omega$, we define parabolic balls $Q(R) := \{X \in \Omega; |X - X_0| <$

$R, t < t_0\} \equiv (t_0 - R^2, t_0) \times \{z \in \mathbb{T}^n; |z - z_0| < R\}$ in the continuous setting, and $Q_N(R) := Q(R) \cap \Omega_N$ in the discrete setting.

Several Hölder norms are introduced in Section 3.3 for functions F on Ω and Ω_N. Let us briefly summarize. For a function $F = F(X)$ on Ω, set

$$|F|_0 \equiv \|F\|_\infty := \sup_{X \in \Omega} |F(X)|, \tag{1.22}$$

see (3.32) below. For $\alpha \in (0, 1]$, define the parabolic Hölder seminorms by

$$\begin{aligned} [F]_\alpha &:= \sup_{X \neq Y \in \Omega} \frac{|F(X) - F(Y)|}{|X - Y|^\alpha}, \\ [F]_{1+\alpha} &:= \sup_{X \neq Y \in \Omega} \frac{|\nabla^N F(X) - \nabla^N F(Y)|}{|X - Y|^\alpha}, \end{aligned} \tag{1.23}$$

see (3.33) below. For $\alpha = 0$, $[F]_0 = \operatorname*{osc}_\Omega(F) := \sup_{X,Y \in \Omega} |F(X) - F(Y)|$ is the oscillation of F on Ω. For $\beta \in (0, 2]$, define

$$\langle F \rangle_\beta := \sup_{X \neq Y \in \Omega, x = y} \frac{|F(X) - F(Y)|}{|X - Y|^{\frac{\beta}{2}}}, \tag{1.24}$$

see (3.34) below. Adding all these, we define for $a \in (0, 2]$, the unweighted Hölder norm

$$|F|_a := [F]_a + \langle F \rangle_a + |F|_0, \tag{1.25}$$

see (3.35) below. We also introduce weighted norms to take care of diverging effects near $t = 0$. For $a = 0$ and $b \geq 0$, define

$$|F|_0^{(b)} := \sup_{X \in \Omega} d(X)^b |F(X)|. \tag{1.26}$$

For $0 < a = k + \alpha \leq 2$ where $k = 0, 1$ and $\alpha \in (0, 1]$, and $b \geq 0$, let

$$[F]_a^{(b)} := \sup_{X \neq Y \in \Omega} (d(X) \wedge d(Y))^{a+b} \frac{|(\nabla^N)^k F(X) - (\nabla^N)^k F(Y)|}{|X - Y|^\alpha}. \tag{1.27}$$

$$\langle F \rangle_a^{(b)} = \sup_{X \neq Y \in \Omega, x = y} (d(X) \wedge d(Y))^{a+b} \frac{|F(X) - F(Y)|}{|X - Y|^{\frac{a}{2}}}, \tag{1.28}$$

see (3.36), (3.37), (3.38) below. Replacing Ω by Ω_N in the definitions (1.26), (1.27), (1.28), we define, for a function F on Ω_N, seminorms on $\Omega_N = [0, T] \times \frac{1}{N}\mathbb{T}_N^n$:

$$|F|_0^{(b),N}, \; [F]_a^{(b),N}, \; \langle F \rangle_a^{(b),N},$$

see (3.39), (3.40), (3.41) below. Seminorms with $*$ means (0) so that for $a \in (0, 2]$,

$$[F]_a^* := [F]_a^{(0)}, \; \langle F \rangle_a^* := \langle F \rangle_a^{(0)}, \; [F]_a^{*,N} := [F]_a^{(0),N}, \; \langle F \rangle_a^{*,N} := \langle F \rangle_a^{(0),N},$$

see (3.42) below. We will use $\|F\|_\infty$ rather than $|F|_0^*$ or $|F|_0^{*,N}$.

We will frequently use the notation $r_N = r + \frac{c}{N}$ for $r \geq 0$ with some $c > 0$ which reflects the non-local nature of our discrete problem.

Also, we will have occasion to use the notation $\|h\|_\infty := \|h\|_{L^\infty([u_-, u_+])}$ for real-valued functions h which may be the kth derivative $\varphi^{(k)}$ or f, with respect to $u_\pm$ in (1.2); see for instance Corollary 2.3. On the other hand, for functions h^N defined on Ω_N, such as u^N, we set $\|h^N\|_\infty := \sup_{\Omega_N} |h^N(t, \frac{x}{N})|$; see Theorem 4.1.

2 Hölder estimate for the solution of a discrete PDE

We first define spatial gradients and other notation in Section 2.1. Then, in Section 2.2, recalling Appendix B of Giacomin, Olla and Spohn [23], we formulate a Nash estimate for the solution of a discrete linear PDE (1.6) with $g \equiv 0$ on $\frac{1}{N}\mathbb{Z}^n$ instead of $\frac{1}{N}\mathbb{T}_N^n$. Based on this, we derive Hölder regularity of the solution u^N of (1.6) with g and therefore (1.1) on $\frac{1}{N}\mathbb{T}_N^n$ in Section 2.3. In Section 2.4, we show that the singularity in Hölder estimate at $t=0$ can be removed under a regularity assumption on the initial value $u^N(0)$. In Section 2.5, we show (1.2) and formulate the maximum principle.

2.1 Discrete derivatives and Laplacian

For functions u and v on $\frac{1}{N}\mathbb{T}_N^n$, we define the inner product by

$$\langle u, v\rangle_N = \frac{1}{N^n} \sum_{x\in\mathbb{T}_N^n} u(\tfrac{x}{N})v(\tfrac{x}{N}). \tag{2.1}$$

For $e \in \mathbb{Z}^n : |e| = 1$, we define the discrete derivative ∇_e^N to the direction e and $\nabla_e^{N,*}$ by

$$\nabla_e^N u(\tfrac{x}{N}) := N(u(\tfrac{x+e}{N}) - u(\tfrac{x}{N})), \quad \nabla_e^{N,*} u(\tfrac{x}{N}) := N(u(\tfrac{x-e}{N}) - u(\tfrac{x}{N})). \tag{2.2}$$

Note that $\nabla_e^{N,*} = \nabla_{-e}^N$ is the dual operator of ∇_e^N with respect to the inner product $\langle\ ,\ \rangle_N$.

We also define the discrete Laplacian by

$$\Delta^N := -\tfrac{1}{2} \sum_{|e|=1} \nabla_e^{N,*}\nabla_e^N = - \sum_{|e|=1,e>0} \nabla_e^{N,*}\nabla_e^N = N \sum_{|e|=1} \nabla_e^N. \tag{2.3}$$

Here, $e > 0$ for $e : |e| = 1$ means that its nonzero component is 1. To see the second and third identities in (2.3), we note the following simple fact. Let $\tau_y, y \in \mathbb{Z}^n$, be the shift operator defined by $\tau_y u(\frac{x}{N}) = u(\frac{x+y}{N})$. Then, we have

$$\nabla_e^{N,*} = \nabla_{-e}^N = -\tau_{-e}\nabla_e^N, \quad [\tau_y, \nabla_e^N] = [\nabla_e^N, \nabla_{e'}^N] = [\nabla_e^N, \nabla_{e'}^{N,*}] = 0 \tag{2.4}$$

for every $e, e' : |e| = |e'| = 1$, where $[A, B] = AB - BA$ denotes the commutator of two operators A and B.

The second identity in (2.3) follows from $\nabla_e^{N,*}\nabla_e^N = \nabla_{-e}^N\nabla_{-e}^{N,*} = \nabla_{-e}^{N,*}\nabla_{-e}^N$, while the third identity $\Delta^N = N\sum_{|e|=1}\nabla_e^N$ follows from $\Delta^N u(\frac{x}{N}) = N\sum_{|e|=1,e>0}\big(\nabla_e^N u(\frac{x}{N}) - \nabla_e^N u(\frac{x-e}{N})\big)$ and $\nabla_e^N u(\frac{x-e}{N}) = \tau_{-e}\nabla_e^N u(\frac{x}{N}) = -\nabla_{-e}^N u(\frac{x}{N})$.

Given $a = \{a_{x,e}\}_{x\in\mathbb{T}_N^n, e\in\mathbb{Z}^n:|e|=1} \in \mathcal{A}(c_-, c_+)$ (recall two conditions (1.3) and (1.4)), we consider the discrete diffusion operator L_a^N defined by (1.5). As an extension of (2.3), with the relations (2.4) in hand and by (1.3), it can be rewritten as

$$L_a^N u(\tfrac{x}{N}) \Big(= N \sum_{|e|=1} a_{x,e}\nabla_e^N u(\tfrac{x}{N})\Big) = - \sum_{|e|=1,e>0} \nabla_e^{N,*}\big(a_{x,e}\nabla_e^N u\big)(\tfrac{x}{N}), \tag{2.5}$$

since we have

$$\nabla_{-e}^{N,*}\big(a_{x,-e}\nabla_{-e}^N u\big)(\tfrac{x}{N}) = \nabla_{-e}^{N,*}\big(a_{x-e,e}\nabla_{-e}^N u\big)(\tfrac{x}{N})$$

$$= -\nabla_{-e}^{N,*}\big(\tau_{-e}a_{x,e}\tau_{-e}\nabla_e^N u\big)(\tfrac{x}{N})$$
$$= -\nabla_{-e}^{N,*}\big(\tau_{-e}(a_{x,e}\nabla_e^N u)\big)(\tfrac{x}{N})$$
$$= \nabla_e^{N,*}\big(a_{x,e}\nabla_e^N u\big)(\tfrac{x}{N}).$$

Moreover, L_a^N is symmetric with respect to $\langle\ ,\ \rangle_N$ and the corresponding Dirichlet form is given by

$$\mathcal{E}_a^N(u,v) := -\langle L_a^N u, v\rangle_N = \frac{1}{2N^n}\sum_{x\in\mathbb{T}_N^n}\sum_{|e|=1} a_{x,e}\nabla_e^N u(\tfrac{x}{N})\nabla_e^N v(\tfrac{x}{N}). \tag{2.6}$$

Given $a(t)\in\mathcal{A}(c_-,c_+)$ for $t\ge 0$, we will consider the time-dependent operator $L_{a(t)}^N$.

The following will be used in Section 3 and later. We denote $\Delta_a^N := L_a^N$ in case that $a=\{a_e\}_{|e|=1}$ does not depend on x and call Δ_a^N the discrete Laplacian with constant coefficients. Namely, for given constants $a=\{a_e\}_{|e|=1}$ satisfying

$$a_e = a_{-e}, \quad c_- \le a_e \le c_+, \tag{2.7}$$

we set

$$\Delta_a^N := -\sum_{|e|=1,e>0} a_e\nabla_e^{N,*}\nabla_e^N \;=\; -\frac{1}{2}\sum_{|e|=1}\nabla_e^{N,*}a_e\nabla_e^N \;=\; N\sum_{|e|=1}a_e\nabla_e^N. \tag{2.8}$$

The following formulas for discrete derivatives will be useful:

$$\begin{aligned}\nabla_e^N(uv)(\tfrac{x}{N}) &= v(\tfrac{x+e}{N})\nabla_e^N u(\tfrac{x}{N}) + u(\tfrac{x}{N})\nabla_e^N v(\tfrac{x}{N})\\ &= \tfrac{1}{2}(v(\tfrac{x}{N})+v(\tfrac{x+e}{N}))\nabla_e^N u(\tfrac{x}{N}) + \tfrac{1}{2}(u(\tfrac{x}{N})+u(\tfrac{x+e}{N}))\nabla_e^N v(\tfrac{x}{N}).\end{aligned} \tag{2.9}$$

In particular,

$$\nabla_e^N u^2(\tfrac{x}{N}) = (u(\tfrac{x}{N})+u(\tfrac{x+e}{N}))\nabla_e^N u(\tfrac{x}{N}). \tag{2.10}$$

Note also that

$$\Delta_a^N(uv)(\tfrac{x}{N}) = v(\tfrac{x}{N})\Delta_a^N u(\tfrac{x}{N}) + u(\tfrac{x}{N})\Delta_a^N v(\tfrac{x}{N}) + \sum_{|e|=1} a_e\nabla_e^N u(\tfrac{x}{N})\nabla_e^N v(\tfrac{x}{N}). \tag{2.11}$$

2.2 Nash continuity estimate for linear discrete PDEs on $\frac{1}{N}\mathbb{Z}^n$

In order to apply the results in [23] and [42], we consider the linear discrete PDE (1.6) with $g\equiv 0$ for $\frac{x}{N}\in\frac{1}{N}\mathbb{Z}^n$ instead of $\frac{1}{N}\mathbb{T}_N^n$:

$$\partial_t u = L_{a(t)}^N u, \quad \tfrac{x}{N}\in\tfrac{1}{N}\mathbb{Z}^n. \tag{2.12}$$

Here, the coefficient $a(t)=\{a_{x,e}(t)\}_{x\in\mathbb{Z}^n,e\in\mathbb{Z}^n:|e|=1}\in\mathcal{A}(c_-,c_+)$ satisfying (1.3) and (1.4) is given for $t\ge 0$ and for $x\in\mathbb{Z}^n$ instead of $x\in\mathbb{T}_N^n$. We assume $a(t)$ is continuous in t. Then, the operator $L_{a(t)}^N$ is defined by (1.5) (or (2.5)) for $x\in\mathbb{Z}^n$. The inner product $\langle\ ,\ \rangle_N$ in (2.1), discrete gradients $\nabla_e^N, \nabla_e^{N,*}$ in (2.2) and discrete Laplacian Δ^N in (2.3) are easily extended to $\frac{1}{N}\mathbb{Z}^n$.

We take $\alpha = N^{-1}$ as the scaling parameter α in [23], p.1167– and in [42]. We note that the factor $N^2(=\alpha^{-2})$ from the time change is hidden in ∇_e^N and $\nabla_e^{N,*}$ in our case.

The temporally inhomogeneous semigroup generated by $L_{a(t)}^N$ is denoted by $P_{s,t}^N$ for $0 \le s < t < \infty$, that is, $u(t, \frac{x}{N}) = P_{s,t}^N \bar{u}(\frac{x}{N})$ is the solution of (2.12) for $t \ge s$ satisfying $u(s,\cdot) = \bar{u}(\cdot)$. The corresponding fundamental solution is given by

$$p^N(s, \tfrac{y}{N}; t, \tfrac{x}{N}) = N^n (P_{s,t}^N 1_{\frac{y}{N}})(\tfrac{x}{N}).$$

Note that p^N satisfies the forward equation $\partial_t p^N = L_{a(t),x}^N p^N$ for $t \in (s,\infty)$ with $p^N(s, \frac{y}{N}; s, \frac{x}{N}) = N^n \delta_{x,y}$ and the backward equation $\partial_s p^N = -L_{a(s),y}^N p^N$, for $s \in (0,t)$ with $p^N(t, \frac{y}{N}; t, \frac{x}{N}) = N^n \delta_{x,y}$; see Definition 6.1 below and recall the symmetry of $L_{a(t)}^N$.

The following proposition is a consequence of the parabolic Harnack inequality.

Proposition 2.1. *(Nash continuity estimate) There exist $\sigma \in (0,1)$ and $C = C(n, c_\pm)$ (in particular, uniform in N) such that for every $N \in \mathbb{N}$ and $u_0 \in L^\infty(\frac{1}{N}\mathbb{Z}^n)$*

$$|P_{0,t}^N u_0(\tfrac{x}{N}) - P_{0,s}^N u_0(\tfrac{y}{N})| \le C \|u_0\|_\infty \Big(\frac{|t-s|^{\frac{1}{2}} \vee |\frac{x}{N} - \frac{y}{N}|}{(t \wedge s)^{\frac{1}{2}}} \Big)^\sigma,$$

for $t, s > 0$ and $x, y \in \mathbb{Z}^n$, where $\|u_0\|_\infty = \sup_{x \in \mathbb{Z}^n} |u_0(\frac{x}{N})|$. In particular, we have

$$\big| p^N(0, \tfrac{y'}{N}; t', \tfrac{x'}{N}) - p^N(0, \tfrac{y}{N}; t, \tfrac{x}{N}) \big| \le C \Big(|t'-t|^{\frac{1}{2}} + |\tfrac{x}{N} - \tfrac{x'}{N}| + |\tfrac{y}{N} - \tfrac{y'}{N}| \Big)^\sigma (t' \wedge t)^{-\frac{n+\sigma}{2}}.$$

Proof. See Proposition B.6 of [23] and also Theorem 1.31 of Stroock and Zheng [42]. Note that [42] deals with the temporally homogeneous case. Note also that the symmetry of $L_{a(t)}^N$ implies that of p^N in $\frac{x}{N}$ and $\frac{y}{N}$ by Lemma 6.1 below. □

Remark 2.1. The Hölder exponent $\sigma \in (0,1)$ is implicitly determined. Its lower bound is discussed in [31] in a continuous setting. We note that, in the semilinear case with $\varphi(u) = cu$ with $c > 0$, the bound (1.16) is shown with $\sigma = 1$ under the assumption $\sup_N \|u^N(0)\|_{C_N^1} \le CK$; see [22], Proposition 4.3. See also Remark 3.1 below.

2.3 Hölder estimate for discrete PDEs with an external term on $\frac{1}{N}\mathbb{T}_N^n$

In this subsection, we make use of the Nash estimate (Proposition 2.1) to show Hölder regularity for the solution $u^N(t, \frac{x}{N})$ of (1.6) with the external term $g = g(t, \frac{x}{N})$ and accordingly (1.1) in $(t, \frac{x}{N})$, $t > 0$, $\frac{x}{N} \in \frac{1}{N}\mathbb{T}_N^n$, uniformly in N. Recall that g is bounded. To apply Proposition 2.1 with respect to $\frac{1}{N}\mathbb{Z}^n$, we extend the solution $u^N(t, \frac{x}{N})$ to $\frac{1}{N}\mathbb{Z}^n$ periodically in x, setting $u^N(t, \frac{x}{N} + e) = u^N(t, \frac{x}{N})$ for every $e \in \mathbb{Z}^n$. Alternatively, we may solve the discrete PDE (1.6) on $\frac{1}{N}\mathbb{Z}^n$ with N-periodically extended coefficients $a(t) = \{a_{x,e}(t)\}$, $g(t, \frac{x}{N})$ and initial value $u^N(0, \frac{x}{N})$, $x \in \mathbb{Z}^n$. Then, by uniqueness of solution, $u^N(t, \frac{x}{N})$ is also periodic.

We now derive a Hölder estimate for the solution u^N of (1.6). The following theorem holds on $\frac{1}{N}\mathbb{Z}^n$ for non-periodic $a(t)$ and g. But, we state it only on $\frac{1}{N}\mathbb{T}_N^n$. We define the distance on $\mathbb{T}^n \cong [0,1)^n$ and therefore on $\frac{1}{N}\mathbb{T}_N^n$ as follows. For $z_1, z_2 \in \mathbb{T}^n$ and for $x_1, x_2 \in \mathbb{T}_N^n$,

$$\big|z_1 - z_2\big| := \min_{e\in\mathbb{Z}^n} \big|z_1 - z_2 + e\big|, \quad \big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big| := \min_{x_2' = x_2 \bmod N} \big|\tfrac{x_1}{N} - \tfrac{x_2'}{N}\big|. \tag{2.13}$$

Theorem 2.2. *Let $u^N(t, \frac{x}{N})$ be the solution of (1.6) on $\frac{1}{N}\mathbb{T}_N^n$ with $a(t) \in \mathcal{A}(c_-, c_+)$ being continuous in t and initial value $u^N(0)$ satisfying $\|u^N(0)\|_\infty < \infty$. Let $\sigma \in (0,1)$ be as in Proposition 2.1. Then, we have*

$$|u^N(t_1, \tfrac{x_1}{N}) - u^N(t_2, \tfrac{x_2}{N})| \le C(\|g\|_\infty + \|u^N(0)\|_\infty) \frac{\big|t_2 - t_1\big|^{\frac{\sigma}{2}} + \big|\frac{x_1}{N} - \frac{x_2}{N}\big|^\sigma}{(t_1 \wedge t_2)^{\frac{\sigma}{2}}}, \tag{2.14}$$

for $0 < t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{T}_N^n$, where $C = C(n, c_\pm, T)$ and $\|g\|_\infty := \|g\|_{L^\infty([0,T]\times\frac{1}{N}\mathbb{T}_N^n)}$.

Proof. Step 1. Recall that $P_{s,t}^N$ is the semigroup associated to the discrete linear PDE (2.12) with periodic $a(t)$. By Duhamel's formula, the periodically extended solution u^N of (1.6) with g satisfies

$$\begin{aligned} u^N(t, \tfrac{x}{N}) &= \big(P_{0,t}^N u^N(0)\big)(\tfrac{x}{N}) + \int_0^t ds \big(P_{s,t}^N g(s,\cdot)\big)(\tfrac{x}{N}) \\ &=: I_1^N(t, \tfrac{x}{N}) + I_2^N(t, \tfrac{x}{N}), \end{aligned} \tag{2.15}$$

where I_1^N and I_2^N satisfy

$$\begin{aligned} &\partial_t I_1^N = L_{a(t)}^N I_1^N, \quad I_1^N(0) = u^N(0), \\ &\partial_t I_2^N = L_{a(t)}^N I_2^N + g(t, \tfrac{x}{N}), \quad I_2^N(0) = 0. \end{aligned}$$

Observe that the first term $I_1^N(t, \frac{x}{N})$ is $(\frac{\sigma}{2}, \sigma)$-Hölder continuous and the difference $|I_1(t_1, \frac{x_1}{N}) - I_1(t_2, \frac{x_2}{N})|$ is bounded, with front factor $C(n, c_\pm)\|u^N(0)\|_\infty (t_1 \wedge t_2)^{-\frac{\sigma}{2}}$, using Proposition 2.1.

Step 2. The second term $I_2^N(t, \frac{x}{N})$ can be estimated as follows. First for $x_1, x_2 \in \mathbb{T}_N^n$ (embedded in $\mathbb{Z}^n$), by Proposition 2.1, shifting time by s,

$$\begin{aligned} |I_2^N(t, \tfrac{x_1}{N}) - I_2^N(t, \tfrac{x_2}{N})| &\le \int_0^t ds \big|\big(P_{s,t}^N g(s,\cdot)\big)(\tfrac{x_1}{N}) - \big(P_{s,t}^N g(s,\cdot)\big)(\tfrac{x_2}{N})\big| \\ &\le C(n, c_\pm)\|g\|_\infty \big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big|^\sigma \int_0^t (t-s)^{-\frac{\sigma}{2}} ds \le C(n, c_\pm, T)\|g\|_\infty \big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big|^\sigma, \end{aligned}$$

for $t \in [0,T]$, since $\sigma \in (0,1)$.

Step 3. Next, let us show the Hölder estimate for $I_2^N(t, \frac{x}{N})$ in t. The proof is similar. For $0 < t_1 < t_2$,

$$\begin{aligned} I_2^N(t_2, \tfrac{x}{N}) - I_2^N(t_1, \tfrac{x}{N}) = &\int_0^{t_1} ds \Big\{ \big(P_{s,t_2}^N g(s,\cdot)\big)(\tfrac{x}{N}) - \big(P_{s,t_1}^N g(s,\cdot)\big)(\tfrac{x}{N}) \Big\} \\ &+ \int_{t_1}^{t_2} ds \big(P_{s,t_2}^N g(s,\cdot)\big)(\tfrac{x}{N}) \end{aligned}$$

$$=: I_{2,1}^N + I_{2,2}^N.$$

Here, $P_{s,t}^N 1 \equiv 1$ since $u(t) \equiv 1$ solves $\partial_t u = L_{a(t)}^N u$ for $t > s$ with $u(s) \equiv 1$. Also, $\big|\big(P_{s,t_2}^N g(s,\cdot)\big)(\frac{x}{N})\big| \le \|g\|_\infty \big|P_{s,t_2}^N 1\big| = \|g\|_\infty$. Hence,

$$|I_{2,2}^N| \le \|g\|_\infty |t_2 - t_1|.$$

For $I_{2,1}^N$, by Proposition 2.1 shifting time by s again,

$$|I_{2,1}^N| \le C(n, c_\pm)\|g\|_\infty \big|t_2 - t_1\big|^{\frac{\sigma}{2}} \int_0^{t_1} (t_1 - s)^{-\frac{\sigma}{2}} ds \le C(n, c_\pm, T)\|g\|_\infty \big|t_2 - t_1\big|^{\frac{\sigma}{2}}.$$

Thus, we obtain

$$|I_2^N(t_2, \tfrac{x}{N}) - I_2^N(t_1, \tfrac{x}{N})| \le C(n, c_\pm, T)\|g\|_\infty \big|t_2 - t_1\big|^{\frac{\sigma}{2}}, \quad 0 \le t_1, t_2 \le T.$$

The theorem is shown by combining these estimates. $\square$

The diverging denominator in the estimate in Theorem 2.2 comes from $I_1^N(t,x)$. In the next subsection, we remove this singularity when $u^N(0,\cdot)$ has C^2-regularity.

Theorem 2.2 shows the following corollary for the equation (1.1).

Corollary 2.3. *Let $u^N(t, \frac{x}{N})$ be the solution of* (1.1) *on $\frac{1}{N}\mathbb{T}_N^n$ satisfying* (1.2)*. Then, we have, with $C = C(n, c_\pm, T)$ that*

$$(2.16) \qquad |u^N(t_1, \tfrac{x_1}{N}) - u^N(t_2, \tfrac{x_2}{N})| \le C(K\|f\|_\infty + \|u^N(0)\|_\infty) \frac{\big|t_2 - t_1\big|^{\frac{\sigma}{2}} + \big|\frac{x_1}{N} - \frac{x_2}{N}\big|^\sigma}{(t_1 \wedge t_2)^{\frac{\sigma}{2}}},$$

for $0 < t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{T}_N^n$, where $\|f\|_\infty = \|f\|_{L^\infty([u_-,u_+])}$.

Moreover, $a_{x,e}(t) := a_{x,e}(u^N(t))$ defined by (1.8) *with $u = u^N(t)$ satisfies*

$$(2.17) \qquad |a_{x_1,e}(t_1) - a_{x_2,e}(t_2)| \le C(K\|f\|_\infty + \|u^N(0)\|_\infty) \frac{\big|t_2 - t_1\big|^{\frac{\sigma}{2}} + \big|\frac{x_1}{N} - \frac{x_2}{N}\big|^\sigma}{(t_1 \wedge t_2)^{\frac{\sigma}{2}}},$$

for $0 < t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{T}_N^n$ where $C = C(n, c_\pm, T, \|\varphi''\|_\infty)$ and $\|\varphi''\|_\infty = \|\varphi''\|_{L^\infty([u_-,u_+])}$.

Proof. The first estimate (2.16) is immediate from Theorem 2.2 by noting (1.7), (1.9) and $\|g\|_\infty \le K\|f\|_{L^\infty([u_-,u_+])}$. Note that the continuity of $a(t)$ in t follows from that of $u^N(t)$, (1.8) and the next Lemma 2.4. The second estimate (2.17) follows from (1.8), Lemma 2.4 and (2.16). $\square$

We state the 'mean-value' lemma used in the proof of Corollary 2.3.

Lemma 2.4. *If $\varphi \in C^2([u_-, u_+])$, we have for every $a, b, c, d \in [u_-, u_+]$ such that $a \ne b$, $c \ne d$,*

$$\left|\tfrac{\varphi(a)-\varphi(b)}{a-b} - \tfrac{\varphi(c)-\varphi(d)}{c-d}\right| \le C(|a-c| + |b-d|),$$

and also

$$\left|\tfrac{\varphi(a)-\varphi(b)}{a-b} - \varphi'(c)\right| \le C(|a-c| + |b-c|).$$

where we can take $C = \frac{1}{2}\|\varphi''\|_{L^\infty([u_-,u_+])}$ and recall (1.2) *for $u_\pm$.*

Proof. We prove the first statement as the second follows from the first by letting $d \to c$.

The left hand side is bounded by

$$\left|\frac{\varphi(a)-\varphi(b)}{a-b} - \frac{\varphi(a)-\varphi(d)}{a-d}\right| + \left|\frac{\varphi(a)-\varphi(d)}{a-d} - \frac{\varphi(c)-\varphi(d)}{c-d}\right|,$$

so that we may bound each term by $C|b-d|$ and $C|a-c|$, respectively. But these are essentially the same by symmetry so that we may prove the first one. Fix a and d, and set

$$\theta(b) := \frac{\varphi(a)-\varphi(b)}{a-b} - \frac{\varphi(a)-\varphi(d)}{a-d}.$$

Then, $\theta(d) = 0$ and by Taylor's formula

$$\theta'(b) = \frac{-\varphi'(b)(a-b)+(\varphi(a)-\varphi(b))}{(a-b)^2} = \frac{1}{2}\varphi''(A)$$

for some $A \in [u_-, u_+]$, which implies $|\theta(b)| = |\theta(b) - \theta(d)| \le C|b-d|$. $\square$

Remark 2.2. Under the N-periodic situation, the fundamental solution $p^N(s, \frac{y}{N}; t, \frac{x}{N})$, $\frac{y}{N}, \frac{x}{N} \in \frac{1}{N}\mathbb{T}_N^n$, on $\frac{1}{N}\mathbb{T}_N^n$ can be constructed from the fundamental solution $\tilde{p}^N(s, \frac{y}{N}; t, \frac{x}{N})$ on $\frac{1}{N}\mathbb{Z}^n$ as follows:

$$p^N(s, \tfrac{y}{N}; t, \tfrac{x}{N}) := \sum_{x' \in \mathbb{Z}^n : x' = x \bmod N} \tilde{p}^N(s, \tfrac{y}{N}; t, \tfrac{x'}{N}).$$

2.4 Regularity at $t = 0$

Here, we improve the Hölder regularity near $t = 0$ of the solution $u = u^N(t, \frac{x}{N})$ for $x \in \mathbb{T}_N^n$ to the equation (1.6) given in Theorem 2.2, when the initial condition $u^N(0, \frac{x}{N})$ has uniformly bounded second derivatives:

$$\sup_N \max_{x \in \mathbb{T}_N^n, |e_1|=|e_2|=1} |\nabla_{e_1}^N \nabla_{e_2}^N u^N(0, \tfrac{x}{N})| \le C_0 < \infty. \tag{2.18}$$

A sufficient condition is that $u^N(0, \cdot) = u_0(\cdot)$ for $u_0 \in C^2(\mathbb{T}^n)$. Recall that $a = \{a_{x,e}^N(t)\}$ satisfies (1.3) and (1.4) for $t \ge 0$.

We show the following theorem. The idea of the proof is that the regularity of $u^N(0, \cdot)$ allows to extend the solution backwards in time, say for $t \in [-1, 0]$. Then, after a time-shift by $+1$, one formulates a new discrete PDE, whose diffusion coefficient still satisfies symmetry and nondegeneracy/boundedness as in (1.3) and (1.4), but which matches equation (1.6) for times $t \ge 1$. From the Hölder estimate (Theorem 2.2) to this new formulation for times $t \ge 0$, one can get a Hölder estimate for the original equation above for times $t \ge 1$ but without a diverging divisor as before.

Theorem 2.5. *Let $\sigma \in (0,1)$ be as in Proposition 2.1 and assume $u^N(0, \cdot)$ satisfies* (2.18) *with $C_0 < \infty$. Then, for the solution $u^N(t, \frac{x}{N})$ of* (1.6) *with initial value $u^N(0, \cdot)$, we have*

$$|u^N(t_1, \tfrac{x_1}{N}) - u^N(t_2, \tfrac{x_2}{N})| \le C(\|g\|_\infty + \|u^N(0)\|_\infty + C_0)\Big\{ \big|t_2 - t_1\big|^{\frac{\sigma}{2}} + \big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big|^\sigma \Big\}, \tag{2.19}$$

for $0 \le t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{T}_N^n$, where $C = C(n, c_\pm, T)$.

Proof. Consider the discrete heat equation on $\frac{1}{N}\mathbb{T}_N^n$:

$$\partial_s v = \Delta^N v, \quad s \in (0,1], \tag{2.20}$$

with initial condition $v(0, \frac{x}{N}) = u^N(0, \frac{x}{N})$ for $x \in \mathbb{T}_N^n$. Define $\hat{v}(t) := v(1-t)$ for $0 \le t < 1$ and $\hat{h}(t, \frac{x}{N}) := -\Delta^N \hat{v}(t, \frac{x}{N})$. Note, for $0 \le t < 1$, that $\hat{v}$ satisfies

$$\partial_t \hat{v} = -\Delta^N \hat{v} = \Delta^N \hat{v} + 2\hat{h}.$$

However, by (2.20), $h(t) := \hat{h}(1-t) = -\Delta^N v(t)$ satisfies the discrete heat equation $\partial_t h(t) = \Delta^N h(t)$ with initial value $h(0) = -\Delta^N u^N(0)$ and thus, by the maximum principle for the discrete heat equation (see Lemma 2.7 below) and by the condition (2.18) for $u^N(0)$, we have

$$|h(t, \tfrac{x}{N})| \le \max_y |\Delta^N u^N(0, \tfrac{y}{N})| \le C_0,$$

which implies $\|\hat{h}\|_\infty \le C_0$.

Define now

$$\hat{a}_{x,e}(t) = \begin{cases} a_{x,e}(t-1) & \text{for } t \ge 1 \\ 1 & \text{for } 0 \le t < 1, \end{cases} \tag{2.21}$$

and

$$\hat{g}(t, \tfrac{x}{N}) = \begin{cases} g(t-1, \frac{x}{N}) & \text{for } t \ge 1 \\ 2\hat{h}(t, \frac{x}{N}) & \text{for } 0 \le t < 1. \end{cases} \tag{2.22}$$

Consider the extended system, for $t \ge 0$,

$$\partial_t \hat{u}^N = L^N_{\hat{a}(t)} \hat{u}^N + \hat{g}(t, \tfrac{x}{N}). \tag{2.23}$$

Since $\hat{a}$ satisfies symmetry and nondegeneracy/boundedness, conditions (1.3) and (1.4) (modify $c_\pm$ if necessary), and $\|\hat{g}\|_\infty \le \|g\|_\infty + 2C_0$, Theorem 2.2 yields the following statement:

$$|\hat{u}(t_1, \tfrac{x_1}{N}) - \hat{u}(t_2, \tfrac{x_2}{N})| \le C(\|g\|_\infty + \|u^N(0)\|_\infty + C_0) \frac{\left|t_2 - t_1\right|^{\frac{\sigma}{2}} + \left|\frac{x_1}{N} - \frac{x_2}{N}\right|^\sigma}{(t_1 \wedge t_2)^{\frac{\sigma}{2}}},$$

for $0 < t_1, t_2 \le T+1$ and $x_1, x_2 \in \mathbb{T}_N^n$, where $C = C(n, c_\pm) > 0$ depends on n, $c_\pm$ given in (1.4), and T. We obtain, by specializing to times $1 \le t \le T+1$ and noting that $\hat{u}(t, \cdot) = u^N(t-1, \cdot)$, the theorem for the solution $u = u^N$ of (1.6). $\square$

Theorem 2.5 immediately implies the following corollary for the equation (1.1). The proof is similar to that of Corollary 2.3.

Corollary 2.6. *Let $u^N(t, \frac{x}{N})$ be the solution of* (1.1) *satisfying* (1.2). *We assume that the initial value $u^N(0, \cdot)$ satisfies* (2.18). *Then, we have with $C = C(n, c_\pm, T)$ that*

$$|u^N(t_1, \tfrac{x_1}{N}) - u^N(t_2, \tfrac{x_2}{N})| \le C(K\|f\|_\infty + \|u^N(0)\|_\infty + C_0)\Big\{\left|t_2 - t_1\right|^{\frac{\sigma}{2}} + \left|\tfrac{x_1}{N} - \tfrac{x_2}{N}\right|^\sigma\Big\}, \tag{2.24}$$

for $0 \le t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{T}_N^n$. Moreover, $a_{x,e}(t) := a_{x,e}(u^N(t))$ satisfies

$$|a_{x_1,e}(t_1) - a_{x_2,e}(t_2)| \le C(K\|f\|_\infty + \|u^N(0)\|_\infty + C_0)\Big\{\left|t_2 - t_1\right|^{\frac{\sigma}{2}} + \left|\tfrac{x_1}{N} - \tfrac{x_2}{N}\right|^\sigma\Big\}, \tag{2.25}$$

for $0 \le t_1, t_2 \le T$ and $x_1, x_2 \in \mathbb{T}_N^n$ where $C = C(n, c_\pm, T, \|\varphi''\|_\infty)$.

2.5 Comparison argument and maximum principle

Here, we show (1.2) for the quasilinear discrete PDE (1.1) and formulate the maximum principle for linear discrete PDEs.

To show (1.2), take $-\infty < u_- < u_+ < \infty$ such that $f(u_-) > 0, f(u_+) < 0$ and $u_- < u^N(0, \frac{x}{N}) < u_+$ for all N, x, K. This is possible by our assumptions for $f(u)$ and $u^N(0)$; in particular, recall that $u \cdot f(u) < 0$ for large enough $|u|$. We only show the upper bound in (1.2), since the lower bound is similar.

For a fixed $K > 0$, take $\varepsilon > 0$ small enough such that $Kf(u_+) + \varepsilon < 0$ and let $u_+(t)$ be the solution of the ODE $\frac{du_+}{dt}(t) = Kf(u_+(t)) + \varepsilon$ with the initial value $u_+(0) = u_+$. We easily see that $u_+(t) \le u_+$ holds for all $t \ge 0$.

For the solution $u^N(t)$ of (1.1), set $\tau := \inf\{t > 0; u^N(t, \frac{x}{N}) = u_+(t)$ for some $x \in \mathbb{T}_N^n\}$ and $\tau = \infty$ if the set $\{t > 0; \cdots\} = \emptyset$. If $\tau = \infty$, by definition, we have $u^N(t, \frac{x}{N}) < u_+(t) \le u_+$ for all $t \ge 0$ and $x \in \mathbb{T}_N^n$ so that the upper bound in (1.2) holds. Therefore, we may assume $\tau < \infty$ and clearly $\tau > 0$. Then, at $t = \tau$, $u^N(\tau, \frac{x}{N}) \le u_+(\tau)$ for all $x \in \mathbb{T}_N^n$ and $u^N(\tau, \frac{y}{N}) = u_+(\tau)$ for some $y \in \mathbb{T}_N^n$. However, by the equation (1.1), we have

$$\begin{aligned}\partial_t(u_+ - u^N)(\tau, \tfrac{y}{N}) &= \Delta^N\{\varphi(u_+) - \varphi(u^N)\}(\tau, \tfrac{y}{N}) + Kf(u_+(\tau)) + \varepsilon - Kf(u^N(\tau, \tfrac{y}{N})) \\ &= N^2 \sum_{|e|=1} \Big\{(\varphi(u_+) - \varphi(u^N))(\tau, \tfrac{y+e}{N}) - (\varphi(u_+) - \varphi(u^N))(\tau, \tfrac{y}{N})\Big\} + \varepsilon \\ &= N^2 \sum_{|e|=1} (\varphi(u_+) - \varphi(u^N))(\tau, \tfrac{y+e}{N}) + \varepsilon \ge \varepsilon,\end{aligned}$$

where we set $u_+(t, \frac{x}{N}) = u_+(t)$ for all x. Note $\Delta^N\varphi(u_+) = 0$ and that we have used the increasing property of φ. This, however, implies that $u^N(t, \frac{y}{N}) > u_+(t)$ for some $t \in (0, \tau)$ close to τ, which contradicts the definition of τ. Thus, the upper bound in (1.2) is shown.

Now we state the maximum principle. Let $a_{x,e}(t) \ge 0$ be given and consider the linear discrete PDEs of divergence form

$$\partial_t u = L^N_{a(t)} u, \quad \tfrac{x}{N} \in \tfrac{1}{N}\mathbb{T}_N^n, \tag{2.26}$$

and also non-divergence form

$$\partial_t u = \bar{L}^N_{a(t)} u := \sum_{|e|=1} a_{x,e}(t) \nabla_e^{N,*} \nabla_e^N u, \quad \tfrac{x}{N} \in \tfrac{1}{N}\mathbb{T}_N^n. \tag{2.27}$$

Note that the symmetry condition (1.3) is not assumed for $a_{x,e}(t)$. For completeness, we give a proof.

Lemma 2.7. *(1) Let $\Lambda \subset \frac{1}{N}\mathbb{T}_N^n$ be given. Assume that $u(t, \frac{x}{N})$ is defined on $[0, T] \times \bar{\Lambda}$ and satisfy $(L^N_{a(t)} - \partial_t)u \ge 0$ or $(\bar{L}^N_{a(t)} - \partial_t)u \ge 0$ on $(0, T] \times \Lambda$. Then, we have*

$$\max_{[0,T]\times\bar{\Lambda}} u = \max_{\Gamma_T} u, \tag{2.28}$$

where $\Gamma_T = \{0\} \times \bar{\Lambda} \cup (0, T] \times \partial_N^+\Lambda$, and see (3.52) for $\partial_N^+\Lambda$ and $\bar{\Lambda}$.
(2) For the solutions of the equations (2.26) or (2.27), we have $\max_{t\ge0, x\in\mathbb{T}_N^n} u(t, \frac{x}{N}) = \max_{x\in\mathbb{T}_N^n} u(0, \frac{x}{N})$ and $\min_{t\ge0, x\in\mathbb{T}_N^n} u(t, \frac{x}{N}) = \min_{x\in\mathbb{T}_N^n} u(0, \frac{x}{N})$.

Proof. (1) First note that, if the function u takes local maximum at $\frac{y}{N}$ in the sense that $u(\frac{y}{N}) \geq u(\frac{y+e}{N})$ for every $e : |e| = 1$, then we have $L_a^N u(\frac{y}{N}) \leq 0$ and $\bar{L}_a^N u(\frac{y}{N}) \leq 0$. Indeed, the first inequality is shown as

$$\begin{aligned} L_a^N u(\tfrac{y}{N}) &= -\tfrac{1}{2} \sum_{|e|=1} \nabla_e^{N,*}(a_{y,e}\nabla_e^N u)(\tfrac{y}{N}) \\ &= -\tfrac{N}{2} \sum_{|e|=1} \left\{ a_{y-e,e}\nabla_e^N u(\tfrac{y-e}{N}) - a_{y,e}\nabla_e^N u(\tfrac{y}{N}) \right\} \leq 0, \end{aligned}$$

since $a_{y-e,e}, a_{y,e} \geq 0$ and $\nabla_e^N u(\frac{y-e}{N}) \geq 0$, $\nabla_e^N u(\frac{y}{N}) \leq 0$. The second is similar.

Now, set $u^\varepsilon = u - \varepsilon t$ for $\varepsilon > 0$ and prove (2.28) for u^ε instead of u. Once this is done, (2.28) is shown for u by letting $\varepsilon \downarrow 0$. If (2.28) does not hold for u^ε, there exist $\frac{y}{N} \in \Lambda$ and $\tau \in (0, T]$ such that $u^\varepsilon(\tau, \frac{y}{N}) = \max_{[0,T]\times\bar{\Lambda}} u^\varepsilon(t, \frac{x}{N})$. Then, by the above observation, $L_{a(\tau)}^N u^\varepsilon(\tau, \frac{y}{N}) \leq 0$ and $\bar{L}_{a(\tau)}^N u(\tau, \frac{y}{N}) \leq 0$ hold. Thus, if the first condition is satisfied by u,

$$\begin{aligned} \partial_t u^\varepsilon(\tau, \tfrac{y}{N}) &= \partial_t u(\tau, \tfrac{y}{N}) - \varepsilon \leq L_{a(\tau)}^N u(\tau, \tfrac{y}{N}) - \varepsilon \\ &= L_{a(\tau)}^N u^\varepsilon(\tau, \tfrac{y}{N}) - \varepsilon \leq -\varepsilon. \end{aligned} \tag{2.29}$$

The same bound holds if the second condition is satisfied.

However, by the definition of τ and y, we have $\partial_t u^\varepsilon(\tau, \frac{y}{N}) = 0$ in case of $0 < \tau < T$ or $\partial_t u^\varepsilon(\tau, \frac{y}{N}) \geq 0$ in case of $\tau = T$. Both contradict (2.29) so that (2.28) is shown for u^ε.

The assertion (2) follows from (1) by taking $\Lambda = \frac{1}{N}\mathbb{T}_N^n$. The statement for the minimum follows by considering $-u$. □

2.6 Comments on the probabilistic method to show the Hölder estimate

We consider the equation (1.6) in the case $g \equiv 0$. At least in a temporally homogeneous case and for the generator of the form

$$Lu(x) = \mu_x^{-1} \sum_{y:|x-y|=1} \mu_{xy}(u(y) - u(x)), \quad \mu_x = \sum_y \mu_{xy}, \tag{2.30}$$

the parabolic Harnack inequality is shown ([5], [11]) and, based on it, Hölder continuity, which is uniform in N, of $u^N(t, \frac{x}{N})$ in $(t, \frac{x}{N})$ and the corresponding fundamental solution is shown; see Proposition 3.2 and below (4.12) of [5] and also p.100 below Theorem 8.1.5 of [37]. Note that they consider the analysis on a percolation cluster $\mathcal{C}_\infty$ or disordered media, which is quite complicated, but if we take the percolation probability $p = 1$, we have $\mathcal{C}_\infty = \mathbb{Z}^d$.

The differences from our situation are the following:
(1) Our generator $L_{a(t)}^N$ is temporally inhomogeneous
(2) L in (2.30) is defined by dividing by μ_x (cf. CSRW and VSRW, for example in [37], for the relations of two generators, one not divided by μ_x).

As we see in Section 6, if we have the Hölder continuity in $(t, \frac{x}{N})$ of the fundamental solution $p^N(s, \frac{y}{N}; t, \frac{x}{N})$ of $L_{a(t)}^N - \partial_t$, by the symmetry of $L_{a(t)}^N$, it is Hölder continuous also in $(s, \frac{y}{N})$ (with diverging front factor for $|t-s|$ small as in (4.14) of [5]). Then, for the solution of (1.6) with g, one can apply Duhamel's formula.

A Hölder estimate of the fundamental solution is available in the quenched sense. However, in general, a gradient estimate of the fundamental solution is not available–as discussed in [12] an annealed gradient estimate holds but not in the quenched sense. Note that the parabolic Caccioppoli inequality, formulated in Proposition 4.1 of [12], is applied to obtain the gradient estimate. See also [3], [27] in a continuous setting.

3 Preliminary estimates for the discrete heat equation, polylinear interpolation and some norms

Once the Hölder estimate is established, one can move to the next stage to prove the Schauder estimate. We adapt the approach in Chapter IV of [41] originally given for continuous PDEs. This section summarizes some preparatory facts.

In Section 3.1, we give bounds on the space-time oscillations of the solution of the discrete heat equation (1.20); see Lemmas 3.1 and 3.2. Then, in Section 3.2, we extend the approach to continuous space by polylinear interpolations, and rewrite these bounds in the continuous setting; see Corollary 3.3 and Proposition 3.4.

With Proposition 3.4 in hand, we can consider versions of the arguments in [41]. In Section 3.3, we introduce space-time Hölder norms of u and $\nabla^N u$ with weights which control of diverging factors near $t=0$. Then, in Section 3.4, we state Lemmas 3.6 and 3.7, which correspond to Lemmas 4.3 and 4.6 in [41], respectively, with a proper modification for Lemma 3.6 in the discrete setting. Finally, in Section 3.5, we state the summation by parts formula, which will be used in the next section.

We comment that the interior estimate for the discrete heat equation obtained in Proposition 3.4 is useful, especially, to derive (4.42) for the gradient of the solution later. Note that the discrete heat equation (1.20) has a simple but special feature, namely that if v^N is a solution then its discrete derivative $\nabla_e^N v^N$ is also its solution for every e, since the two operators ∇_e^N and Δ_a^N commute with each other.

3.1 Interior estimates for the discrete heat equation

Let $T>0$ be a fixed time horizon. Define $\Omega = [0,T]\times \mathbb{T}^n$ and $\Omega_N = [0,T]\times \frac{1}{N}\mathbb{T}_N^n$. For $X=(t,z)\in\Omega$, set $|X| = \max\{t^{\frac{1}{2}}, |z|\}$, where $|z|$ is usually the L^2 (Euclidean) norm on $\mathbb{T}^n \cong [-\frac{1}{2},\frac{1}{2})^n$. In context, to fit in the lattice structure, we will sometimes use the L^∞ norm, which is denoted by $|z|_{L^\infty} := \max_{1\le i\le n}|z_i|$ for $z=(z_i)_{i=1}^n$, instead of L^2 norm.

Define the domain $Q(R)=Q(X_0,R)$ in Ω for $X_0=(t_0,z_0)\in\Omega$ and $0<R\,(<\frac{1}{2}\wedge\sqrt{t_0})$ as

$$
\begin{aligned}
Q(R) :=& \{X\in\Omega; |X-X_0|<R, t<t_0\} \\
\equiv& (t_0-R^2,t_0)\times\{z\in\mathbb{T}^n; |z-z_0|<R\}.
\end{aligned}
\tag{3.1}
$$

We may regard the spatial part of $Q(R)$ as $\mathbb{T}^n$ when $R\ge\frac{1}{2}$ and the temporal part as $[0,T]$ when $R\ge\sqrt{t_0}$. We also define $Q_N(R)=Q(R)\cap\Omega_N$.

We now show a discrete analog of Lemma 4.4 in [41] for the solution of the discrete heat equation (1.20); see Lemma 3.1 and Corollary 3.3. If a linear function is defined on a region of width R and takes values in $[-M, M]$, its slope behaves as $\frac{2M}{R}$. The next lemma roughly claims that the solution v^N of (1.20) has a similar property.

Lemma 3.1. *(cf. Proof of Lemma 4.4 of [41]) Let $v^N = v^N(t, \frac{x}{N})$ be a solution of the discrete heat equation (1.20) on $Q_N(R)$ with coefficients a such that (2.7) holds, that is,*

$$\partial_t v^N = \Delta_a^N v^N, \quad (t, \tfrac{x}{N}) \in Q_N(R), \tag{3.2}$$

satisfying $|v^N| \le M_R$ on $Q_N(R)$ with $R > \frac{c_0}{N}$ for a constant $c_0 > \sqrt{n}+1$. (The condition $R > \frac{c_0}{N}$ will be removed in the setting of Corollary 3.3). Then, we have

$$|\nabla_e^N v^N(t, \tfrac{x}{N})| \le \frac{CM_R}{R} \quad \text{in } \ Q_N(r),$$

for $r \in (0, \frac{1}{2}c_1 R)$ and $e \in \mathbb{Z}^n : |e| = 1$, where $c_1 = \frac{1}{\sqrt{n}}\big(1 - \frac{\sqrt{n}+1}{c_0}\big) \in (0,1)$ and $C = C(n, c_\pm)$. Here, we take the spatial center z_0 of $Q(R)$ (and therefore $Q(r)$) in $\frac{1}{N}\mathbb{T}_N^n$.

Note that, for $\Delta_a^N v^N$ in (3.2) to be defined on $Q_N(R)$, v^N should be given at least on its closure $\overline{Q_N(R)}$ defined as in (4.29) below.

Proof. The main method is to apply the maximum principle for the equation (3.2). Note that $\nabla_e^N v^N$ satisfies the same equation, on a smaller domain, due to the commuting property $[\Delta_a^N, \nabla_e^N] = 0$.

Step 1. Let $\mathcal{L} = \Delta_a^N - \partial_t$ and let $v = v^N(t, \frac{x}{N})$ be the solution of $\mathcal{L}v = 0$ on $Q_N(R)$. Then, by a simple computation and writing $v(\frac{x}{N})$ for $v(t, \frac{x}{N})$ for simplicity, we have

$$\mathcal{L}(\nabla_e^N v(\tfrac{x}{N}))^2 = \sum_{|e'|=1} a_{e'} (\nabla_{e'}^N \nabla_e^N v(\tfrac{x}{N}))^2, \tag{3.3}$$

$$\mathcal{L} v^2(\tfrac{x}{N}) = \sum_{|e|=1} a_e (\nabla_e^N v(\tfrac{x}{N}))^2, \tag{3.4}$$

on $Q_N(R - \frac{1}{N})$. Indeed, by (2.8), (2.10) and $\mathcal{L}\nabla_e^N v = 0$ on $Q_N(R - \frac{1}{N})$, which follows by noting that $(t, \frac{x+e}{N}) \in Q_N(R)$ for $(t, \frac{x}{N}) \in Q_N(R - \frac{1}{N})$,

$$\begin{aligned}
\mathcal{L}(\nabla_e^N v(\tfrac{x}{N}))^2 &= \Delta_a^N (\nabla_e^N v(\tfrac{x}{N}))^2 - \partial_t (\nabla_e^N v(\tfrac{x}{N}))^2 \\
&= N \sum_{|e'|=1} a_{e'} \nabla_{e'}^N (\nabla_e^N v(\tfrac{x}{N}))^2 - 2\nabla_e^N v(\tfrac{x}{N})\, \partial_t \nabla_e^N v(\tfrac{x}{N}) \\
&= N \sum_{|e'|=1} a_{e'} (\nabla_e^N v(\tfrac{x}{N}) + \nabla_e^N v(\tfrac{x+e'}{N})) \nabla_{e'}^N \nabla_e^N v(\tfrac{x}{N}) \\
&\qquad - 2\nabla_e^N v(\tfrac{x}{N}) N \sum_{|e'|=1} a_{e'} \nabla_{e'}^N \nabla_e^N v(\tfrac{x}{N}) \\
&= N \sum_{|e'|=1} a_{e'} (\nabla_e^N v(\tfrac{x+e'}{N}) - \nabla_e^N v(\tfrac{x}{N})) \nabla_{e'}^N \nabla_e^N v(\tfrac{x}{N})
\end{aligned}$$

$$= \sum_{|e'|=1} a_{e'}(\nabla_{e'}^N \nabla_e^N v(\tfrac{x}{N}))^2.$$

This shows (3.3). Equation (3.4) follows, again by (2.8) and (2.10), from

$$\begin{aligned}\mathcal{L}v^2(\tfrac{x}{N}) &= \Delta_a^N v^2(\tfrac{x}{N}) - \partial_t v^2(\tfrac{x}{N}) \\ &= N\sum_{|e|=1} a_e \nabla_e^N v^2(\tfrac{x}{N}) - 2v(\tfrac{x}{N})N\sum_{|e|=1} a_e\nabla_e^N v(\tfrac{x}{N}) \\ &= N\sum_{|e|=1} a_e(v(\tfrac{x+e}{N}) - v(\tfrac{x}{N}))\nabla_e^N v(\tfrac{x}{N}) = \sum_{|e|=1} a_e(\nabla_e^N v(\tfrac{x}{N}))^2.\end{aligned}$$

Step 2. To accommodate the discrete setting, especially to have a proper discrete boundary in Step 5, instead of $Q(R)$, we consider a domain $\tilde{Q}(R)$, defined similarly to $Q(R) = Q(X_0, R)$ but where we use the L^∞ norm $|z|_{L^\infty}$ instead of $|z|$, to define $|X - X_0|$. Note that $Q_N(R) \subset \tilde{Q}_N(R) \subset \overline{\tilde{Q}_N(R)} \subset Q_N(\sqrt{n}(R + \frac{1}{N}))$ holds, where $\tilde{Q}_N(R) = \tilde{Q}(R) \cap \Omega_N$ and $\overline{\tilde{Q}_N(R)} = \tilde{Q}_N(R) \cup \partial_N^+ \tilde{Q}_N(R)$ is the domain added the discrete outer boundary; see below. We will take a smaller $c_1 R$ with $c_1 = \frac{1}{\sqrt{n}}\big(1 - \frac{\sqrt{n}+1}{c_0}\big)$ instead of R such that $Q_N(\sqrt{n}(c_1R + \frac{1}{N})) \subset Q_N(R - \frac{1}{N})$ holds. In particular, $Q_N(c_1R) \subset \overline{\tilde{Q}_N(c_1R)} \subset Q_N(R - \frac{1}{N})$ holds.

Define a cut off function $\zeta(X) = ((c_1R)^2 - |z - z_0|^2_{L^\infty})^+((c_1R)^2 - |t_0 - t|)^+$ for $X = (t, z) \in \overline{\tilde{Q}_N(c_1R)}$ and $X_0 = (t_0, z_0)$. Actually for ζ, we only use the properties: (3.6) below, $\zeta \le C(n)R^4$ and $|\partial_t \zeta| \le C(n)R^2$. Note that $\zeta = 0$ at the discrete outer boundary $\partial_N^+ \tilde{Q}(c_1R) = \partial\big([0, t_0] \times \frac{1}{N}\mathbb{T}_N^n \cap \tilde{Q}(c_1R)^c\big)$.

By (3.3), (2.10) and (2.11), for each $e \in \mathbb{Z}^n : |e| = 1$, we have on $\tilde{Q}_N(c_1R)$

$$\begin{aligned}\text{(3.5)} \quad \mathcal{L}(\zeta\nabla_e^N v)^2(\tfrac{x}{N}) &= \zeta^2(\tfrac{x}{N})\mathcal{L}(\nabla_e^N v(\tfrac{x}{N}))^2 + (\nabla_e^N v(\tfrac{x}{N}))^2\mathcal{L}\zeta^2(\tfrac{x}{N}) \\ &\quad + \sum_{|e'|=1} a_{e'}\nabla_{e'}^N\zeta^2(\tfrac{x}{N}) \cdot \nabla_{e'}^N(\nabla_e^N v)^2(\tfrac{x}{N}) \\ &= \zeta^2(\tfrac{x}{N})\sum_{|e'|=1} a_{e'}(\nabla_{e'}^N\nabla_e^N v(\tfrac{x}{N}))^2 + (\nabla_e^N v(\tfrac{x}{N}))^2\mathcal{L}\zeta^2(\tfrac{x}{N}) \\ &\quad + \sum_{|e'|=1} a_{e'}\nabla_{e'}^N\zeta^2(\tfrac{x}{N}) \cdot (\nabla_e^N v(\tfrac{x}{N}) + \nabla_e^N v(\tfrac{x+e'}{N}))\nabla_{e'}^N\nabla_e^N v(\tfrac{x}{N}) \\ &=: I_1 + I_2 + I_3,\end{aligned}$$

where $I_1 \ge 0$. This is a discrete analog of the formula in line -9 in p.34 (proof of Lemma 3.18) of [41] ($H = \mathcal{L}, W = \zeta$).

Step 3. The continuous ζ (defined with L^2-norm for $|z - z_0|$) satisfies $\zeta \le CR^4$, $|D\zeta| \le CR^3$ and $|D^2\zeta| \le CR^2$; see [41], p.52. We have a similar property here for ζ defined above with the L^∞-norm. Since the space singularity is not a problem given the discrete setting, noting $R > \frac{c_0}{N}$, comparable to the mesh size $\frac{1}{N}$, we have the same estimates for the discrete derivatives of ζ by the mean value theorem:

$$\text{(3.6)} \qquad |\nabla_e^N\zeta| \le C(n)R^3, \quad |\nabla_{e'}^N\nabla_e^N\zeta| \le C(n)R^2.$$

Therefore, noting $|\partial_t\zeta| \le CR^2$ and (2.7), we have

$$|\mathcal{L}\zeta^2(\tfrac{x}{N})| = \Big|2\zeta(\tfrac{x}{N})\Delta_a^N\zeta(\tfrac{x}{N}) + \sum_{|e|=1} a_e\big(\nabla_e^N\zeta(\tfrac{x}{N})\big)^2 - 2\zeta(\tfrac{x}{N})\partial_t\zeta(\tfrac{x}{N})\Big| \le C(n,c_\pm)R^6. \tag{3.7}$$

In particular, we have

$$I_2 \ge -\,C(n,c_\pm)R^6(\nabla_e^N v(\tfrac{x}{N}))^2.$$

Step 4. For I_3 in the right hand side of (3.5), we first note from (2.10) that

$$\nabla_{e'}^N\zeta^2(\tfrac{x}{N}) = (\zeta(\tfrac{x}{N}) + \zeta(\tfrac{x+e'}{N}))\nabla_{e'}^N\zeta(\tfrac{x}{N}) = (2\zeta(\tfrac{x}{N}) + \tfrac{1}{N}\nabla_{e'}^N\zeta(\tfrac{x}{N}))\nabla_{e'}^N\zeta(\tfrac{x}{N}).$$

Then, since $\frac{1}{N}\nabla_{e'}^N F(\frac{x}{N}) = F(\frac{x+e'}{N}) - F(\frac{x}{N})$, and using this for $F = \nabla_e^N u$, we can rewrite I_3 as

$$\begin{aligned}
I_3 = &\sum_{|e'|=1} 2a_{e'}\zeta(\tfrac{x}{N})\nabla_{e'}^N\zeta(\tfrac{x}{N})\cdot(\nabla_e^N v(\tfrac{x}{N}) + \nabla_e^N v(\tfrac{x+e'}{N}))\nabla_{e'}^N\nabla_e^N v(\tfrac{x}{N})\\
&+ \sum_{|e'|=1} a_{e'}(\nabla_{e'}^N\zeta(\tfrac{x}{N}))^2\cdot(\nabla_e^N v(\tfrac{x}{N}) + \nabla_e^N v(\tfrac{x+e'}{N}))(\nabla_e^N v(\tfrac{x+e'}{N}) - \nabla_e^N v(\tfrac{x}{N}))\\
=:&I_{3,1} + I_{3,2}.
\end{aligned}$$

Here, by a simple bound $2bc \ge -(b^2+c^2)$ and then by (3.6), $I_{3,1}$ is estimated from below as

$$\begin{aligned}
I_{3,1} \ge &-\zeta^2(\tfrac{x}{N})\sum_{|e'|=1} a_{e'}(\nabla_{e'}^N\nabla_e^N v(\tfrac{x}{N}))^2\\
&-\sum_{|e'|=1} a_{e'}(\nabla_{e'}^N\zeta(\tfrac{x}{N}))^2(\nabla_e^N v(\tfrac{x}{N}) + \nabla_e^N v(\tfrac{x+e'}{N}))^2\\
\ge &-I_1 - C(n,c_\pm)R^6\sum_{|e'|=1,0}(\nabla_e^N v(\tfrac{x+e'}{N}))^2.
\end{aligned}$$

The other term $I_{3,2}$ is also estimated from below by using (3.6) as

$$I_{3,2} \ge -\sum_{|e'|=1} a_{e'}(\nabla_{e'}^N\zeta(\tfrac{x}{N}))^2(\nabla_e^N v(\tfrac{x}{N}))^2 \ge -C(n,c_\pm)R^6(\nabla_e^N v(\tfrac{x}{N}))^2.$$

Step 5. Summarizing these estimates, we obtain

$$\mathcal{L}(\zeta\nabla_e^N v)^2(\tfrac{x}{N}) \ge -C(n,c_\pm)R^6\sum_{|e'|=1,0}(\nabla_e^N v(\tfrac{x+e'}{N}))^2,$$

on $\tilde{Q}_N(c_1R)$. This combined with (3.4) shows

$$\mathcal{L}\Big((\zeta\nabla_e^N v)^2(\tfrac{x}{N}) + \tfrac{C(n,c_\pm)}{c_-}R^6\sum_{|e'|=1,0} v^2(\tfrac{x+e'}{N})\Big) \ge 0, \tag{3.8}$$

on $\tilde{Q}_N(c_1R)$, where c_- is as in (2.7).

However, since $\zeta = 0$ at the discrete outer boundary $\partial_N^+ \tilde{Q}(c_1R)$, this function, that is the function acted upon by $\mathcal{L}$ in the formula (3.8), at $\partial_N^+ \tilde{Q}(c_1R)$ is bounded from above by $\frac{C}{c_-}(2n+1)R^6 M_R^2$ by recalling that $\overline{\tilde{Q}_N(c_1R)} \subset Q_N(R - \frac{1}{N})$. Thus, the maximum principle (Lemma 2.7) for (3.2) shows

$$\left((\zeta \nabla_e^N v)(\tfrac{x}{N})\right)^2 \leq \tfrac{C(n,c_\pm)}{c_-}(2n+1)R^6 M_R^2,$$

for each e on $\tilde{Q}_N(c_1R)$. Since $\zeta \geq C(c_1)R^4$ on $Q_N(r)$ from our initial assumption $r < \frac{1}{2}c_1R$, we conclude the proof. □

The next lemma shows that the interior modulus of continuity estimate in $\frac{x}{N}$ implies that in t.

Lemma 3.2. *(cf. Theorem 2.13 and (2.27) in [41]) Let $v^N(t, \frac{x}{N})$ be the solution of the discrete heat equation* (3.2) *on $Q = (t_1, t_1 + \frac{R^2}{16a_*}) \times \{|\frac{x}{N} - z_0| < R\}\,(\subset \Omega_N)$, $z_0 \in \frac{1}{N}\mathbb{T}_N^n$, where $a_* := \sum_{|e|=1, e>0} a_e$. Assume that $R > \frac{c_0}{N}$ with some $c_0 > 2$ and*

$$|v^N(t, \tfrac{x}{N}) - v^N(t, z_0)| \leq \omega \tag{3.9}$$

holds for $|\frac{x}{N} - z_0| < R$ and $t \in [t_1, t_1 + \frac{R^2}{16a_*}]$. Then, we have

$$|v^N(t, z_0) - v^N(t_1, z_0)| \leq 2\omega,$$

for $t \in [t_1, t_1 + \frac{R^2}{16a_*}]$.

Proof. The proof is essentially the same as that of Theorem 2.13 of [41] (in our case, $b = c = f = 0$ and $\Lambda_0 = a_*$). Set

$$s = \sup_{t \in (t_1, t_1 + \frac{R^2}{16a_*})} |v^N(t, z_0) - v^N(t_1, z_0)| \tag{3.10}$$

and define

$$\nu^\pm(t, \tfrac{x}{N}) = \tfrac{8sa_*}{R^2}(t - t_1) + \tfrac{4s}{R^2}|\tfrac{x}{N} - z_0|^2 + \omega \pm (v^N(t, \tfrac{x}{N}) - v^N(t_1, z_0)),$$

with L^2-norm for $|\frac{x}{N} - z_0|$. Then, recalling $\mathcal{L} = \Delta_a^N - \partial_t$ and noting $\Delta_a^N|\frac{x}{N} - z_0|^2 = 2a_*$ and $\mathcal{L}v^N = 0$, we have

$$\mathcal{L}\nu^\pm(t, \tfrac{x}{N}) = -\tfrac{8sa_*}{R^2} + \tfrac{8sa_*}{R^2} = 0$$

in Q. We remark that $\mathcal{L}\nu^\pm \leq 0$ would be enough to apply the maximum principle below.

Let $\mathcal{P}_N Q$ be the discrete parabolic inner boundary of Q defined by

$$\mathcal{P}_N Q := \{t_1\} \times \{|\tfrac{x}{N} - z_0| < R\} \bigcup (t_1, t_1 + \tfrac{R^2}{16a_*}) \times \partial_N^- \{|\tfrac{x}{N} - z_0| < R\},$$

where $\partial_N^- E$ is the inner boundary of the set $E \subset \frac{1}{N}\mathbb{T}_N^n$, that is,

$$\partial_N^- E := \{\tfrac{x}{N} \in E; |\tfrac{x}{N} - \tfrac{y}{N}| = \tfrac{1}{N} \text{ for some } \tfrac{y}{N} \notin E\}.$$

On $\mathcal{P}_N Q$, at $t = t_1$, we have

$$\nu^\pm(t_1, \tfrac{x}{N}) \geq \omega \pm (v^N(t_1, \tfrac{x}{N}) - v^N(t_1, z_0)) \geq 0$$

by (3.9). For $\frac{x}{N} \in \partial_N^- \{|\frac{x}{N} - z_0| < R\}$, we have

$$|\tfrac{x}{N} - z_0| \geq |\tfrac{y}{N} - z_0| - |\tfrac{x}{N} - \tfrac{y}{N}| \geq R - \tfrac{1}{N} > \tfrac{1}{2}R,$$

since $R > \frac{2}{N}$, so that

$$\nu^\pm(t, \tfrac{x}{N}) \geq s + \omega \pm \big((v^N(t, \tfrac{x}{N}) - v^N(t, z_0)) + (v^N(t, z_0) - v^N(t_1, z_0)\big) \geq 0$$

by (3.9) and (3.10). This shows $\nu^\pm \geq 0$ on $\mathcal{P}Q$.

We now apply the maximum principle and see that $\nu^\pm \geq 0$ on Q. Thus, we have obtained

$$\tfrac{8sa_*}{R^2}(t - t_1) + \omega \geq |v^N(t, z_0) - v^N(t_1, z_0)|$$

at $\frac{x}{N} = z_0 \in \frac{1}{N}\mathbb{T}_N^n$. The conclusion follows by taking the supremum in t. □

3.2 Interior oscillation estimate for polylinear interpolations

We now reformulate the discrete space and time estimates, Lemmas 3.1 and 3.2 respectively, to the continuous setting via polylinear interpolation; see Corollary 3.3 below.

Let $u^N = \{u^N(\frac{x}{N}); x \in \mathbb{T}_N^n\}$ be given and define $\tilde{u}^N(z), z = (z_i)_{i=1}^n \in \mathbb{T}^n$ $(= [0,1)^n$ or $[-\frac{1}{2}, \frac{1}{2})^n)$ as a polylinear interpolation of u^N:

$$\tilde{u}^N(z) = \sum_{v \in \{0,1\}^n} \vartheta^N(v, z) u^N\big(\tfrac{[Nz]+v}{N}\big), \qquad \vartheta^N(v, z) = \prod_{i=1}^n \vartheta^N(v_i, z_i), \tag{3.11}$$

where $\vartheta^N(a, b) = \{Nb\}1_{\{a=1\}} + (1 - \{Nb\})1_{\{a=0\}} \in [0,1]$ for $a = 0, 1, b \in [0,1)$, $[Nz] = ([Nz_i])_{i=1}^n$, and $[Nz_i]$ and $\{Nb\}$ denote the integer and the fractional parts of Nz_i and Nb, respectively, and $v = (v_i)_{i=1}^n$. The polylinear interpolation $\tilde{u}^N(z)$ is determined as $\tilde{u}^N(\frac{x}{N}) = u^N(\frac{x}{N})$ for every $\frac{x}{N} \in \frac{1}{N}\mathbb{T}_N^n$ with nonnegative weights $\{\vartheta^N(v, z)\}_{v \in \{0,1\}^n}$ satisfying $\sum_{v \in \{0,1\}^n} \vartheta^N(v, z) = 1$ to fill inside each box with size $\frac{1}{N}$. It is an extension of the polygonal interpolation in one dimension. The polylinear interpolation is used to reduce the derivation of the Hölder estimate to a Campanato-type integral estimate in a continuous setting. However, in our discrete setting, this method is valid only outside of a short distance regime. This gap in the short distance regime is however resolved, indeed, by the form of the polylinear interpolation. This shows the uniform Hölder estimate in the whole region; see Proposition 4.10, its proof and Lemma 4.11.

In particular, we have

$$\partial_{z_i} \tilde{u}^N(z) = \sum_{v \in \{0,1\}^n} \vartheta_i^N(v, z) \nabla_{e_i}^N u^N\big(\tfrac{[Nz]+\hat{v}_i}{N}\big), \tag{3.12}$$

where $e_i \in \mathbb{Z}^n : |e_i| = 1, e_i > 0$ is the ith unit vector, $\vartheta_i^N(v, z) = \frac{1}{2}\prod_{j \neq i} \vartheta^N(v_j, z_j)$ and $\hat{v}_i = (v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_n)$; see [13]. Since $\sum_{v \in \{0,1\}^n} \vartheta_i^N(v, z) = 1$ for each i, (3.12) implies

$$|\partial_{z_i} \tilde{u}^N(z)| \leq \max_{\hat{v}_i : v \in \{0,1\}^n} \big|\nabla_{e_i}^N u^N\big(\tfrac{[Nz]+\hat{v}_i}{N}\big)\big|. \tag{3.13}$$

Lemma 3.1 roughly shows the spatial slope of the solution v^N of the discrete heat equation (3.2) is bounded by $\frac{CM_R}{R}$ if $R > \frac{c_0}{N}$ and this combined with Lemma 3.2 controls the oscillation in space and time. For a function F on Ω and $Q \subset \Omega$, we set

$$[F]_0 \equiv [F]_0^* := \operatorname*{osc}_{\Omega} F, \quad \operatorname*{osc}_{Q} F := \sup_{X,Y \in Q} |F(X) - F(Y)|. \tag{3.14}$$

Then the following corollary holds. This will be used later to show Proposition 3.4.

Before stating the corollary, let us examine the definability of the polylinear interpolation $\tilde{v}$ on a subdomain. In general for a domain $D \subset \mathbb{T}^n$, we define its continuous minimal cover by $\frac{1}{N}$-boxes (with all vertices in $\frac{1}{N}\mathbb{T}_N^n$) by

$$D^* := \bigcup_{x \in \mathbb{T}_N^n : B(\frac{x}{N}) \cap D \neq \emptyset} B(\tfrac{x}{N}) \ (\supset D),$$

where $B(\frac{x}{N}) = \prod_{i=1}^n [\frac{x_i}{N}, \frac{x_i+1}{N}]$. Set $D_N^* := D^* \cap \frac{1}{N}\mathbb{T}_N^n$. When $\{v(\frac{x}{N}); \frac{x}{N} \in D_N^*\}$ are given, its polylinear interpolation $\tilde{v}(z)$ is definable on D, since $v(\frac{x}{N})$ is defined at every vertex of the $\frac{1}{N}$-box containing $z \in D$. Note that D_N^* is slightly wider than $\overline{D_N} = D_N \cup \partial_N^+ D_N$ defined below (4.27) or (3.52), since D_N^* contains "diagonal" boundary points.

Denote by $D(r) = \{z \in \mathbb{T}^n; |z - z_0| < r\}$ the spatial part of $Q(r)$. If $r > 0$ satisfies $r + \frac{\sqrt{n}}{N} \le R$, then $(D(r))_N^* \subset D_N(R) := D(R) \cap \frac{1}{N}\mathbb{T}_N^n$ holds, accordingly, if v^N is defined on $Q_N(R)$, $\tilde{v}^N$ is definable on $Q(r)$. Indeed, if $\frac{x}{N} \in (D(r))_N^*$, then $\mathrm{dist}(\frac{x}{N}, D(r)) \le \frac{\sqrt{n}}{N}$ so that $|z_0 - \frac{x}{N}| \le r + \frac{\sqrt{n}}{N} \le R$ and thus $\frac{x}{N} \in D_N(R)$.

In other words, the interior estimates on $Q(r)$, given the behavior of v^N on $Q_N(R)$, can be discussed only under the condition $r + \frac{\sqrt{n}}{N} \le R$. If r and R are close, we need some outer condition on v^N (which, in application, is the solution of original PDE (1.6) and not that of simpler discrete heat equation (3.2)) to have the estimate on $Q(r)$ (though, as noted below Lemma 3.1, v^N is given on $\overline{Q_N(R)}$). We need a band area to separate $Q(r)$ and $Q(R)$. This comes from the non-local property of the polylinear interpolation and represents a difference from the continuous case.

We now state the corollary. Differently from the continuous setting, we need to consider three different ranges $0 < r \le R_1 < R$. Especially, R_1 and R should be distinguished with a gap of at least $\frac{\sqrt{n}}{N}$ due to the non-local nature of our problem.

Corollary 3.3. *(cf. Lemma 4.4 of [41]) Let $v^N(t, \frac{x}{N})$ be the solution of the discrete heat equation* (3.2) *on $Q_N(R) (\neq \emptyset)$. Assume $R_1 + \frac{\sqrt{n}}{N} \le R$ for $R_1 > 0$ (in particular, $R > \frac{\sqrt{n}}{N}$) so that the polylinear interpolation $\tilde{v}^N(t, z)$ of $v^N(t, \frac{x}{N})$ as in* (3.11) *in the spatial variable is well-defined on $Q(R_1)$.*

We assume $|\tilde{v}^N| \le M_{R_1}$ on $Q(R_1)$. Then, we have

$$\operatorname*{osc}_{Q(r)} \tilde{v}^N \le \frac{CrM_{R_1}}{R_1}, \tag{3.15}$$

for every $0 < r \le R_1 \le R - \frac{\sqrt{n}}{N}$. Moreover, we have

$$\sup_{Q(r)} \max_i |\partial_{z_i} \tilde{v}^N(t, z)| \le \frac{CM_{R_1}}{R_1}, \tag{3.16}$$

where $r \in (0, \varepsilon R_1/2)$ and $0 < \varepsilon < 1/2$ is fixed in the proof. Here, the constants $C = C(n, c_\pm) > 0$.

Note that the spatial center z_0 of $Q(r) = Q(X_0, r)$, $X_0 = (t_0, z_0)$, may not be on $\frac{1}{N}\mathbb{T}_N^n$. Note also that we don't require the gap $\frac{\sqrt{n}}{N}$ between r and R_1. The estimate (3.16) will not be used later, but it might be useful for the reader. It holds due to the polylinearity in our setting.

Proof. The condition $|\tilde{v}^N| \le M_{R_1}$ on $Q(R_1)$ implies $\operatorname*{osc}_{Q(r)} \tilde{v}^N \le 2M_{R_1}$, so that (3.15) is trivial (with $C = \frac{2}{\varepsilon}$) if $r \ge \varepsilon R_1$ for any $\varepsilon > 0$. Therefore, we assume $r < \varepsilon R_1$ in the following with a fixed $\varepsilon > 0$ small. See cases 1-4 below which specify different regions in terms of $c_0 > \sqrt{n} + 1$, so that Lemma 3.1 applies, and also $c_2 \ll c_0$. We now estimate the oscillation of $\tilde{v}^N$ on $Q(r)$ as

$$
\begin{aligned}
(3.17)\qquad \operatorname*{osc}_{Q(r)} \tilde{v}^N &= \sup_{X=(t,z), X'=(t',z') \in Q(r)} |\tilde{v}^N(X) - \tilde{v}^N(X')| \\
&\le \sup_{t,z} |\tilde{v}^N(t,z) - \tilde{v}^N(t,z_0)| + \sup_{t,t'} |\tilde{v}^N(t,z_0) - \tilde{v}^N(t',z_0)| \\
&\qquad + \sup_{t',z'} |\tilde{v}^N(t',z_0) - \tilde{v}^N(t',z')| \\
&= 2 \sup_{(t,z)\in Q(r)} |\tilde{v}^N(t,z) - \tilde{v}^N(t,z_0)| + \sup_{t,t' \in (t_0 - r^2, t_0)} |\tilde{v}^N(t,z_0) - \tilde{v}^N(t',z_0)| \\
&=: 2I_1 + I_2.
\end{aligned}
$$

We divide the analysis into four cases: (1) $R_1 \le \frac{c_0}{N}$, $r + \frac{\sqrt{n}+1}{N} \le R_1$, (2) $R_1 > \frac{c_0}{N}$, $0 < r \le \frac{c_2}{N}$ ($c_2 \ll c_0$), (3) $R_1 > \frac{c_0}{N}$, $\frac{c_2}{N} < r < \varepsilon R_1$, (4) $R_1 \le \frac{c_0}{N}$, $r + \frac{\sqrt{n}+1}{N} > R_1$.

Case 1. We prepare two rough estimates when R_1 satisfies $R_1 > \frac{\sqrt{n}+1}{N}$:

$$
(3.18)\qquad |\nabla_e^N v^N(t, \tfrac{x}{N})| \le 2NM_{R_1} \qquad \text{on} \quad Q_N(R_1 - \tfrac{1}{N}),
$$

$$
(3.19)\qquad |\partial_{z_i} \tilde{v}^N(t,z)| \le 2NM_{R_1} \qquad \text{on} \quad Q(R_1 - \tfrac{\sqrt{n}+1}{N}).
$$

Indeed, (3.18) is shown from $|\nabla_e^N v^N(t, \frac{x}{N})| = \big|N\big(v^N(t, \frac{x+e}{N}) - v^N(t, \frac{x}{N})\big)\big| \le 2NM_{R_1}$ by noting $\frac{x+e}{N} \in Q_N(R_1) \subset Q(R_1)$ for $\frac{x}{N} \in Q_N(R_1 - \frac{1}{N})$, and $v^N = \tilde{v}^N$ on $\frac{1}{N}\mathbb{T}_N^n$. Equation (3.19) follows from (3.18) and (3.13) by noting $\big(D(R_1 - \frac{1}{N} - \frac{\sqrt{n}}{N})\big)_N^* \subset D_N(R_1 - \frac{1}{N})$.

First consider the case that $R_1 \le \frac{c_0}{N}$ for some large enough $c_0 > \sqrt{n} + 1$ determined later and $r + \frac{\sqrt{n}+1}{N} \le R_1$ is satisfied, where c_0 is the same constant as in Lemma 3.1 but note that one can make c_0 larger in Lemma 3.1. In this case, since $Q(r) \subset Q(R_1 - \frac{\sqrt{n}+1}{N})$, (3.19) is applicable on $Q(r)$. Thus, for I_1, from (3.19) and noting $|z - z_0| < r$ and then $N \le \frac{c_0}{R_1}$, we have

$$
I_1 \le \sqrt{n} 2NM_{R_1} r \le 2\sqrt{n} c_0 M_{R_1} \tfrac{r}{R_1}.
$$

For I_2, by the discrete heat equation (3.2) and (3.18), noting $r + \frac{\sqrt{n}}{N} \le R_1 - \frac{1}{N}$, and recalling $a_* = \sum_{|e|=1, e>0} a_e \le nc_+$, we have

$$
(3.20)\qquad |\partial_t v^N(t, \tfrac{x}{N})| = \Big| N \sum_{|e|=1} a_e \nabla_e^N v^N(t, \tfrac{x}{N}) \Big| \le 2a_* \cdot 2M_{R_1} N^2 \qquad \text{on} \quad Q_N(r + \tfrac{\sqrt{n}}{N}).
$$

Further, note $\tilde{v}^N(t,z_0)$ is a convex combination of $\{v^N(t,\frac{x}{N})\}$ around z_0 as in (3.11). Observe also that $X, X' \in Q(r)$ implies $t, t' \in (t_0 - r^2, t_0)$ so that $|t-t'| < r^2$. These facts show that

$$I_2 \le 4a_* M_{R_1} N^2 |t-t'| \le 4a_* M_{R_1} N^2 r^2.$$

Since $Nr \le \varepsilon N R_1 \le \varepsilon c_0$ and $R_1 \le \frac{c_0}{N}$, we obtain

$$I_2 \le 4a_* \varepsilon c_0^2 \frac{M_{R_1} r}{R_1}.$$

Thus, we obtain (3.15) in case that $R_1 \le \frac{c_0}{N}$ and $r + \frac{\sqrt{n}+1}{N} \le R_1$ is satisfied.

Also, in this case, (3.16) holds from (3.19) as $N \le \frac{c_0}{R_1}$.

Case 2. Second, we consider the case that $R_1 > \frac{c_0}{N}$ and $0 < r \le \frac{c_2}{N}$ with $c_2 \ll c_0$, such that $r + \frac{\sqrt{n}+1}{N} \le R_1$ is satisfied, by choosing c_2, c_0 so that $c_2 + \sqrt{n} + 1 < c_0$. In this case, one can apply Lemma 3.1 with the pair $(r + \frac{\sqrt{n}}{N}, R_1)$ in place of (r, R) in this lemma, noting $r + \frac{\sqrt{n}}{N} < \frac{1}{2} c_1 R_1$ if $c_0 c_1 > 2(c_2 + \sqrt{n})$, to get that $|\nabla_e^N v^N(t, \frac{x}{N})| \le \frac{C(n,c_\pm) M_{R_1}}{R_1}$ on $Q_N(r + \frac{\sqrt{n}}{N})$, recall that c_1 is the constant given in Lemma 3.1. By (3.13), this shows (3.16), namely $|\partial_{z_i} \tilde{v}^N(t,z)| \le \frac{C(n,c_\pm) M_{R_1}}{R_1}$ on $Q(r)$. From these estimates, we obtain bounds for I_1 and I_2 as above, and therefore (3.15) holds.

Case 3. Third, we consider the case that $R_1 > \frac{c_0}{N}$ and $\frac{c_2}{N} < r < \varepsilon R_1$. Recall that $z_0 \in \mathbb{T}^n$ is the spatial center of $Q(r)$. Take $\bar{z}_0 \in \frac{1}{N}\mathbb{T}_N^n$ such that $|z_0 - \bar{z}_0| \le \frac{\sqrt{n}}{N}$ (or $\frac{\sqrt{n}}{2N}$ is enough). Then, we claim

$$Q(r) = Q_{z_0}(r) \subset Q_{\bar{z}_0}(\bar{r}) \subset Q_{\bar{z}_0}(\bar{R}_1) \subset Q_{z_0}(R_1) = Q(R_1)$$

where $\bar{r} := r + \frac{\sqrt{n}}{N}$, $\bar{R}_1 := R_1 - \frac{\sqrt{n}}{N}$ and the subscript $\bar{z}_0$ in $Q_{\bar{z}_0}(\cdot)$ and $Q_{N,\bar{z}_0}(\cdot)$ below means that the spatial center is $\bar{z}_0$. Indeed, note that $\bar{r} \le (1 + \frac{\sqrt{n}}{c_2}) r$. Note also that $\bar{r} + \frac{\sqrt{n}}{N} < \frac{1}{2} c_1 \bar{R}_1$ automatically holds as $R_1 > \frac{c_0}{N}$, under the choice $c_0 > \frac{(4+c_1)\sqrt{n}}{c_1 - 2\varepsilon}$ (here, ε should be taken as $\varepsilon \in (0, \frac{1}{2} c_1)$), and $r < \varepsilon R_1$. In particular, $\bar{r} < \bar{R}_1$ so that the middle inclusion indicated above holds.

We now investigate the oscillation of $\tilde{v}^N$ on the wider space $Q_{\bar{z}_0}(\bar{r})$. Since $|v^N| \le M_{R_1}$ on $Q_N(R_1)$, this holds also on $Q_{N,\bar{z}_0}(\bar{R}_1)$. Therefore, by Lemma 3.1 applied for the pair $(r, R) = (\bar{r} + \frac{\sqrt{n}}{N}, \bar{R}_1)$ (recall $\bar{r} + \frac{\sqrt{n}}{N} < \frac{1}{2} c_1 \bar{R}_1$), we have $|\nabla_e^N v^N(t, \frac{x}{N})| \le \frac{C(n,c_\pm) M_{R_1}}{\bar{R}_1}$ on $Q_{N,\bar{z}_0}(\bar{r} + \frac{\sqrt{n}}{N})$.

This shows (3.16), namely $|\partial_{z_i} \tilde{v}^N(t,z)| \le \frac{C(n,c_\pm) M_{R_1}}{\bar{R}_1}$ on $Q_{\bar{z}_0}(\bar{r})$ by (3.13), in the present case. Note that $\frac{1}{\bar{R}_1} < \frac{c_0}{c_0 - \sqrt{n}} \frac{1}{R_1}$ holds from $R_1 > \frac{c_0}{N}$. Therefore, we also obtain

$$I_1 \le \sqrt{n} C(n,c_\pm) M_{R_1} \frac{r}{\bar{R}_1}.$$

For I_2, from Lemma 3.2 applied on $Q_{\bar{z}_0}(\bar{r})$ with $\omega = \sqrt{n} C(n,c_\pm) M_{R_1} \frac{\bar{r}}{\bar{R}_1}$ in (3.9), we have $|\tilde{v}^N(t,\bar{z}_0) - \tilde{v}^N(t_1,\bar{z}_0)| = |v^N(t,\bar{z}_0) - v^N(t_1,\bar{z}_0)| \le 2\omega = 2\sqrt{n} C(n,c_\pm) M_{R_1} \frac{\bar{r}}{\bar{R}_1}$ if $|t - t_1| \le \bar{r}^2$ since $\bar{z}_0 \in \frac{1}{N}\mathbb{T}_N^n$. Note that $X, X' \in Q_{\bar{z}_0}(\bar{r})$ implies $|t - t'| < \bar{r}^2$. Take the smaller one of $\{t,t'\}$ as t_1. Thus, recalling $\bar r \le (1+\frac{\sqrt{n}}{c_2})r$, we obtain (3.15) in case that $R_1 > \frac{c_0}{N}$ and $\frac{c_2}{N} < r < \varepsilon R_1$.

Case 4. Fourth and finally, we consider the remaining case that $R_1 \le \frac{c_0}{N}$ and $r + \frac{\sqrt{n}+1}{N} > R_1$. It is then sufficient to show the conclusion when $R_1 < \frac{\sqrt{n}+1}{(1-\varepsilon)N}$ and $r < \varepsilon R_1$, for an $\varepsilon > 0$ taken small enough. Polylinearity will play a role in the following proof in a short distance regime.

First, we consider the oscillation in z by deriving estimates on the slope of $\tilde v^N$ on $Q(r)$ in terms of M_{R_1} and R_1. Recall $D(r) = \{|z-z_0| < r\}$, which is the spatial part of $Q(r)$. By a spatial shift, we may assume $|z_0|_{L^\infty} \le \frac{1}{2N}$ for the center $z_0 \in \frac{1}{N}\mathbb{T}_N^n$ of $D(r)$ and $D(R_1)$. We divide the box $E_{\frac{1}{N}} := \{|z|_{L^\infty} < \frac{1}{N}\}$ as $E_{\frac{1}{N}} = \cup_{v\in\{\pm1\}^n} E_{v,\frac{1}{N}}$ into 2^n unit orthants $E_{v,\frac{1}{N}} := \{z \in E_{\frac{1}{N}}; \operatorname{sgn} z_i = v_i, 1 \le i \le n\}$, where $v = (v_i)_{i=1}^n$.

From (3.12), the slope $\partial_{z_i}\tilde v^N(z)$ of $\tilde v^N$ is constant in z_i on each unit orthant $E_{v,\frac{1}{N}}$ and depends only on $\check z_{(i)} := (z_1,\dots,z_{i-1},z_{i+1}\dots,z_n)$. We need to consider only v such that $E_{v,\frac{1}{N}} \cap D(r) \ne \emptyset$, that is those orthants that $D(r)$ touches. On such an $E_{v,\frac{1}{N}}$, we work on $\tilde D_{v,\frac{1}{N}}(R_1) := E_{v,\frac{1}{N}} \cap D(R_1)$.

Define $R_i^*(\check z_{(i)})$ by

$$\begin{aligned} R_i^* \equiv R_i^*(\check z_{(i)}) := \max\{|z_i^1 - z_i^2|\,;\, z^1 = (\check z_{(i)}, z_i^1), z^2 = (\check z_{(i)}, z_i^2) \in \tilde D_{v,\frac{1}{N}}(R_1) \\ \text{and line connecting } z^1, z^2 \text{ intersects } D(r)\}. \end{aligned}$$

Then, we see $R_i^* \ge \{(\sqrt{1-\varepsilon^2}-\varepsilon)R_1\} \wedge \frac{1}{N}$.

Indeed, "the line intersects $D(r)$" implies "there exists b such that $(\check z_{(i)}, b) \in D(r)$", that is, $|z_0 - (\check z_{(i)}, b)|^2 = A + |z_{0,i} - b|^2 < r^2$, where $z_0 = (\check z_{0,(i)}, z_{0,i})$ and $A := |\check z_{0,(i)} - \check z_{(i)}|^2$. By the definition of R_i^*, we see $R_i^* \ge |a-b| \wedge \frac{1}{N}$, where a is taken as $|z_0 - (\check z_{(i)}, a)|^2 = R_1^2$. Then, since $|z_{0,i} - b| \le \sqrt{r^2 - A}$, $0 \le A \le r^2$ and also $r < \varepsilon R_1$, we see

$$\begin{aligned} |a-b| &\ge |z_{0,i} - a| - |z_{0,i} - b| \\ &\ge \sqrt{R_1^2 - A} - \sqrt{r^2 - A} \\ &\ge \sqrt{R_1^2 - r^2} - r \\ &\ge (\sqrt{1-\varepsilon^2} - \varepsilon)R_1. \end{aligned}$$

Thus, we obtain $R_i^* \ge \{(\sqrt{1-\varepsilon^2}-\varepsilon)R_1\} \wedge \frac{1}{N}$.

Now, choosing $\varepsilon > 0$ small so that $\sqrt{1-\varepsilon^2} - \varepsilon$ is close to 1, we see $R_i^* \ge \frac{1}{c_0}R_1$ (note $\frac{1}{N} \ge \frac{R_1}{c_0}$ and $c_0 \ge \sqrt{n}+1$). Since the slope $\partial_{z_i}\tilde v^N(t,z)$ is constant in z_i, this leads to the bound on $Q(r) \cap E_{v,\frac{1}{N}}$,

$$|\partial_{z_i}\tilde v^N(t,z)| \le \frac{2M_{R_1}}{R_i^*} \le c_0 \frac{2M_{R_1}}{R_1}, \tag{3.21}$$

establishing (3.16) in the present case. Therefore, we also obtain

$$I_1 \le c_0\sqrt{n}\frac{2M_{R_1}}{R_1} r.$$

For I_2, we can use the same idea as in Case 1. Indeed, note that, from (3.12), $\partial_{z_i}\tilde{v}^N(t,z)$ is a convex combination of $\nabla^N_{e_i}v^N(t,\frac{x}{N})$ on $Q_N(r+\frac{\sqrt{n}}{N})$ so that, by (3.21), we have $|\nabla^N_e v^N(t,\frac{x}{N})| \le c_0\frac{2M_{R_1}}{R_1}$ on $Q_N(r+\frac{\sqrt{n}}{N})$. Therefore, one can first apply a similar argument to (3.20) noting $|t-t'|\le r^2$, and then since $Nr < \frac{(\sqrt{n}+1)\varepsilon}{1-\varepsilon}$ recall $r<\varepsilon R_1$ and $R_1 < \frac{\sqrt{n}+1}{(1-\varepsilon)N}$), we obtain

$$I_2 \le 2a_* c_0 \frac{2M_{R_1}}{R_1} N r^2 \le 4a_* c_0 \frac{M_{R_1}}{R_1}\frac{(\sqrt{n}+1)\varepsilon}{1-\varepsilon} r.$$

This completes the proof of the corollary. □

We now obtain the following integral estimates for the polylinear interpolation $\tilde{v}^N$ of the solution v^N of the discrete heat equation (3.2) on $Q_N(R)$. These estimates will be applied in the proof of Lemma 4.8 below. As in Corollary 3.3, we consider three different ranges $0<\rho<r<R$, especially, by distinguishing r and R. Otherwise, we would get estimate (4.42) in the proof of Lemma 4.8 only for $r\ge\frac{\sqrt{n}}{N}$, which is insufficient.

Proposition 3.4. *(cf. Lemma 4.5 of [41]) Let $v^N = v^N(t,\frac{x}{N}) (\equiv v^{N,R}(t,\frac{x}{N}))$ be given on $Q_N(R)=Q_N(X_0,R)$ as in Corollary 3.3 for any fixed $X_0=(t_0,z_0)\in\Omega$ and $R>\frac{\sqrt{n}}{N}$. Let $0<\rho<r\le R-\frac{\sqrt{n}}{N}$. In particular, $\tilde{v}^N$ is defined on $Q(r)=Q(X_0,r)$. Then, there is a constant $C=C(n,c_\pm)$ such that*

$$\int_{Q(\rho)} (\tilde{v}^N)^2 dX \le C\big(\tfrac{\rho}{r}\big)^{n+2}\int_{Q(r)} (\tilde{v}^N)^2 dX, \tag{3.22}$$

$$\int_{Q(\rho)} |\tilde{v}^N - \{\tilde{v}^N\}_\rho|^2 dX \le C\big(\tfrac{\rho}{r}\big)^{n+4}\int_{Q(r)} |\tilde{v}^N - \{\tilde{v}^N\}_r|^2 dX, \tag{3.23}$$

where $Q(\rho)=Q(X_0,\rho)$ and $\{v\}_\rho = \frac{1}{|Q(\rho)|}\int_{Q(\rho)} v\,dX$. Moreover, assuming $R>\frac{\sqrt{n}+1}{N}$, for $0<\rho<r\le R-\frac{\sqrt{n}+1}{N}$, we have

$$\int_{Q(\rho)} (\nabla^N_e\tilde{v}^N)^2 dX \le C\big(\tfrac{\rho}{r}\big)^{n+2}\int_{Q(r)} (\nabla^N_e\tilde{v}^N)^2 dX, \tag{3.24}$$

$$\int_{Q(\rho)} |\nabla^N_e\tilde{v}^N - \{\nabla^N_e\tilde{v}^N\}_\rho|^2 dX \le C\big(\tfrac{\rho}{r}\big)^{n+4}\int_{Q(r)} |\nabla^N_e\tilde{v}^N - \{\nabla^N_e\tilde{v}^N\}_r|^2 dX. \tag{3.25}$$

The power, for example, $n+4$ in (3.25) could be understood as follows. If $\nabla^N_e\tilde{v}^N$ would behave as a linear or Lipschitz function uniform in N, its oscillation in $Q(\rho)$ is like $C\rho$. Therefore, recalling $|Q(\rho)|=C\rho^{n+2}$, the integral in the left hand side of (3.25) would behave as $C\rho^{n+4}$. Similarly, the integral in the right hand side would behave as Cr^{n+4}, intuitively explaining the bound (3.25).

Proof. The proof follows that of Lemma 4.5 of [41], properly modified in our discrete setting. Indeed, to establish (3.22), we may assume $\rho<r/4$ as otherwise the statement follows straightforwardly. Recall that $\tilde{v}^N$ is well-defined on $Q(R-\frac{\sqrt{n}}{N})=Q(X_0,R-\frac{\sqrt{n}}{N})$ and, in particular, on $Q(r)=Q(X_0,r)$.

Let now $U = \sup_{X \in Q(r/2)} |\tilde{v}^N(X)| d_r(X)^{1+n/2}$ (the factor $d_r(X)^{1+n/2}$ leads to r^{-n-2} in (3.27), (3.28)) and find $X_1 = (t_1, z_1) \in Q(r/2)$ so that $d_r(X_1)^{1+n/2} |\tilde{v}^N(X_1)| > U/2$, where $Q(r/2) = Q(X_0, r/2)$ and $d_r(X_1) := \inf\{|Y - X_1|; Y \in \mathcal{P}Q(\frac{r}{2})\}$ is the (parabolic) distance from X_1 to the parabolic boundary of $Q(\frac{r}{2})$: $\mathcal{P}Q(\frac{r}{2}) = \{t_0 - (\frac{r}{2})^2\} \times D_{z_0}(\frac{r}{2}) \cup (t_0 - (\frac{r}{2})^2, t_0) \times \partial D_{z_0}(\frac{r}{2})$. Recall $|X| = \max\{\sqrt{|t|}, |z|\}$ for $X = (t, z) \in \mathbb{R} \times \mathbb{T}^n$ (including $t < 0$). Note that $|\cdot|$ satisfies the triangular inequality (viewing $\mathbb{T}^n$ as $\mathbb{R}^n$ by periodic extension).

By the oscillation bound (3.15) in Corollary 3.3 applied on the region $Q_N(X_1, R - \frac{r}{2})$ (i.e., take X_1 for X_0 and $R - \frac{r}{2}$ for R, and note $Q_N(X_1, R - \frac{r}{2}) \subset Q_N(X_0, R)$), with $r_1 = \gamma d_r(X_1)$ (take r_1 for r in Corollary 3.3) and $R_1 = d_r(X_1)/2$, we obtain

$$\operatorname*{osc}_{Q(X_1, \gamma d_r(X_1))} \tilde{v}^N \le 2C(n, c_\pm)\gamma \sup_{Q(X_1, d_r(X_1)/2)} |\tilde{v}^N|, \tag{3.26}$$

where we take $\gamma \in (0, 1/4)$. Note that this bound is applicable, since the condition of Corollary 3.3, $R_1 + \frac{\sqrt{n}}{N} \le R - \frac{r}{2}$, holds. Indeed, from $d_r(X_1) \le \frac{r}{2}$, we see $R_1 \le \frac{r}{4}$ so that the condition follows from our assumption: $r + \frac{\sqrt{n}}{N} \le R$.

Since $d_r(X) \ge d_r(X_1)/2$ for $X \in Q(X_1, d_r(X_1)/2)$ and $Q(X_1, d_r(X_1)/2) \subset Q(X_0, \frac{r}{2})$ holds, the right-hand side of (3.26) is bounded by

$$\begin{aligned}
&2C(n, c_\pm)\gamma 2^{1+n/2} d_r(X_1)^{-1-n/2} \sup_{Q(X_1, d_r(X_1)/2)} |\tilde{v}^N(X)| d_r(X)^{1+n/2} \\
&\le C(n, c_\pm) 2^{2+n/2} \gamma d_r(X_1)^{-1-n/2} U.
\end{aligned}$$

Note, as well consequently, for $X \in Q(X_1, \gamma d_r(X_1))$, that

$$\begin{aligned}
|\tilde{v}^N(X)| &\ge |\tilde{v}^N(X_1)| - \operatorname*{osc}_{Q(X_1, \gamma d_r(X_1))} \tilde{v}^N \\
&\ge |\tilde{v}^N(X_1)| - C(n, c_\pm) 2^{2+n/2} \gamma d_r(X_1)^{-1-n/2} U \\
&\ge |\tilde{v}^N(X_1)| \big(1 - 2^{3+n/2} C(n, c_\pm)\gamma\big).
\end{aligned}$$

The last line follows by the choice of X_1. We now choose $\gamma > 0$ small enough so that $C_1 := 1 - 2^{3+n/2} C(n, c_\pm)\gamma > 0$. Then, taking infimum, and by the choice of X_1 again,

$$\inf_{Q(X_1, \gamma d_r(X_1))} |\tilde{v}^N|^2 \ge C_1^2 |\tilde{v}^N(X_1)|^2 \ge C_1^2 U^2 (d_r(X_1)^{-2-n}/4).$$

Thus, we have

$$\begin{aligned}
U^2 &\le 4C_1^{-2} d_r(X_1)^{2+n} \inf_{Q(X_1, \gamma d_r(X_1))} |\tilde{v}^N|^2 \\
&\le C(n, C_1)\gamma^{-n-2} \int_{Q(X_1, \gamma d_r(X_1))} |\tilde{v}^N|^2 dX \ \le C(n, c_\pm) \int_{Q(r)} |\tilde{v}^N|^2 dX,
\end{aligned} \tag{3.27}$$

since $d_r(X_1)^{2+n} / |Q(X_1, \gamma d_r(X_1))| = C(n)\gamma^{-n-2}$ and then $Q(X_1, \gamma d_r(X_1)) \subset Q(r) = Q(X_0, r)$.

Hence, if $\rho \le r/4$,

$$\int_{Q(\rho)} |\tilde{v}^N|^2 dX \le C(n)\rho^{n+2} \sup_{Q(\rho)} |\tilde{v}^N|^2 \tag{3.28}$$

$$
\begin{aligned}
&\le C(n)\rho^{n+2} \sup_{Q(r/4)} |\tilde{v}^N(X)|^2 d_r(X)^{2+n} d_r(X)^{-2-n} \\
&\le C(n)(\tfrac{\rho}{r})^{n+2} U^2,
\end{aligned}
$$

since $d_r(X) \ge r/4$ for $X \in Q(r/4) = Q(X_0, r/4)$ as $Q(r/4) \subset Q(r/2)$ belongs to the interior.

Using the bound (3.27) on U, we obtain (3.22). In particular, we have also shown

$$
\sup_{Q(\rho)} |\tilde{v}^N|^2 \le \frac{C(n, c_\pm)}{r^{n+2}} \int_{Q(r)} |\tilde{v}^N|^2 dX. \tag{3.29}
$$

To establish (3.23), for a fixed $r \le R - \frac{\sqrt{n}}{N}$, we may assume $\rho < r/4$ as above. Indeed,

$$
\int_{Q(\rho)} |\tilde{v}^N - \{\tilde{v}^N\}_\rho|^2 dX \le 2\int_{Q(\rho)} |\tilde{v}^N - \{\tilde{v}^N\}_r|^2 dX + 2|Q(\rho)| \cdot |\{\tilde{v}^N\}_\rho - \{\tilde{v}^N\}_r|^2
$$

and the second term is rewritten and then bounded as

$$
2|Q(\rho)| \Big| \tfrac{1}{|Q(\rho)|} \int_{Q(\rho)} (\tilde{v}^N - \{\tilde{v}^N\}_r) dX \Big|^2 \le 2 \int_{Q(\rho)} |\tilde{v}^N - \{\tilde{v}^N\}_r|^2 dX,
$$

by applying Schwarz's inequality.

Write now

$$
\int_{Q(\rho)} |\tilde{v}^N - \{\tilde{v}^N\}_\rho|^2 dX \ \le C(n)\rho^{n+2} \sup_{Q(\rho)} |\tilde{v}^N - \{\tilde{v}^N\}_\rho|^2. \tag{3.30}
$$

Note that

$$
\begin{aligned}
\tilde{v}^N(X) - \{\tilde{v}^N\}_\rho &= \tfrac{1}{|Q(\rho)|} \int_{Q(\rho)} \{\tilde{v}^N(X) - \tilde{v}^N(Y)\} dY \\
&= \tfrac{1}{|Q(\rho)|} \int_{Q(\rho)} \{\big(\tilde{v}^N(X) - \{\tilde{v}^N\}_r\big) - \big(\tilde{v}^N(Y) - \{\tilde{v}^N\}_r\big)\} dY
\end{aligned}
$$

and that $\tilde{v}^N - \{\tilde{v}^N\}_r$, as $\{\tilde{v}^N\}_r$ is a constant, also satisfies the discrete heat equation (3.2) on $Q_N(R)$. Then, applying the oscillation estimate Corollary 3.3 to $\tilde{v}^N - \{\tilde{v}^N\}_r$ (with there $r = \rho$, $R_1 = r/4 < r < R - \sqrt{n}N$), we obtain for $X \in Q(\rho)$ that

$$
|\tilde{v}^N(X) - \{\tilde{v}^N\}_\rho| \le C(n, c_\pm)\big(\frac{\rho}{r}\big) \sup_{Q(r/4)} |\tilde{v}^N - \{\tilde{v}^N\}_r|.
$$

Moreover, applying (3.29) (with $\rho = r/4$) to the discrete heat equation solution $\tilde{v}^N - \{\tilde{v}^N\}_r$, the right-hand side of (3.30), is bounded by

$$
\frac{C(n, c_\pm)\rho^{n+4}}{r^2} \sup_{Q(r/4)} |\tilde{v}^N - \{\tilde{v}^N\}_r|^2 \le C(n, c_\pm) \left(\frac{\rho}{r}\right)^{n+4} \int_{Q(r)} |\tilde{v}^N - \{\tilde{v}^N\}_r|^2 dX,
$$

finishing the proof of (3.23).

The remaining two statements, (3.24) and (3.25), follow from (3.22) and (3.23), respectively, by taking $R - \frac{1}{N}$ instead of R, since $\nabla_e^N v^N$ also solves the discrete heat equation (3.2) on $Q_N(R - \frac{1}{N})$ by noting $\frac{x+e}{N} \in Q_N(R)$ for $\frac{x}{N} \in Q_N(R - \frac{1}{N})$. $\square$

3.3 Hölder norms

To set the stage for the 'weighted' norms that we will use, we first define 'unweighted' Hölder norms or seminorms analogous to that in [41], but tailored to our setting. Recall $\Omega = [0,T] \times \mathbb{T}^n$ and for $X = (t,z) \in \Omega$ that $|X| = \max\{t^{\frac{1}{2}}, |z|\}$. Consider, for a function F on $\mathbb{T}^n$ (and therefore on Ω), the continuous space gradient $\nabla^N F(z) = \{\nabla^N_e F(z)\}_{|e|=1,e>0} \in \mathbb{R}^n$ for $z \in \mathbb{T}^n$ where

$$\nabla^N_e F(z) := N\big(F(z + \tfrac{e}{N}) - F(z)\big). \tag{3.31}$$

For a function $F = F(X)$ on Ω, set

$$|F|_0 \equiv \|F\|_\infty := \sup_{X\in\Omega} |F(X)|. \tag{3.32}$$

For $\alpha \in (0,1]$, define the parabolic Hölder seminorms by

$$\begin{aligned} [F]_\alpha &:= \sup_{X\neq Y\in\Omega} \frac{|F(X)-F(Y)|}{|X-Y|^\alpha}, \\ [F]_{1+\alpha} &:= \sup_{X\neq Y\in\Omega} \frac{|\nabla^N F(X) - \nabla^N F(Y)|}{|X-Y|^\alpha}, \end{aligned} \tag{3.33}$$

where $|\nabla^N F(X) - \nabla^N F(Y)| := \max_{|e|=1,e>0} |\nabla^N_e F(X) - \nabla^N_e F(Y)|$. For $\alpha = 0$, $[F]_0 := \underset{\Omega}{\operatorname{osc}}(F)$ is the oscillation of F on Ω, recall (3.14). For $\beta \in (0,2]$, define

$$\langle F\rangle_\beta := \sup_{X\neq Y\in\Omega, x=y} \frac{|F(X)-F(Y)|}{|X-Y|^{\frac{\beta}{2}}}, \tag{3.34}$$

where the spatial coordinates x, y of X and Y are in common. Adding all these, we define for $a \in (0,2]$, the unweighted Hölder norm

$$|F|_a := [F]_a + \langle F\rangle_a + |F|_0. \tag{3.35}$$

We now introduce several weighted norms to take care of possible diverging effects near $t = 0$. Define the parabolic boundary $\mathcal{P}\Omega$ of Ω by

$$\mathcal{P}\Omega = \{t = 0\} \times \mathbb{T}^n.$$

Since we work on the torus, for $X = (t,z) \in \Omega$, the (parabolic) distance $d(X)$ to the boundary is defined by

$$d(X) := \inf\{|X-Y|; Y = (0,y) \in \mathcal{P}\Omega\} \equiv \sqrt{t}.$$

We will sometimes write d instead of $d(X)$ to simplify notation when the context is clear.

For $a = 0$ and $b \geq 0$, define

$$|F|_0^{(b)} := \sup_{X\in\Omega} d(X)^b |F(X)|. \tag{3.36}$$

For $0 < a = k + \alpha \le 2$ where $k = 0, 1$ and $\alpha \in (0, 1]$, and $b \ge 0$, let

$$[F]_a^{(b)} := \sup_{X \ne Y \in \Omega} (d(X) \wedge d(Y))^{a+b} \frac{|(\nabla^N)^k F(X) - (\nabla^N)^k F(Y)|}{|X - Y|^\alpha}. \tag{3.37}$$

$$\langle F \rangle_a^{(b)} = \sup_{X \ne Y \in \Omega, x = y} (d(X) \wedge d(Y))^{a+b} \frac{|F(X) - F(Y)|}{|X - Y|^{\frac{a}{2}}}. \tag{3.38}$$

Here, $(\nabla^N)^0$ is the identity operator.

The norm and seminorms without weights defined in (3.32), (3.33), (3.34) can be expressed as $|F|_0 = |F|_0^{(0)}$, $[F]_a = [F]_a^{(-a)}$ and $\langle F \rangle_a = \langle F \rangle_a^{(-a)}$ in terms of those defined in (3.36), (3.37), (3.38), respectively, taking $b = -a$ though we have restricted $b \ge 0$.

We will also have occasion to to use Hölder norms with respect to the continuous time/discrete space $\Omega_N = [0, T] \times \frac{1}{N}\mathbb{T}_N^n$. Replacing Ω by Ω_N in the definitions (3.36), (3.37), (3.38) of seminorms, we define, for a function F on Ω_N and $b \ge 0$,

$$|F|_0^{(b),N} := \sup_{X \in \Omega_N} d(X)^b |F(X)|, \tag{3.39}$$

and also, for $a = k + \alpha$ where $k = 0, 1$ and $\alpha \in (0, 1]$, and $b \ge 0$ that

$$[F]_a^{(b),N} := \sup_{X \ne Y \in \Omega_N} (d(X) \wedge d(Y))^{a+b} \frac{|(\nabla^N)^k F(X) - (\nabla^N)^k F(Y)|}{|X - Y|^\alpha} \tag{3.40}$$

$$\langle F \rangle_a^{(b),N} := \sup_{X \ne Y \in \Omega_N, x = y} (d(X) \wedge d(Y))^{a+b} \frac{|F(X) - F(Y)|}{|X - Y|^{\frac{a}{2}}}. \tag{3.41}$$

In the following, the superscript $*$ for seminorms means (0) so that for $a \in (0, 2]$,

$$[F]_a^* := [F]_a^{(0)}, \quad \langle F \rangle_a^* := \langle F \rangle_a^{(0)}, \quad [F]_a^{*,N} := [F]_a^{(0),N}, \quad \langle F \rangle_a^{*,N} := \langle F \rangle_a^{(0),N}. \tag{3.42}$$

We will use $\|F\|_\infty$ rather than $|F|_0^*$ or $|F|_0^{*,N}$.

The seminorms defined by taking the supremum over $Q \subset \Omega$ instead of Ω, such as $\sup_{X \in Q}$ or $\sup_{X \ne Y \in Q}$, are denoted by adding Q in the subscript. [We actually do not use this notation in this article.]

We finally note that, via polylinear interpolation, seminorms for discrete and continuous functions are mutually equivalent.

Lemma 3.5. *For a function $F = F(t, \frac{x}{N})$ on Ω_N, let $\tilde{F}$ be its polylinear interpolation in spatial variable defined by* (3.11) *taking $F(t, \cdot)$ instead of u^N. Then, we have the following for some $C = C(n) > 0$,*

$$|F|_0^{(b),N} = |\tilde{F}|_0^{(b)}, \tag{3.43}$$

$$[F]_a^{(b),N} \le [\tilde{F}]_a^{(b)} \le C[F]_a^{(b),N}, \tag{3.44}$$

$$\langle F \rangle_a^{(b),N} \le \langle \tilde{F} \rangle_a^{(b)} \le C \langle F \rangle_a^{(b),N}. \tag{3.45}$$

Proof. The first inequalities in (3.44), (3.45) and " $\le$" in (3.43) are obvious as $\tilde{F}(X) = F(X)$ and $\nabla_e^N \tilde{F}(X) = \nabla_e^N F(X)$ for $X \in \Omega_N$, the latter of which follows from

$$\widetilde{\nabla_e^N F}(X) = \nabla_e^N \tilde{F}(X), \quad X \in \Omega. \tag{3.46}$$

This equality is shown from $\vartheta^N(v, z + \frac{e}{N}) = \vartheta^N(v, z)$ in (3.11). The converse inequality "$\geq$" in (3.43) follows from $|\tilde{F}(t,z)| \leq \sup_{x\in\mathbb{T}_N^n} |F(t, \frac{x}{N})|$ for every $z \in \mathbb{T}^n$ and $t \in [0,T]$.

The second inequality in (3.44) in the case $k = 0$ is shown as follows. For $X = (t,z), Y = (s,y) \in \Omega$, first take $Y_0 = (s, y_0) \in \Omega$ such that $z \equiv y_0$ modulo $\frac{1}{N}$ and $[Ny] = [Ny_0]$ (i.e. y and y_0 belong to the same $\frac{1}{N}$-box) hold both componentwise, in particular, $|Y_0 - Y| \leq \frac{\sqrt{n}}{N}$. Then,

$$|\tilde{F}(X) - \tilde{F}(Y)| \leq |\tilde{F}(X) - \tilde{F}(Y_0)| + |\tilde{F}(Y_0) - \tilde{F}(Y)|,$$

and the first term is written, noting that the ϑ-parts are in common, as

$$\Big| \sum_{v\in\{0,1\}^n} \vartheta^N(v,z) \Big\{ F\big(t, \tfrac{[Nz]+v}{N}\big) - F\big(s, \tfrac{[Ny_0]+v}{N}\big) \Big\} \Big|.$$

This is bounded by $[F]_a^{(b),N} (d(X) \wedge d(Y))^{-(a+b)} |X - Y_0|^\alpha$. On the other hand, the second term is rewritten as $|\{\tilde{F}(Y_0) - F(Z_0)\} - \{\tilde{F}(Y) - F(Z_0)\}|$ taking $Z_0 = (s, z_0)$ with $z_0 = \frac{1}{N}[Ny] \in \frac{1}{N}\mathbb{T}_N^n$ so that it is further rewritten as

$$\Big| \sum_{v\in\{0,1\}^n} \Big\{ \vartheta^N(v, y_0) - \vartheta^N(v,y) \Big\} \Big\{ F\big(s, z_0 + \tfrac{v}{N}\big) - F\big(s, z_0\big) \Big\} \Big|.$$

This is bounded by

$$C(n) N |y_0 - y| [F]_a^{(b),N} d(Y)^{-(a+b)} (\tfrac{\sqrt{n}}{N})^\alpha \leq C(n)\sqrt{n} |Y_0 - Y|^\alpha [F]_a^{(b),N} d(Y)^{-(a+b)}$$

by noting (3.11) and $|y_0 - y| \leq \frac{\sqrt{n}}{N}$.

In the case $|X - Y| \geq \frac{1}{10N}$, since both $|X - Y_0|, |Y_0 - Y| \leq C(n)|X - Y|$, we obtain the second inequality in (3.44) in the case $k = 0$.

On the other hand, in the case $|X - Y| < \frac{1}{10N}$, we connect $X_0 = (s,z)$ and $Y = (s,y)$ by a line. Then, it touches the boundary of $\frac{1}{N}$-boxes at most 2^n times, so that, the line is divided as $X_0 \to Z_1 \to \cdots \to Z_\ell \to Y$ with $\ell \leq 2^n$. As we saw above, in the same $\frac{1}{N}$-box, our estimate picks up factors $|Z_{i+1} - Z_i|^\alpha$, all of them are bounded by $|X_0 - Y|^\alpha$. We also have the estimate on $|\tilde{F}(X) - \tilde{F}(X_0)|$ as above; see the first term with $Y_0 = X_0$. From these, we can derive the second inequality in (3.44) for $k = 0$ also in the case $|X - Y| < \frac{1}{10N}$.

The second inequalities in (3.44) in case $k = 1$ (i.e. $\nabla_e F$ in place of F) and in (3.45) are similar. □

3.4 Two basic lemmas

We now consider and adapt some of the results from Chapter 4 of [41] with respect to our discrete setting, especially in terms of polylinear interpolations.

The first is Lemma 4.3 of [41] that a Campanato-type integral estimate implies Hölder continuity. Recall the definition of the parabolic ball $Q(R) = Q(X_0, R)$ in (3.1). Due to the discrete nature of the problem in our setting, especially, the non-locality of the polylinear interpolation, the assumption (3.47) below can be given only with r_N instead of r, which is weaker than that of Lemma 4.3 of [41]. As a result, we obtain the Hölder property (3.48) only for $|Y - Y_1| \geq \frac{1}{MN}$, excluding the short distance regime, with an additional term $\delta U_{F,1+\alpha}\big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)}$, which can be made arbitrary small by taking $C = C_{\delta,M}$ large. This will be applied in the proofs of Propositions 4.10 and 5.4 below. The short distance regime $|Y - Y_1| \leq \frac{1}{MN}$ will be covered separately by Lemmas 4.11 and 5.3 later.

Lemma 3.6. *(cf. Lemma 4.3 of [41]) Let $F \in L^1(Q(X_0, 2R))$ and suppose there are constants $\alpha \in (0, 1]$ and $H > 0$ along with a function G defined on $Q(X_0, 2R) \times (0, R)$ such that*

$$\int_{Q(Y,r)} |F(X) - G(Y, r)| dX \leq H r_N^{n+2+\alpha}, \tag{3.47}$$

for any $Y \in Q(X_0, R)$ and any $r \in (0, R)$, where $r_N = r + \frac{c}{N}$ with some $c > 0$. Then, for every $\delta > 0$ and $M > 0$, there exists $C = C_{\delta,M}(n, \alpha) > 0$ such that

$$|F(Y) - F(Y_1)| \leq \Big(CH + \delta U_{F,1+\alpha}\big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)}\Big)|Y - Y_1|^\alpha, \tag{3.48}$$

holds if $Y, Y_1 \in Q(X_0, R)$ satisfy $|Y - Y_1| \geq \frac{1}{MN}$, where

$$U_{F,1+\alpha} := \sup_{X \neq Y} \big(d(X) \wedge d(Y)\big)^{1+\alpha} \tfrac{|F(X)-F(Y)|}{|X-Y|^\alpha}.$$

Note that, for $F = \nabla^N \tilde{u}^N$ (vector-valued), $U_{F,1+\alpha} = U_{1+\alpha}$ defined in (4.2) below. [We will choose $\delta > 0$ small enough such that $\delta U_{F,1+\alpha}$ will be eventually absorbed by $U_{F,1+\alpha}$ itself.]

Proof. As in the proof of Lemma 4.3 of [41], we take two nonnegative convolution kernels $\varphi = \varphi(z) \in C^1(\mathbb{R}^n)$ supported on $\{|z| \leq 1\}$ and $\eta = \eta(\sigma) \in C^1(\mathbb{R})$ supported on $[0, 1]$ such that $\int_{\mathbb{R}^n} \varphi dz = \int_{\mathbb{R}} \eta d\sigma = 1$. Define for $(Y, \tau) \in Q(X_0, R) \times (0, R)$

$$\bar{F}(Y, \tau) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} F(y + \tau z, s - \tau^2 \sigma)\varphi(z)\eta(\sigma) dz d\sigma, \quad Y = (s, y).$$

Then, as in [41], setting $K = \|D\varphi\|_{L^\infty}, L = \|\eta'\|_{L^\infty}$, by the condition (3.47),

$$\begin{aligned} |\bar{F}_y(Y, \tau)| &\leq \tfrac{K}{\tau^{n+3}} \int_{Q(Y,\tau)} |F(X) - g(Y, \tau)| \eta(\tfrac{s-t}{\tau^2}) dX \\ &\leq HKL \frac{\tau_N^{n+2+\alpha}}{\tau^{n+3}}. \end{aligned}$$

Similarly,

$$\begin{aligned} |\bar{F}_s(Y, \tau)| &\leq HKL \frac{\tau_N^{n+2+\alpha}}{\tau^{n+4}}, \\ |\bar{F}_\tau(Y, \tau)| &\leq (n + K + 2 + 2L) H \frac{\tau_N^{n+2+\alpha}}{\tau^{n+3}}. \end{aligned}$$

Now, fix Y and Y_1 in $Q(X_0, R)$ and set $\tau = |Y - Y_1|$. If $\varepsilon \in (0, \tau)$, then similarly to [41] p.51,

$$|\bar{F}(Y, \varepsilon) - \bar{F}(Y_1, \varepsilon)| \leq 2(n + K + 2 + 2L) H \int_\varepsilon^\tau \frac{\rho_N^{n+2+\alpha}}{\rho^{n+3}} d\rho \tag{3.49}$$

$$+ HKL\tfrac{\tau_N^{n+2+\alpha}}{\tau^{n+3}}|y-y_1| + HKL\tfrac{\tau_N^{n+2+\alpha}}{\tau^{n+4}}|s-s_1|.$$

Recall that $\tau \geq \frac{1}{MN}$ by our assumption so that we have

$$\tfrac{\tau_N}{\tau} = 1 + \tfrac{c}{\tau N} \leq 1 + cM.$$

Thus, the sum of the second and third terms is bounded as

$$HKL\Big(\tfrac{\tau_N^{n+2+\alpha}}{\tau^{n+3}}|y-y_1| + \tfrac{\tau_N^{n+2+\alpha}}{\tau^{n+4}}|s-s_1|\Big) \leq C_M H\tau^\alpha,$$

since $|y-y_1| \leq \tau, |s-s_1| \leq \tau^2$.

Let us estimate the integral $\int_\varepsilon^\tau \frac{\rho_N^{n+2+\alpha}}{\rho^{n+3}} d\rho$ in the first term in the right hand side of (3.49). (This diverges as $\varepsilon \downarrow 0$, while the integral $\int_\varepsilon^\tau \frac{\rho^\alpha}{\rho} d\rho$ in [41] in continuous case converges.)

$$\int_\varepsilon^\tau \frac{\rho_N^{n+2+\alpha}}{\rho^{n+3}} d\rho = \int_\varepsilon^\tau \frac{(\frac{\rho_N}{\rho})^{n+2+\alpha}}{\rho^{1-\alpha}} d\rho \leq (1 + \tfrac{c}{N\varepsilon})^{n+2+\alpha} \tfrac{1}{\alpha}\tau^\alpha,$$

since $\frac{\rho_N}{\rho} = 1 + \frac{c}{N\rho} \leq 1 + \frac{c}{N\varepsilon}$.

Moreover, the convergence rate of $\bar{F}(Y,\varepsilon)$ to $F(Y)$ is estimated as

$$\begin{aligned}
|F(Y) - \bar{F}(Y,\varepsilon)| &= \Big| \int_{\mathbb{R}^n}\int_{\mathbb{R}} \big\{F(y,s) - F(y+\varepsilon z, s-\varepsilon^2\sigma)\big\}\varphi(z)\eta(\sigma)dzd\sigma \Big| \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}} \big(|\varepsilon z|^\alpha + |\varepsilon^2\sigma|^{\frac{\alpha}{2}}\big)\varphi(z)\eta(\sigma)dzd\sigma \times U_{F,1+\alpha} d(Y)^{-(1+\alpha)} \\
&\leq 2\varepsilon^\alpha U_{F,1+\alpha} d(Y)^{-(1+\alpha)}.
\end{aligned}$$

Summarizing these, if $\tau \geq \frac{1}{MN}$, we have

$$\begin{aligned}
|F(Y) - F(Y_1)| &\leq |F(Y) - \bar{F}(Y,\varepsilon)| + |\bar{F}(Y,\varepsilon) - \bar{F}(Y_1,\varepsilon)| + |\bar{F}(Y_1,\varepsilon) - F(Y_1)| \\
&\leq 4\varepsilon^\alpha U_{F,1+\alpha}\big(d(Y)\wedge d(Y_1)\big)^{-(1+\alpha)} + C_M H\tau^\alpha + C(n,\alpha)H(1+\tfrac{c}{N\varepsilon})^{n+2+\alpha}\tau^\alpha,
\end{aligned}$$

for every $\varepsilon \in (0,\tau)$. We now choose $\varepsilon = \tilde{\delta}\cdot\tau$ with small enough $\tilde{\delta} > 0$ such that $4\tilde{\delta}^\alpha \leq \delta$. Then, since $\frac{c}{N\varepsilon} = \frac{c}{\tilde{\delta}N\tau} \leq \frac{Mc}{\tilde{\delta}}$, we obtain (3.48) for $\tau \geq \frac{1}{MN}$. □

Remark 3.1. To derive the uniform Hölder continuity of u^N, we applied the results of [23] and [42] in Section 2.2. We will then use Lemma 3.6 to show the Hölder continuity of $\nabla^N u^N$ in Section 4. However, it might be possible to apply Lemma 3.6 to derive the Hölder continuity of u^N as in [31] in a continuous setting.

We state the iteration Lemma 4.6 of [41]. This will be modified later in Lemma 4.9 with $\sigma(r)$ changed to $\sigma(r_N)$ as in Lemma 3.6 to adjust to our discrete setting and used to prove Propositions 4.10 and 5.4 below.

Lemma 3.7. *(cf. Lemma 4.6 of [41]) Let ω and σ be increasing functions on an interval $(0,R_0]$ and suppose there are positive constants $\bar{\alpha}, \delta$, and τ with $\tau < 1$ and $\delta < \bar{\alpha}$ such that*

$$r^{-\delta}\sigma(r) \leq s^{-\delta}\sigma(s) \qquad \text{if } 0 < s \leq r \leq R_0 \tag{3.50}$$

and

$$\omega(\tau r) \le \tau^{\bar\alpha}\omega(r) + \sigma(r) \qquad \textit{if } \ 0 < r \le R_0. \tag{3.51}$$

Then, there is a constant $C = C(\bar\alpha, \delta, \tau)$ such that

$$\omega(r) \le C\Big[\big(\tfrac{r}{R_0}\big)^{\bar\alpha}\omega(R_0) + \sigma(r)\Big] \le C\Big[\big(\tfrac{r}{R_0}\big)^{\delta}\omega(R_0) + \sigma(r)\Big],$$

for $0 < r \le R_0$.

We remark, as noted in [41], if (4.15)' in [41] holds, that is $\omega(\tau r) \le C\tau^\beta\omega(r) + \sigma(r)$ with $\beta > \delta$ for all small τ, then (3.51) is satisfied with $\bar\alpha \in (\delta, \beta)$ and τ chosen so that $C\tau^\beta \le \tau^{\bar\alpha}$.

Proof. Since the argument is short, for the convenience of the reader, we give it here. The lemma is clear when $r \ge \tau R_0$. Hence, suppose $0 < r < \tau R_0$. Let k be the smallest integer such that $r < \tau^k R_0$. Consider the base case $k = 1$: Write

$$\begin{aligned}\omega(r) &\le \omega(\tau R_0) \le \tau^{\bar\alpha}\omega(R_0) + \sigma(R_0)\\ &\le \tau^{\bar\alpha}\omega(R_0) + \tau^{-\delta}\sigma(\tau R_0).\end{aligned}$$

Suppose now for $j \le k-1$ that

$$\omega(\tau^j R_0) \le \tau^{j\bar\alpha}\omega(R_0) + \tau^{-\delta}\sigma(\tau^j R_0)\sum_{\ell=1}^{j}\tau^{(\bar\alpha-\delta)(\ell-1)}.$$

Then, when $k > 1$, we have

$$\begin{aligned}\omega(\tau^k R_0) &\le \tau^{\bar\alpha}\omega(\tau^{k-1}R_0) + \sigma(\tau^{k-1}R_0)\\ &\le \tau^{\bar\alpha}\big[\tau^{(k-1)\bar\alpha}\omega(R_0) + \tau^{-\delta}\sigma(\tau^{k-1}R_0)\sum_{\ell=1}^{k-1}\tau^{(\bar\alpha-\delta)(\ell-1)}\big] + \tau^{-\delta}\sigma(\tau^k R_0)\\ &\le \tau^{k\bar\alpha}\omega(R_0) + \tau^{-\delta}\sigma(\tau^k R_0)\sum_{\ell=2}^{k}\tau^{(\bar\alpha-\delta)(\ell-1)} + \tau^{-\delta}\sigma(\tau^k R_0)\\ &= \tau^{k\bar\alpha}\omega(R_0) + \tau^{-\delta}\sigma(\tau^k R_0)\sum_{\ell=1}^{k}\tau^{(\bar\alpha-\delta)(\ell-1)}\\ &\le \tau^{k\bar\alpha}\omega(R_0) + \sigma(\tau^k R_0)\frac{1}{\tau^\delta - \tau^{\bar\alpha}}.\end{aligned}$$

Since $\tau^k R_0 < r/k < R_0$, we have $\sigma(\tau^k R_0) \le \sigma(r/\tau) \le \tau^{-\delta}\sigma(r)$. Then, as $\tau^k \le r/R_0$, we obtain

$$\omega(r) \le \omega(\tau^k R_0) \le C\Big[\tau^{k\bar\alpha}\omega(R_0) + \sigma(r)\Big] \le C\Big[\big(\frac{r}{R_0}\big)^{\bar\alpha}\omega(R_0) + \sigma(r)\Big]$$

where $C = \max\{\tau^{\bar\alpha}, \tau^{-\delta}(\tau^\delta - \tau^{\bar\alpha})^{-1}\}$. □

3.5 Summation by parts formula

Finally, we prepare a summation by parts formula, that is, Green's identity in discrete setting. For $\Lambda \subset \frac{1}{N}\mathbb{T}_N^n$, we denote the outer boundary and closure of Λ by

$$\partial_N^+\Lambda = \{\tfrac{x}{N} \in \tfrac{1}{N}\mathbb{T}_N^n \setminus \Lambda; \mathrm{dist}(\tfrac{x}{N}, \Lambda) = \tfrac{1}{N}\}, \quad \overline{\Lambda} = \Lambda \cup \partial_N^+\Lambda, \tag{3.52}$$

respectively.

Lemma 3.8. *Let $\Lambda \subset \frac{1}{N}\mathbb{T}_N^n$ and let $e \in \mathbb{Z}^n : |e| = 1$ be given. For functions F, G on $\overline{\Lambda}$ satisfying $F = 0$ at $\partial_N^+\Lambda$, we have*

$$\sum_{\frac{x}{N}\in\Lambda} F(\tfrac{x}{N})\nabla_e^{N,*}G(\tfrac{x}{N}) = \sum_{\frac{x}{N} \text{ or } \frac{x+e}{N}\in\Lambda} \nabla_e^N F(\tfrac{x}{N}) \cdot G(\tfrac{x}{N}), \tag{3.53}$$

where the sum in the right hand side is taken over x satisfying $\frac{x}{N} \in \Lambda$ or $\frac{x+e}{N} \in \Lambda$. If we replace $\nabla_e^{N,*}$ by ∇_e^N in the left hand side, ∇_e^N and $\frac{x+e}{N}$ in the right hand side should be $\nabla_e^{N,*}$ and $\frac{x-e}{N}$, respectively (since $\nabla_e^{N,*} = \nabla_{-e}^N$).

In particular, for $a = \{\bar{a}_x\}$, we have

$$\sum_{\frac{x}{N}\in\Lambda} F(\tfrac{x}{N})\nabla_e^{N,*}\big(\bar{a}_x G\big)(\tfrac{x}{N}) = \sum_{\frac{x}{N} \text{ or } \frac{x+e}{N}\in\Lambda} \bar{a}_x \nabla_e^N F(\tfrac{x}{N}) \cdot G(\tfrac{x}{N}). \tag{3.54}$$

Moreover, for $a = \{a_{x,e}\}$

$$\begin{aligned} &\sum_{\frac{x}{N}\in\Lambda} F(\tfrac{x}{N}) \sum_{|e|=1, e>0} \nabla_e^{N,*}\big(a_{x,e}\nabla_e^N G\big)(\tfrac{x}{N}) \\ &\quad = \sum_{\frac{x}{N} \text{ or } \frac{x+e}{N}\in\Lambda; |e|=1, e>0} a_{x,e}\nabla_e^N F(\tfrac{x}{N})\nabla_e^N G(\tfrac{x}{N}). \end{aligned} \tag{3.55}$$

Proof. The left hand side of (3.53) is rewritten as

$$\begin{aligned} &\sum_{\frac{x}{N}\in\Lambda} F(\tfrac{x}{N})N\Big\{G(\tfrac{x-e}{N}) - G(\tfrac{x}{N})\Big\} \\ &\quad = N\sum_{\frac{x+e}{N}\in\Lambda} F(\tfrac{x+e}{N})G(\tfrac{x}{N}) - N\sum_{\frac{x}{N}\in\Lambda} F(\tfrac{x}{N})G(\tfrac{x}{N}) \\ &\quad = \sum_{\frac{x}{N}, \frac{x+e}{N}\in\Lambda} \nabla_e^N F(\tfrac{x}{N})G(\tfrac{x}{N}) + \sum_{\frac{x}{N}\notin\Lambda, \frac{x+e}{N}\in\Lambda} NF(\tfrac{x+e}{N})G(\tfrac{x}{N}) - \sum_{\frac{x}{N}\in\Lambda, \frac{x+e}{N}\notin\Lambda} NF(\tfrac{x}{N})G(\tfrac{x}{N}) \\ &\quad = \sum_{\{\frac{x}{N}, \frac{x+e}{N}\in\Lambda\}\cup\{\frac{x}{N}\in\Lambda, \frac{x+e}{N}\notin\Lambda\}\cup\{\frac{x}{N}\notin\Lambda, \frac{x+e}{N}\in\Lambda\}} \nabla_e^N F(\tfrac{x}{N}) \cdot G(\tfrac{x}{N}). \end{aligned}$$

The last equality holds since $F(\frac{x}{N}) = 0$ if $\frac{x}{N} \notin \Lambda$ for the second term and $F(\frac{x+e}{N}) = 0$ if $\frac{x+e}{N} \notin \Lambda$ for the third term. This shows (3.53). The identity (3.54) is immediate from (3.53) by taking $\bar{G}(\frac{x}{N}) := \bar{a}_x G(\frac{x}{N})$ instead of $G(\frac{x}{N})$. For (3.55), first consider $\bar{G}(\frac{x}{N}) := a_{x,e}G(\frac{x}{N})$ instead of $G(\frac{x}{N})$ for each fixed e, and then take the sum in e. □

4 Schauder estimate for the first discrete derivatives

We are now ready to show the Schauder estimate for the first discrete derivatives. We consider the linear discrete PDE (1.6) under the assumption that the coefficients $a(t) = \{a_{x,e}(t)\}_{|e|=1}$ satisfy the α-Hölder continuity condition (A.2) stated below. By (2.17) in Corollary 2.3, the nonlinear discrete PDE (1.1) falls in this class with $\alpha = \sigma$.

We preview the main results. In Section 4.1, we study the equation (1.6) and state Theorem 4.1, which is an analog of Theorem 4.8 of [41]. Then, in Theorem 4.2, we improve the regularity at $t = 0$ for C^2-initial values. In Section 4.2, these results are applied for the nonlinear equation (1.1); see Corollaries 4.3 and 4.4. The outline of the proof of Theorem 4.1 is given in Section 4.1 and the main part of the proof is postponed to later. In Section 4.3, we show the discrete version of the interpolation inequality. In Section 4.4, we apply Proposition 3.4 and two basic lemmas (Lemmas 3.6 and 3.7 or its variant 4.9) together with the summation by parts formula (Lemma 3.8) to show a useful energy inequality and a main Hölder estimate. Finally, Section 4.5 gives an estimate on the time varying norm.

We will make use of the solution of the discrete heat equation (1.20) or (3.2), which will be denoted by v^N to distinguish it from the solution u^N of (1.6). We will compare u^N to v^N as in (4.31) or (4.34) and will apply the estimates for v^N obtained in Proposition 3.4.

4.1 Schauder estimate for (1.6) with Hölder continuous coefficients

We consider the linear discrete PDE (1.6) with coefficients $a(t)$ and $g(t)$ continuous in t, that is,

$$\partial_t u^N = L^N_{a(t)} u^N + g(t, \tfrac{x}{N}), \quad \tfrac{x}{N} \in \tfrac{1}{N}\mathbb{T}^n_N. \tag{4.1}$$

Our basic assumptions for $a(t)$ and $g(t)$ are:

(A.1) (symmetry, nondegeneracy, boundedness) $\quad a(t) \in \mathcal{A}(c_-, c_+)$, $t \geq 0$,

(A.2) (Hölder continuity) $\quad [a]^{*,N}_\alpha \leq A < \infty$, $\alpha \in (0,1)$,

(A.3) (boundedness of g) $\quad \|g\|_\infty \leq G_\infty < \infty$.

The assumption (A.2) means that $[a_e]^{*,N}_\alpha \leq A$ holds for each e and for the seminorm defined in (3.42) by regarding $a_{x,e}(t)$ as a function of $(t, \frac{x}{N}) \in \Omega_N$. Note that, by (2.17), the coefficient $a(t)$ determined from the nonlinear Laplacian as in (1.7) and (1.8) in the equation (1.1) satisfies (A.2) with $\alpha = \sigma$ and $A = C(K+1)$; see Corollary 4.3. The bound on $\|g\|_\infty$ in (A.3) is used in the proof of Lemma 4.11, but a weaker bound on $\sup_{t\in[0,T]} \|g(t)\|_{L^1(\mathbb{T}^n_N)}$ is sufficient at most places.

With respect to the solution $u^N(t, \frac{x}{N})$ of (4.1), define the polylinear spatial interpolation $\tilde{u}^N(t,z)$ following the prescription in (3.11).

Set also, for $\alpha \in (0,1)$,

$$U_{1+\alpha} := [\tilde{u}^N]^*_{1+\alpha} = \sup_{X \neq Y \in \Omega} (d(X) \wedge d(Y))^{1+\alpha} \frac{|\nabla^N \tilde{u}^N(X) - \nabla^N \tilde{u}^N(Y)|}{|X-Y|^\alpha},$$

$$
\begin{aligned}
(4.2)\qquad & U_1 := |\nabla^N \tilde{u}^N|_0^{(1)} \equiv \max_{|e|=1, e>0} |\nabla_e^N \tilde{u}^N|_0^{(1)}, \\
& \mathcal{U} := U_{1+\alpha} + \langle u^N \rangle_{1+\alpha}^{*,N},
\end{aligned}
$$

and the seminorm

$$
|\tilde{u}^N|_{1+\alpha}^* := U_1 + \mathcal{U} \equiv [\tilde{u}^N]_{1+\alpha}^* + \langle u^N \rangle_{1+\alpha}^{*,N} + |\nabla^N \tilde{u}^N|_0^{(1)}. \tag{4.3}
$$

Here, we may recall Lemma 3.5 for the consistency of two seminorms $\langle u^N \rangle_{1+\alpha}^{*,N}$ and $\langle \tilde{u}^N \rangle_{1+\alpha}^*$.

We now come to the main estimate of this section.

Theorem 4.1. *(First Schauder estimate, cf. Theorem 4.8 of [41]) For the equation* (1.6), *assume (A.1), (A.2), (A.3) and*

$$
\sup_N \|u^N\|_\infty \equiv \sup_N \sup_{\Omega_N} |u^N(t, \tfrac{x}{N})| < \infty. \tag{4.4}
$$

Then, we have

$$
|\tilde{u}^N|_{1+\alpha}^* \le C\big[(A+1)^{1+\frac{1}{\alpha}} \|u^N\|_\infty + G_\infty\big]. \tag{4.5}
$$

In particular, we have

$$
|\nabla_e^N u^N(t, \tfrac{x}{N})| \le C t^{-\frac{1}{2}} \big[(A+1)^{\frac{1}{\alpha}} \|u^N\|_\infty + (A+1)^{-1} G_\infty\big] \tag{4.6}
$$

Here, the constants $C = C(n, c_\pm, T, \alpha)$.

Proof. We outline the proof of Theorem 4.1, referring to results proved later.

We first prove (4.5). Recalling $\mathcal{U} = U_{1+\alpha} + \langle u^N \rangle_{1+\alpha}^{*,N}$, we show in Proposition 4.10 that

$$
U_{1+\alpha} = [\tilde{u}^N]_{1+\alpha}^* \le \bar{C}[(A+1)U_1 + G_\infty] + \delta \mathcal{U},
$$

for every $\delta > 0$, where $\bar{C} = \bar{C}(n, c_\pm, \alpha, \delta)$. Moreover, in Proposition 4.12, we estimate that

$$
\langle u^N \rangle_{1+\alpha}^{*,N} \le \widehat{C}[AU_1 + U_{1+\alpha} + G_\infty]
$$

where $\widehat{C} = \widehat{C}(n, c_+, T, \alpha)$. Thus, by these two estimates, we have

$$
\begin{aligned}
\mathcal{U} = U_{1+\alpha} + \langle u^N \rangle_{1+\alpha}^{*,N} &\le \widehat{C} A U_1 + (\widehat{C}+1) U_{1+\alpha} + \widehat{C} G_\infty \\
&\le [\widehat{C} + \bar{C}(\widehat{C}+1)](A+1)U_1 + \delta(\widehat{C}+1)\mathcal{U} + [\widehat{C} + \bar{C}(\widehat{C}+1)] G_\infty.
\end{aligned}
$$

Choosing $\delta > 0$ small, this implies

$$
\mathcal{U} \le C[(A+1)U_1 + G_\infty] \tag{4.7}
$$

and as $|\tilde{u}^N|_{1+\alpha}^* = U_1 + \mathcal{U}$ that

$$
|\tilde{u}^N|_{1+\alpha}^* \le \big(C(A+1)+1\big) U_1 + C G_\infty \tag{4.8}
$$

where $C = C(n, c_\pm, T, \alpha)$.

However, by the interpolation inequality proved in Proposition 4.5 in Section 4.3,

$$U_1 \le 3[\tilde u^N]_0^{\frac{\alpha}{1+\alpha}} \big([\tilde u^N]_0 + [\tilde u^N]^*_{1+\alpha}\big)^{\frac{1}{1+\alpha}}, \tag{4.9}$$

where we recall $[u]_0 = \operatorname{osc}_\Omega u$ in (3.14). Observe also that $[\tilde u^N]_0 \le 2|\tilde u^N|_0 = 2\|u^N\|_\infty$. Thus, from (4.9), we have that

$$\begin{aligned}
&\big(C(A+1)+1\big)U_1 \\
&\le \Big(2^{\frac{1}{\alpha}}\big(3\big[C(A+1)+1\big]\big)^{\frac{1+\alpha}{\alpha}} 2\|u^N\|_\infty\Big)^{\frac{\alpha}{1+\alpha}} \Big(\tfrac12\big(2\|u^N\|_\infty + U_{1+\alpha}\big)\Big)^{\frac{1}{1+\alpha}} \\
&\le 2^{\frac{1}{\alpha}}\big(3\big[C(A+1)+1\big]\big)^{\frac{1+\alpha}{\alpha}} 2\|u^N\|_\infty + \tfrac12\big(2\|u^N\|_\infty + |\tilde u^N|^*_{1+\alpha}\big),
\end{aligned} \tag{4.10}$$

where $C = C(n, c_\pm, T, \alpha)$ and we have used a trivial bound: $ab\,(\le \frac{a^p}{p} + \frac{b^q}{q}) \le a^p + b^q$ for $a, b > 0$ and $\frac1p + \frac1q = 1$. Inserting this into (4.8), we obtain (4.5).

The estimate (4.6) now follows from (4.10) and (4.5). □

When the initial value $u^N(0,\cdot)$ is C^2, or more precisely, if it satisfies the bound

$$\sup_N \|u^N(0)\|_{C^2_N} \le C_0 < \infty, \tag{4.11}$$

we expect a better regularity estimate at $t = 0$. Recall (1.10) for the norm $\|\cdot\|_{C^2_N}$. Indeed to discuss the equation (1.6), we need the following additional assumptions for the coefficient $a_{x,e}(t)$:

$$[a]_\alpha^{(-\alpha),N} \le B < \infty : \quad |a_{x_1,e}(t_1) - a_{x_2,e}(t_2)| \le B\Big\{\big|t_2 - t_1\big|^{\frac{\alpha}{2}} + \big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big|^\alpha\Big\}, \tag{4.12}$$

$$\sup_{N,x,e} |\nabla_e^{N,*} a_{x,e}(0)| \le C_1 < \infty. \tag{4.13}$$

The condition (4.12) is understood in relation to (A.2) for the seminorm $[a_e]_\alpha^{(-\alpha),N}$ for each e defined by (3.40) taking $a = \alpha, k = 0, b = -\alpha$ so that $a + b = 0$. Or $[a_e]_\alpha^{(-\alpha),N}$ is the seminorm $[a_e]_\alpha$ in (3.33) with the supremum taken for $X \ne Y \in \Omega_N$.

Theorem 4.2. *We assume (A.1), (A.3),* (4.4) *in Theorem 4.1 and* (4.11), (4.12), (4.13) *stated above. Then, for the solution u^N of* (1.6), *we have*

$$|\tilde u^N|_{1+\alpha} \le C[(B+1)^{1+\frac{1}{\alpha}}\|u^N\|_\infty + G_\infty + C_2], \tag{4.14}$$

where $|\tilde u^N|_{1+\alpha}$ is the unweighted norm defined in (3.35). *In particular, we have*

$$|\nabla_e^N u^N(t, \tfrac{x}{N})| \le C\big[(B+1)^{\frac{1}{\alpha}}\|u^N\|_\infty + (B+1)^{-1}\big(G_\infty + C_2\big)\big] \tag{4.15}$$

and the α-Hölder seminorm of $\nabla_e^N u^N$ is uniformly bounded in $t \ge 0$:

$$[\nabla_e^N \tilde u^N]_\alpha \le C[(B+1)^{1+\frac{1}{\alpha}}\|u^N\|_\infty + G_\infty + C_2]. \tag{4.16}$$

Here, $C = C(n, c_\pm, T, \alpha)$ and $C_2 = n(C_1 C_0 + c_+ C_0)$.

Proof. The proof is similar to that of Theorem 2.5, but to guarantee the condition (A.2) for the extended system (4.19) below, especially to well-connect at $t = 1$, we replace Δ^N by $L^{N,0}$ defined by

$$L^{N,0} = -\sum_{|e|=1,e>0} \nabla_e^{N,*}(a_{x,e}(0)\nabla_e^N \cdot).$$

Then, consider the discrete linear PDE,

$$\partial_s v = L^{N,0} v, \quad s \in (0,1], \tag{4.17}$$

with initial condition $v(0, \frac{x}{N}) = u^N(0, \frac{x}{N})$ for $x \in \mathbb{T}_N^n$. Define $\hat{v}(t) := v(1-t)$ for $0 \le t < 1$ and $\hat{h}(t, \frac{x}{N}) := -L^{N,0}\hat{v}(t, \frac{x}{N})$. Note, for $0 \le t < 1$, that $\hat{v}$ satisfies

$$\partial_t \hat{v} = -L^{N,0}\hat{v} = L^{N,0}\hat{v} + 2\hat{h}.$$

However, by (4.17), $h(t) := \hat{h}(1-t) = -L^{N,0}v(t)$ satisfies the discrete PDE $\partial_t h(t) = L^{N,0}h(t)$ with initial value $h(0) = -L^{N,0}u^N(0)$ and thus, by the maximum principle (Lemma 2.7) for this equation, we have

$$|h(t, \tfrac{x}{N})| \le \max_y |L^{N,0}u^N(0, \tfrac{y}{N})|.$$

But, by the condition (4.11) for $u^N(0)$ and by (4.13) for $a_{x,e}(0)$, we have, applying (2.9),

$$\begin{aligned}\|\hat{h}\|_\infty &= \Big\| \sum_{|e|=1,e>0} \Big\{ \nabla_e^{N,*} a_{\cdot,e}(0) \cdot \nabla_e^N u^N(0,\cdot) + a_{\cdot-e,e}(0)\nabla_e^{N,*}\nabla_e^N u^N(0,\cdot)\Big\}\Big\|_\infty \\ &\le n(C_1C_0 + c_+C_0) =: C_2.\end{aligned}$$

Define now

$$\hat{a}_{x,e}(t) = \begin{cases} a_{x,e}(t-1) & \text{for } t \ge 1 \\ a_{x,e}(0) & \text{for } 0 \le t < 1, \end{cases} \tag{4.18}$$

and $\hat{g}(t, \frac{x}{N})$ by (2.22) in the present setting. Consider the extended system, for $t \ge 0$,

$$\partial_t \hat{u}^N = L^N_{\hat{a}(t)}\hat{u}^N + \hat{g}(t, \tfrac{x}{N}). \tag{4.19}$$

Note that $\hat{a}$ satisfies (A.1) and $[\hat{a}]_\alpha^{*,N} \le T^{\frac{\alpha}{2}}B < \infty$ from the condition (4.12) (which holds also at $t_1, t_2 = 0$), and $\|\hat{g}\|_\infty \le \|g\|_\infty + 2C_2$. Also, by the maximum principle, v is uniformly bounded by $\|v(s)\|_\infty \le \|u^N(0)\|_\infty$, $s \in [0,1]$. Therefore, $\hat{u}^N$ is bounded by $\|u^N\|_\infty$. Then, Theorem 4.1 yields

$$|\widetilde{\hat{u}^N}|^*_{1+\alpha} \le C\big[(B+1)^{1+\frac{1}{\alpha}}\|u^N\|_\infty + G_\infty + 2C_2\big].$$

where $C = C(n, c_\pm, T, \alpha)$. We observe, by specializing to times $1 \le t \le T+1$ and noting $\hat{u}(t,\cdot) = u^N(t-1,\cdot)$, that (4.14) holds for the solution $u = u^N$ of (1.6). Similarly, (4.15) follows from (4.6). (4.16) is immediate from (4.14). $\square$

4.2 Schauder estimate in the context of (1.1)

We now consider the solution $u^N(t, \frac{x}{N})$ of the discrete nonlinear PDE (1.1). Recall $u_\pm$ in (1.2) and $c_\pm$ in (1.9) in this context.

The following is a corollary of Theorem 4.1.

Corollary 4.3. *For the solution $u^N(t, \frac{x}{N})$ of (1.1) satisfying (1.2), we have*

$$|\tilde{u}^N|^*_{1+\sigma} \le C(K^{1+\frac{1}{\sigma}} + 1),$$

where $\sigma \in (0,1)$ is as in Theorem 2.2, and $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|\varphi''\|_\infty, u_\pm)$. In particular, from the last term in the seminorm $|\tilde{u}^N|^*_{1+\sigma}$ in (4.3), noting $d(X) = t^{\frac{1}{2}}$, we obtain

$$|\nabla^N_e u^N(t, \tfrac{x}{N})| \le Ct^{-\frac{1}{2}}(K^{\frac{1}{\sigma}} + 1), \quad t \in (0, T].$$

From the first term in (4.3), that is $U_{1+\alpha}$ in (4.2), we see that the σ-Hölder seminorm of $\nabla^N_e u^N$ has the singularity $(t^{-\frac{1}{2}})^{1+\sigma} = t^{-\frac{1}{2}(1+\sigma)}$ near $t = 0$.

Proof. We first observe that $g = Kf(u^N)$ is such that $\|g\|_\infty \le \|f\|_\infty K := G_\infty$ by (1.2). Next, we have that $a(t) := a(u^N(t))$, that is $a_{x,e}(u)$ in (1.8) taking $u = u^N(t)$, satisfies (A.1) and by Corollary 2.3, $[a]^{*,N}_\sigma \le C(n, c_\pm, T, \|u^N(0)\|_\infty, \|f\|_\infty, \|\varphi''\|_\infty)(K+1) := A$. The condition (4.4) follows from (1.2). Therefore, the corollary is a consequence of Theorem 4.1. □

We have the following better regularity estimate at $t = 0$ assuming (4.11). Recall the constant C_0 in (4.11).

Corollary 4.4. *For the solution u^N of (1.1) satisfying (1.2), when the initial data satisfies (4.11), we have*

$$|\tilde{u}^N|_{1+\sigma} \le C(K + C_0 + 1)^{1+\frac{1}{\sigma}}, \tag{4.20}$$

where $\sigma \in (0,1)$ is as in Theorem 2.2 and $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|\varphi''\|_\infty, u_\pm)$. Recall $\|f\|_\infty$ and $\|\varphi''\|_\infty$ defined in Corollary 2.3. In particular,

$$|\nabla^N_e u^N(t, \tfrac{x}{N})| \le C(K + C_0 + 1)^{\frac{1}{\sigma}} \tag{4.21}$$

and the σ-Hölder seminorm of $\nabla^N_e u^N$ is uniformly bounded in $t \ge 0$,

$$[\nabla^N_e \tilde{u}^N]_\sigma \le C(K + C_0 + 1)^{1+\frac{1}{\sigma}}. \tag{4.22}$$

Proof. Note that the condition (4.12) follows from (2.25) in Corollary 2.6 with

$$B = C(K\|f\|_\infty + \|u^N(0)\|_\infty + C_0)$$

where $C = C(n, c_\pm, T, \|\varphi''\|_\infty)$. Moreover, since $a_{x,e}(0) = a_{x,e}(u^N(0))$ is determined by (1.8), it satisfies (A.1) and the condition (4.13), $|\nabla^{N,*}_e a_{x,e}(0)| \le C_1$ where $C_1 = C(\|\varphi''\|_\infty)C_0$, due to the 'mean-value' Lemma 2.4 and (4.11). Thus, with $\|g\|_\infty =$

$K\|f\|_\infty = G_\infty$ and $C_2 = n(C_1C_0 + c_+C_0) \le C(n, c_+, \|\varphi''\|_\infty)[C_0^2+1]$, the corollary follows from Theorem 4.2. Note that we obtain the bound

$$|\tilde{u}^N|_{1+\sigma} \le C\big((K + C_0 + 1)^{1+\frac{1}{\sigma}} + C_0^2 + 1\big)$$

and the others. However, since $\sigma \in (0,1)$, both C_0^2 and 1 are bounded by $(K+C_0+1)^{1+\frac{1}{\sigma}}$ so that the desired three estimates (4.20)–(4.22) are shown. □

4.3 Interpolation inequality

We adapt the (4.2c) in Proposition 4.1 of [41] to our discrete context. Recall (3.31) for the continuous space gradient $\nabla^N u(t,z)$ defined for a function u on $\Omega = [0,T] \times \mathbb{T}^n$.

Proposition 4.5. *(cf. (4.2c) in Proposition 4.1 of [41]) Let $\alpha \in (0,1)$. Then, for every function u on Ω,*

$$U_1 := |\nabla^N u|_0^{(1)} \Big(= \max_{|e|=1, e>0} |\nabla_e^N u|_0^{(1)}\Big) \le 3[u]_0^{\frac{\alpha}{1+\alpha}} \big([u]_0 + [u]^*_{1+\alpha}\big)^{\frac{1}{1+\alpha}}. \tag{4.23}$$

Moreover,

$$U_2 := |\nabla^N u|_0^{(2)} \;\le\; 5(|u|_0^{(1)})^{\frac{\alpha}{1+\alpha}} \big(|u|_0^{(1)} + [u]^{(1)}_{1+\alpha}\big)^{\frac{1}{1+\alpha}}. \tag{4.24}$$

Proof. We argue now the first statement and later discuss the second inequality.

Consider $\nabla_e^N u(X)$ for $X = (t,z) \in \Omega$ for fixed e. For simplicity, let us assume that e is the vector $\langle 1, 0, \dots, 0\rangle$. We will write $z = \langle z_1, z_2, \dots, z_n\rangle \in \mathbb{T}^n$.

Let $Y = (t,y)$ be such that $y = (y_1, z_2, z_3, \dots, z_n\rangle$ and $z_1 - y_1$ has the same sign as $\nabla_e^N u(X) \in \mathbb{R}$. Take Y also such that $|z-y| = |z_1 - y_1| = \epsilon d(X)$ for an $\epsilon \in (0, \frac{1}{2}]$. We note that in this argument that $t>0$ is fixed, and so $d(X) = d(Y) = \sqrt{t}$.

Then, since $z_1 - y_1$ has the same sign as $\nabla_e^N u(X)$ and $|z-y| = |z_1 - y_1|$, we have

$$\begin{aligned} |\nabla_e^N u(X)| &= \nabla_e^N u(X) \cdot \frac{z_1 - y_1}{|z-y|} \\ &= \Big(\frac{1}{|z-y|}\int_{z_1}^{y_1} \nabla_e^N u(t,w) dw_1\Big) \cdot \frac{z_1-y_1}{|z-y|} \\ &\quad + \Big(\nabla_e^N u(X) - \frac{1}{|z-y|}\int_{z_1}^{y_1} \nabla_e^N u(t,w)dw_1\Big) \cdot \frac{z_1-y_1}{|z-y|}, \end{aligned} \tag{4.25}$$

where $w = \langle w_1, z_2, \dots, z_n\rangle$. However, the above integral term can be rewritten as

$$\begin{aligned} \int_{z_1}^{y_1} \nabla_e^N u(t,w) dw_1 &= N \int_{z_1}^{y_1} [u(t, w + \tfrac{e}{N}) - u(t,w)] dw_1 \\ &= N \Big\{ \int_{z_1 + \frac{1}{N}}^{y_1 + \frac{1}{N}} u(t,w) dw_1 - \int_{z_1}^{y_1} u(t,w) dw_1 \Big\} \\ &= N \int_{z_1}^{z_1 + \frac{1}{N}} \big[u(t, w + y - z) - u(t,w)\big] dw_1, \end{aligned}$$

Thus, by (4.25) and recalling $[u]_0 = \operatorname*{osc}_{\Omega}(u)$ and $[u]^*_{1+\alpha}$ in (3.37), (3.42), we have

$$
\begin{aligned}
(4.26)\qquad |\nabla^N_e u(X)| &\le \frac{N}{|z-y|}\int_{z_1}^{z_1+\frac{1}{N}} \big|u(t,w+y-z)-u(t,w)\big| dw_1 \\
&\quad + \Big(\frac{1}{|z-y|}\int_{z_1}^{y_1}[\nabla^N_e u(X)-\nabla^N_e u(t,w)]dw_1\Big)\cdot\frac{z_1-y_1}{|z-y|} \\
&\le \frac{[u]_0}{|z-y|}+\frac{[u]^*_{1+\alpha}}{|z-y|}\Big|\int_{z_1}^{y_1}(d(X)\wedge d(W))^{-(1+\alpha)}|z-w|^\alpha dw_1\Big|,
\end{aligned}
$$

where $W=(t,w)$. As remarked earlier, as $t>0$ is fixed, $d(X)=d(W)=\sqrt{t}$ for the points considered above. Also, $\big|\int_{z_1}^{y_1}|z-w|^\alpha dw_1\big| \le \frac{1}{1+\alpha}|z-y|^{1+\alpha}\le |z-y|^{1+\alpha}$.

From these computations, we have

$$
\begin{aligned}
|\nabla^N_e u(X)| &\le \frac{[u]_0}{|z-y|}+d(X)^{-(1+\alpha)}[u]^*_{1+\alpha}|z-y|^\alpha \\
&= \frac{[u]_0}{\epsilon d(X)}+\frac{\epsilon^\alpha [u]^*_{1+\alpha}}{d(X)}.
\end{aligned}
$$

Multiply by $d(X)$ and take supremum in X to get

$$U_1 \le \frac{[u]_0}{\epsilon}+\epsilon^\alpha[u]^*_{1+\alpha}.$$

We would like to choose $\epsilon\in(0,\frac12]$ as follows: If $[u]_0<[u]^*_{1+\alpha}$, then take $\epsilon=\frac12([u]_0/[u]^*_{1+\alpha})^{1/(1+\alpha)}(<\frac12)$. We get in this case

$$
\begin{aligned}
U_1 &\le 2\big([u]^*_{1+\alpha}\big)^{1/(1+\alpha)}[u]_0^{\alpha/(1+\alpha)}+2^{-\alpha}[u]_0^{\alpha/(1+\alpha)}\big([u]^*_{1+\alpha}\big)^{1/(1+\alpha)} \\
&\le 3\big([u]^*_{1+\alpha}\big)^{1/(1+\alpha)}[u]_0^{\alpha/(1+\alpha)},
\end{aligned}
$$

which is bounded by the right hand side of (4.23). Otherwise, when $[u]_0\ge[u]^*_{1+\alpha}$, choose $\epsilon=1/2$. Then,

$$U_1\le 2[u]_0+2^{-\alpha}[u]^*_{1+\alpha}\le 3[u]_0,$$

which is also bounded by the right hand side of (4.23). Thus, the first inequality (4.23) follows.

The second inequality (4.24) follows the same scheme. Indeed, in (4.26), one bounds $|u(t,w+y-z)-u(t,w)|\le|u(t,w+y-z)|+|u(t,w)|\le 2d(X)^{-1}|u|_0^{(1)}$ and $|\nabla^N_e u(X)-\nabla^N_e u(t,w)|\le[u]^{(1)}_{1+\alpha}d(X)^{-(2+\alpha)}|z-w|^\alpha$, recall (3.36) for $|u|_0^{(1)}$. Multiplying through by $d^2(X)=(\sqrt{t})^2$ at this point, we may follow the derivation of the first inequality to obtain the second statement. □

4.4 Energy inequalities and estimate on $U_{1+\alpha}=[\tilde{u}^N]^*_{1+\alpha}$

For an open domain $D\subset\mathbb{T}^n$, we define its discrete interior by

$$D_N := D\cap \tfrac{1}{N}\mathbb{T}^n_N. \qquad (4.27)$$

The outer boundary $\partial_N^+ D_N$ and the closure $\overline{D_N}$ of D_N are defined as in (3.52) taking $\Lambda = D_N$, respectively. Recall $\Omega = [0,T]\times\mathbb{T}^n$ and $\Omega_N = [0,T]\times \frac{1}{N}\mathbb{T}_N^n$.

Take $Y = (t_1, y) \in \Omega$ and $r : 0 < r < \frac{1}{2}d(Y) = \frac{1}{2}t_1^{\frac{1}{2}}$. Recall $Q(r) \equiv Q(Y,r) = (t_1 - r^2, t_1)\times D(y,r)$, where $D(y,r) = \{z \in \mathbb{T}^n; |z-y| < r\}$ with the distance $|z-y|$ as in (2.13), and set $Q_N(r) \equiv Q_N(Y,r) = (t_1 - r^2, t_1)\times D_N(y,r)\,(= Q(r)\cap\Omega_N)$, where $D_N(y,r)$ is the discrete interior of $D(y,r)$. Define the parabolic outer boundary of $Q_N(Y,r)$ by

$$\mathcal{P}_N^+ Q_N(Y,r) := \{t_1 - r^2\}\times\overline{D_N(y,r)} \cup (t_1 - r^2, t_1]\times\partial_N^+ D_N(y,r), \tag{4.28}$$

where $\partial_N^+ D_N(y,r)$ and $\overline{D_N(y,r)}$ are the outer boundary and closure of $D_N(y,r)$ defined as above, respectively. We also denote

$$\overline{Q_N(Y,r)} := [t_1 - r^2, t_1]\times\overline{D_N(y,r)}. \tag{4.29}$$

In the following, Y is fixed until it moves in the proof of Proposition 4.10. All constants will be uniform in Y.

Let $u = u^N$ be the solution of the linear discrete PDE (4.1) or equivalently (1.6) on Ω_N:

$$\mathcal{L}_{a(\cdot)}u \equiv (L_{a(t)}^N - \partial_t)u = -g(t)$$

with $a(t)$ and $g(t)$ satisfying the assumptions (A.1), (A.2) and (A.3).

Take the closest point $\tilde{y} \in \frac{1}{N}\mathbb{T}_N^n$ to y, in particular, $|y-\tilde{y}| \le \frac{\sqrt{n}}{2N}$ holds, and set $\tilde{Y} := (t_1, \tilde{y}) \in \Omega_N$. Note that $d(\tilde{Y}) = d(Y) = t_1^{\frac{1}{2}}$. Let $v = v^N = v^{N,Q(r)}$ be the solution of the discrete heat equation (3.2) or equivalently (1.20) on $Q_N(r) = Q_N(Y,r)$:

$$\mathcal{L}_a v \equiv (\Delta_a^N - \partial_t)v = 0 \tag{4.30}$$

with constant coefficients $a_e := a_e(\tilde{Y}) \equiv a_{N\tilde{y},e}(t_1) \ge c_- > 0$ under the boundary condition $v = u$ at $\mathcal{P}_N^+ Q_N(Y,r)$. We will consider the case that $r > \frac{\sqrt{n}}{N}$ (in the proof of Lemma 4.8), in particular, $Q_N(r) \ne \emptyset$.

Set

$$w \equiv w^N := u - v \equiv u^N - v^N \quad \text{on } \overline{Q_N(Y,r)}. \tag{4.31}$$

Then, the following discrete energy inequality holds.

Lemma 4.6. *Assume $0 < r < \frac{1}{2}d(Y) = \frac{1}{2}t_1^{\frac{1}{2}}$ and $Q_N(r) \ne \emptyset$. Then, we have*

$$\begin{aligned} \int_{t_1-r^2}^{t_1} N^{-n} \sum_{\frac{x}{N} \text{ or } \frac{x+e}{N}\in D_N(y,r);|e|=1,e>0} |\nabla_e^N w|^2(X)dt \\ \le CA^2\Big(\frac{r+\frac{1+\sqrt{n}}{N}}{d(Y)}\Big)^{2\alpha}(r+\tfrac{1}{N})^n r^2 \sup_{X \text{ or } X+\frac{e}{N}\in Q_N(r);|e|=1,e>0} |\nabla_e^N u(X)|^2 \\ + CG_\infty(r+\tfrac{1}{N})^n\Big(\frac{r}{d(Y)}\Big)^{1+\alpha}\sup_{X\in Q_N(r)}|w(X)|, \end{aligned} \tag{4.32}$$

where $X = (t, \frac{x}{N})$ and $X + \frac{e}{N} = (t, \frac{x+e}{N})$, and recall $Q_N(r) \equiv Q_N(Y,r)$. Here, the constants $C = C(n, c_-, T)$.

Proof. The proof is divided into steps.

Step 1. First from $\partial_t w = L^N_{a(t)} u + g(t) - \Delta^N_a v$ on $Q_N(r)$, and then applying (3.55) in Lemma 3.8 since $w(t,\cdot)=0$ at the boundary $\partial^+_N D_N(y,r)$, we have

$$\begin{aligned}
&\tfrac{1}{2}\partial_t\Big(N^{-n} \sum_{\frac{x}{N}\in D_N(y,r)} w^2(X)\Big) \\
&= N^{-n} \sum_{\frac{x}{N}\in D_N(y,r); |e|=1, e>0} w(X)\big(-\nabla_e^{N,*}(a_e(\tilde{Y})\nabla_e^N w)\big)(X) \\
&\quad + N^{-n} \sum_{\frac{x}{N}\in D_N(y,r); |e|=1, e>0} w(X)\big(-\nabla_e^{N,*}((a_e(X)-a_e(\tilde{Y}))\nabla_e^N u)\big)(X) \\
&\quad + N^{-n} \sum_{\frac{x}{N}\in D_N(y,r)} w(X) g(X) \\
&= -N^{-n} \sum_{*} a_e(\tilde{Y}) |\nabla_e^N w|^2(X) \\
&\quad - N^{-n} \sum_{*} (a_e(X)-a_e(\tilde{Y}))\nabla_e^N w(X)\cdot \nabla_e^N u(X) \\
&\quad + N^{-n} \sum_{\frac{x}{N}\in D_N(y,r)} w(X) g(X) \\
&=: -I_1 + I_2 + I_3,
\end{aligned}$$

where $X=(t,\frac{x}{N})$, $a_e(X):=a_{x,e}(t)$ and $\sum_*$ means the sum over x and e such that $\frac{x}{N}$ or $\frac{x+e}{N}\in D_N(y,r)$ and $|e|=1, e>0$. Note that we put a minus sign in I_1 so that $I_1\ge 0$.

Step 2. Integrate both sides in $t\in[t_1-r^2,t_1]$. Since $w(t_1-r^2,\cdot)=0$, the left hand side is

$$\tfrac{1}{2} N^{-n} \sum_{\frac{x}{N}\in D_N(y,r)} w^2(t_1,\tfrac{x}{N}) \ge 0.$$

On the other hand, for every $\varepsilon>0$,

$$\begin{aligned}
\int_{t_1-r^2}^{t_1} |I_2| dt \le \varepsilon \int_{t_1-r^2}^{t_1} N^{-n} \sum_* |\nabla_e^N w|^2(X) dt \\
+ \frac{1}{\varepsilon}\int_{t_1-r^2}^{t_1} N^{-n}\sum_* |a_e(X)-a_e(\tilde{Y})|^2 |\nabla_e^N u|^2(X) dt.
\end{aligned}$$

Here, we estimate $|\nabla_e^N u|^2(X) \le \sup_{X \text{ or } X+\frac{e}{N}\in Q_N(r)} |\nabla_e^N u|^2$, and by $[a]_\alpha^{*,N}\le A$,

$$|a_e(X)-a_e(\tilde{Y})| \le A(d(X)\wedge d(\tilde{Y}))^{-\alpha}|X-\tilde{Y}|^\alpha \le A2^\alpha d(Y)^{-\alpha}(r+\tfrac{1+\sqrt{n}}{N})^\alpha,$$

since $|X-\tilde{Y}|\le |X-Y|+|Y-\tilde{Y}| \le (r+\frac{1}{N})+\frac{\sqrt{n}}{2N}\le r+\frac{1+\sqrt{n}}{N}$ for $X\in\overline{Q_N(r)}$ (note that, when $X+\frac{e}{N}\in Q_N(r)$, it happens that $X\in\overline{Q_N(r)}$) and $r<\frac12 d(Y)=\frac12 d(\tilde{Y})$ implies $d(X)\ge \frac12 d(Y)$ (note the time direction is not enlarged by $\frac{1}{N}$) as

$$d(Y)=t_1^{\frac12}\le t^{\frac12}+|t-t_1|^{\frac12}\le d(X)+r\le d(X)+\tfrac12 d(Y), \tag{4.33}$$

so that $d(X) \wedge d(\tilde{Y}) = d(X) \geq \frac{1}{2}d(Y)$. Moreover,

$$\int_{t_1-r^2}^{t_1} N^{-n} \sum_{\frac{x}{N} \text{ or } \frac{x+e}{N} \in D_N(y,r); |e|=1, e>0} 1dt \leq C(n)(r+\tfrac{1}{N})^n r^2.$$

Note that, even for very small $r > 0$, $D_N(y,r)$ may contain a single point so that we pick up the factor $+\frac{1}{N}$. Therefore, recalling $a_e(\tilde{Y}) \geq c_- > 0$, we obtain

$$\begin{aligned}\int_{t_1-r^2}^{t_1} |I_2| dt \leq & \frac{\varepsilon}{c_-}\int_{t_1-r^2}^{t_1} I_1 dt \\ & + \frac{1}{\varepsilon}C(n)2^{2\alpha}A^2\Big(\frac{r+\frac{1+\sqrt{n}}{N}}{d(Y)}\Big)^{2\alpha}(r+\tfrac{1}{N})^n r^2 \sup_{X \text{ or } X+\frac{e}{N} \in Q_N(r); |e|=1, e>0} |\nabla_e^N u|^2.\end{aligned}$$

Step 3. For I_3, since $|Q_N(r)| \leq C(n)N^n(r+\frac{1}{N})^n r^2$ (we pick up $+\frac{1}{N}$ similarly as above), by $\|g\|_\infty \leq G_\infty$,

$$\begin{aligned}\int_{t_1-r^2}^{t_1} |I_3| dt &\leq G_\infty C(n)(r+\tfrac{1}{N})^n r^2 \sup_{Q_N(r)} |w| \\ &= cG_\infty (r+\tfrac{1}{N})^n \big(\tfrac{r}{d(Y)}\big)^{1+\alpha}\big(\tfrac{r}{d(Y)}\big)^{1-\alpha} d(Y)^2 \sup_{Q_N(r)} |w| \\ &\leq (\tfrac{1}{2})^{1-\alpha} TC(n)G_\infty (r+\tfrac{1}{N})^n \big(\tfrac{r}{d(Y)}\big)^{1+\alpha} \sup_{Q_N(r)} |w|,\end{aligned}$$

since $\frac{r}{d(Y)} \leq \frac{1}{2}$ and $d(Y)^2 \leq T$. Summarizing these estimates, we have

$$\begin{aligned}(1-\tfrac{\varepsilon}{c_-})\int_{t_1-r^2}^{t_1} I_1 dt \leq & \frac{1}{\varepsilon}C(n)2^{2\alpha}A^2\Big(\frac{r+\frac{1+\sqrt{n}}{N}}{d(Y)}\Big)^{2\alpha}(r+\tfrac{1}{N})^n r^2 \sup_{X \text{ or } X+\frac{e}{N} \in Q_N(r); |e|=1, e>0} |\nabla_e^N u|^2 \\ & + (\tfrac{1}{2})^{1-\alpha} TC(n)G_\infty (r+\tfrac{1}{N})^n \big(\tfrac{r}{d(Y)}\big)^{1+\alpha} \sup_{Q_N(r)} |w|.\end{aligned}$$

Choosing $\varepsilon > 0$ small and noting $a_e(\tilde{Y}) \geq c_- > 0$, we have shown (4.32). $\square$

We now give estimates on three terms in (4.32) in terms of polylinear interpolations. First recall that $\tilde{u} = \tilde{u}^N(t,z), (t,z) \in \Omega$ was defined as the polylinear interpolation of $u = u^N(t, \frac{x}{N})$ in Section 4.1.

Next, taking $r_N^1 = r + \frac{\sqrt{n}+1}{N}$, consider the solution $v \equiv v^{N,r}(X) := v^{N,Q(Y,r_N^1)}(X)$, $X \in \overline{Q_N(Y,r_N^1)}$ of the discrete heat equation (4.30) on $Q_N(Y,r_N^1)$ with the boundary condition $v = u$ at $\mathcal{P}_N^+ Q_N(Y,r_N^1)$. Then, set as in (4.31)

$$w \equiv w^{N,r} := u - v \equiv u^N - v^{N,r} \quad \text{on } \overline{Q_N(Y,r_N^1)}. \tag{4.34}$$

We consider $v = v^{N,r}$ on a domain $Q_N(Y,r_N^1)$ slightly enlarging $Q_N(Y,r)$. By this choice, the polylinear interpolations $\tilde{v} = \tilde{v}^{N,r}(X)$ and $\tilde{w} = \tilde{w}^{N,r}(X)$ of v and w, respectively, are well-defined for $X \in Q(r) \equiv Q(Y,r)$. Moreover, $\nabla_e^N \tilde{v}(X)$ and $\nabla_e^N \tilde{w}(X)$ are also defined for $X \in Q(r)$ and $\widetilde{\nabla_e^N w} = \nabla_e^N \tilde{w}$ holds on $Q(r)$ from (3.46); recall the discussions above

Corollary 3.3 and Proposition 3.4. Furthermore, the estimate (3.25) in Proposition 3.4 is applicable for $v = v^{N,r}(X)$ by taking $R = r_N^1$ so that $0 < \rho < r \le r_N^1 - \frac{\sqrt{n}+1}{N} = r$.

Recall (4.2) for U_1 and $\mathcal{U}$ defined from $\tilde{u}^N$ and u^N.

Lemma 4.7. *Assume* $0 < r < \frac{1}{2}d(Y) = \frac{1}{2}t_1^{\frac{1}{2}}$. *Then, we have*

$$\int_{Q(r)} |\widetilde{\nabla^N w}|^2(X)dX \le \int_{t_1-r^2}^{t_1} N^{-n} \sum_{\frac{x}{N}\in D_N(y,r+\frac{\sqrt{n}}{N});|e|=1,e>0} |\nabla_e^N w|^2(t,\tfrac{x}{N})dt, \tag{4.35}$$

$$\sup_{X \text{ or } X+\frac{e}{N}\in Q_N(r);|e|=1,e>0} |\nabla_e^N u(X)|^2 \le \frac{4U_1^2}{d(Y)^2}, \tag{4.36}$$

$$\sup_{X\in Q_N(r)} |w(X)| \le C(n)\mathcal{U}(\tfrac{r}{d(Y)})^{1+\alpha}. \tag{4.37}$$

Note that, in the left hand side of (4.35), differently from $X = (t, \frac{x}{N})$ in its right hand side, that $X = (t,z) \in \Omega$ is a continuous variable. Also recall $|\widetilde{\nabla^N w}(X)| := \max_{|e|=1,e>0} |\widetilde{\nabla_e^N w}(X)|$.

Proof. To show (4.35), noting from (3.11) (with $\nabla_e^N w$ instead of u^N) that $\widetilde{\nabla_e^N w}$ is a convex combination of $\nabla_e^N w$ at neighboring sites, we observe

$$|\widetilde{\nabla_e^N w}(t,z)| \le \max_{v\in\{0,1\}^n} \big|\nabla_e^N w\big(t, \tfrac{[Nz]+v}{N}\big)\big|. \tag{4.38}$$

However, since $\big|z - \frac{[Nz]+v}{N}\big| \le \frac{\sqrt{n}}{N}$, we have $\frac{[Nz]+v}{N} \in D_N(y, r+\frac{\sqrt{n}}{N})$ for $z \in D(y,r)$. Thus, (4.38) proves (4.35).

To show (4.36), since $\nabla_e^N u(X) = \nabla_e^N \tilde{u}(X)$ for $X = (t, \frac{x}{N})$ with $x \in \mathbb{T}_N^n$, recalling (4.2) for U_1, bound for each e that

$$\begin{aligned}\sup_{X \text{ or } X+\frac{e}{N}\in Q_N(r)} |\nabla_e^N u(X)| &\le \sup_{X\in\Omega:d(X)\ge(d(Y)^2-r^2)^{1/2}} |\nabla_e^N \tilde{u}(X)| \\ &\le U_1 \sup_{d(X)\ge(d(Y)^2-r^2)^{1/2}} d(X)^{-1}.\end{aligned}$$

However, as we saw in (4.33) in the proof of Lemma 4.6, $d(X) \ge \frac{1}{2}d(Y)$ and this shows (4.36).

To demonstrate (4.37), we divide $|w(X)|$ into two terms when $D_N(r) \equiv D_N(y,r)$ is non-empty, as otherwise the estimate holds trivially. For $X = (t, \frac{x}{N}), Z = (t', \frac{z}{N}) \in Q_N(r)$,

$$|w(X)| \le |u(X) - u(Z) - \nabla^N u(Z)\cdot \tfrac{x-z}{N}| + |v(X) - u(Z) - \nabla^N u(Z)\cdot\tfrac{x-z}{N}|,$$

where $\cdot$ means the inner product in $\mathbb{R}^n$. The first term, writing $u = u^N$, is further bounded from above by

$$|u^N(X) - u^N(X')| + |u^N(X') - u^N(Z) - \nabla^N u^N(Z)\cdot\tfrac{x-z}{N}|$$

$$\le \langle u^N\rangle^{*,N}_{1+\alpha}(2r)^{1+\alpha}\big(d(X)\wedge d(X')\big)^{-(1+\alpha)} + \tfrac{1}{N}\sum_j |\nabla^N u^N(Z_j) - \nabla^N u^N(Z)|$$
$$\le 2^{2+2\alpha}\langle u^N\rangle^{*,N}_{1+\alpha}\big(\tfrac{r}{d(Y)}\big)^{1+\alpha} + 4^{1+\alpha}\sqrt{n}\,[\tilde u^N]^*_{1+\alpha}(r+\tfrac{\sqrt{n}}{N})^{1+\alpha}\cdot d(Y)^{-(1+\alpha)}$$

where X' is taken as $X'=(t',\frac{x}{N})$ for $X=(t,\frac{x}{N})$ and $Z=(t',\frac{z}{N})$, and $\{Z_j=(t',\frac{z_j}{N})\}$ is a sequence of points in $Q_N(r)$ connecting X' and Z by moving along in nearest-neighbor steps. The sum over j consists of at most $|x-z|_{L^\infty}$ so that $\sqrt{n}|x-z|$ terms. Note that $d(X)\wedge d(X')\ge \frac12 d(Y)$, $d(Z_j)\wedge d(Z)\ge \frac12 d(Y)$ from (4.33) and $|Z_j-Z|^\alpha \le \big(2(r+\frac{\sqrt{n}}{N})\big)^\alpha$ for $Z_j, Z\in Q_N(r)$ (or even $\in \overline{Q_N(r)}$).

When there are two distinct space points $\frac{x}{N},\frac{z}{N}\in D_N(r)$, necessarily $\frac{1}{N}<r$. When there is only one space point in $D_N(r)$, we have $x=z$ in the above sequence. Hence, covering both cases and recalling $\mathcal{U}=[\tilde u^N]^*_{1+\alpha}+\langle u^N\rangle^{*,N}_{1+\alpha}$, we bound the first term, noting $\alpha<1$, as

$$|u(X)-u(Z)-\nabla^N u(Z)\cdot \tfrac{x-z}{N}| \le C(n)\mathcal{U}\big(\tfrac{r}{d(Y)}\big)^{1+\alpha}.$$

The second term $V(X):=v^N(X)-u^N(Z)-\nabla^N u^N(Z)\cdot\frac{x-z}{N}$ on $Q_N(r)$, noting the boundary value is the same as the first term, by applying the maximum principle for the discrete heat equation (3.2), it has the same bound as the first term. Therefore, we have

$$|w|\le 2C(n)\mathcal{U}(\tfrac{r}{d(Y)})^{1+\alpha},$$

on $Q_N(r)$. This shows (4.37). □

We now combine the bounds in Lemmas 4.6, 4.7 and also Proposition 3.4 to obtain the following estimate. Recall $Q(\cdot)=Q(Y,\cdot)$.

Lemma 4.8. *For $\rho\in(0,r)$, when $r>0$ satisfies $r_N=r+\frac{1+2\sqrt{n}}{N}<\frac12 d(Y)$, we have*

$$\omega(\rho)\le \bar C\big(\tfrac{\rho}{r}\big)^{n+4}\omega(r)+\sigma(r_N), \tag{4.39}$$

where

$$\begin{aligned}\omega(r) &= \int_{Q(r)} |\widetilde{\nabla^N u}-\{\widetilde{\nabla^N u}\}_r|^2 dX,\\ \sigma(r) &= \widehat C[A^2U_1^2+G_\infty\mathcal{U}]r^{n+2+2\alpha}d(Y)^{-(2+2\alpha)}\end{aligned} \tag{4.40}$$

and $\bar C=\bar C(n,c_\pm)$, $\widehat C=\widehat C(n,c_\pm,T)$.

Proof. Recall that, setting $r_N^1=r+\frac{\sqrt{n}+1}{N}$ for each $r>0$, we consider the solution $v=v^N\equiv v^{N,r}$ of the discrete heat equation (4.30) on $Q_N(r_N^1)$ with boundary condition $v^N=u^N$ at $\mathcal{P}_N^+Q_N(r_N^1)$ and set $w\equiv w^{N,r}=u^N-v^{N,r}$ on $\overline{Q_N(r_N^1)}$ as in (4.34).

Then, from (4.35) in Lemma 4.7, then applying Lemma 4.6 (with r replaced by $r+\frac{\sqrt{n}}{N}$) and (4.36), (4.37) in Lemma 4.7 (with r replaced by $r+\frac{\sqrt{n}}{N}$), we obtain

$$\int_{Q(r)} |\widetilde{\nabla^N w}|^2(X)dX \le \bar C[A^2U_1^2+G_\infty\mathcal{U}]r_N^n(\tfrac{r_N}{d(Y)})^{2+2\alpha}, \tag{4.41}$$

for all $r > 0$ such that $r_N = r + \frac{1+2\sqrt{n}}{N} < \frac{1}{2}d(Y)$ where $\bar{C} = \bar{C}(n, c_-, T)$.

Moreover, v solves the discrete heat equation (3.2) on $Q_N(r_N^1)$. Thus, from (3.25) in Proposition 3.4 (with $R = r_N^1$), we have

$$\int_{Q(\rho)} |\widetilde{\nabla_e^N v} - \{\widetilde{\nabla_e^N v}\}_\rho|^2 dX \le \widehat{C}\big(\tfrac{\rho}{r}\big)^{n+4} \int_{Q(r)} |\widetilde{\nabla_e^N v} - \{\widetilde{\nabla_e^N v}\}_r|^2 dX, \tag{4.42}$$

for $\rho \in (0, r)$ where $\widehat{C} = \widehat{C}(n, c_\pm)$.

Now, let us show the estimate (4.39) for $\omega(\rho)$ and $\rho \in (0, r)$. We first rewrite u in the integrand $|\widetilde{\nabla^N u} - \{\widetilde{\nabla^N u}\}_\rho|^2$ in $\omega(\rho)$ as $u = w + v \equiv w^{N,r} + v^{N,r}$ (note that we take w and v those determined from r and not by ρ) and estimate it by $3\big(|\widetilde{\nabla^N w}|^2 + \{\widetilde{\nabla^N w}\}_\rho^2 + |\widetilde{\nabla^N v} - \{\widetilde{\nabla^N v}\}_\rho|^2\big)$. Then, we have

$$\begin{aligned}
\omega(\rho) \le & 3 \int_{Q(\rho)} |\widetilde{\nabla^N w}|^2 dX + 3\{\widetilde{\nabla^N w}\}_\rho^2 \cdot |Q(\rho)| + 3 \int_{Q(\rho)} |\widetilde{\nabla^N v} - \{\widetilde{\nabla^N v}\}_\rho|^2 dX \\
\le & 6 \int_{Q(r)} |\widetilde{\nabla^N w}|^2 dX + 3\widehat{C}\big(\tfrac{\rho}{r}\big)^{n+4} \int_{Q(r)} |\widetilde{\nabla^N v} - \{\widetilde{\nabla^N v}\}_r|^2 dX \\
\le & 6\bar{C}[A^2 U_1^2 + G_\infty \mathcal{U}] r_N^n \big(\tfrac{r_N}{d(Y)}\big)^{2+2\alpha} \\
& \qquad + 3\widehat{C}\big(\tfrac{\rho}{r}\big)^{n+4} \Big[3 \int_{Q(r)} |\widetilde{\nabla^N w}|^2 dX + 3\{\widetilde{\nabla^N w}\}_r^2 \cdot |Q(r)| + 3\omega(r)\Big] \\
\le & 6\bar{C}[A^2 U_1^2 + G_\infty \mathcal{U}] r_N^n \big(\tfrac{r_N}{d(Y)}\big)^{2+2\alpha} \\
& \qquad + 3\widehat{C}\big(\tfrac{\rho}{r}\big)^{n+4} \cdot 6\bar{C}[A^2 U_1^2 + G_\infty \mathcal{U}] r_N^n \big(\tfrac{r_N}{d(Y)}\big)^{2+2\alpha} + 9\widehat{C}\big(\tfrac{\rho}{r}\big)^{n+4} \omega(r).
\end{aligned}$$

Here, for the second inequality, we have estimated $\{\widetilde{\nabla^N w}\}_\rho^2 \le \frac{1}{|Q(\rho)|} \int_{Q(\rho)} |\widetilde{\nabla^N w}|^2 dX$ and then simply enlarged the domain of integral to $Q(r)$ for the first and second terms. We also used (4.42) for the last term. For the third inequality, we have used (4.41) for the first term and a similar estimate to the first inequality for the second term. For the fourth inequality, we use (4.41) again. Finally, estimating $\big(\frac{\rho}{r}\big)^{n+4} \le 1$ in the second term, we obtain (4.39). $\square$

We now apply Lemma 3.7 in our setting. This iteration lemma improves $\sigma((R_0)_N)$, in (4.39) with $r = R_0$, to $\sigma(r_N)$.

Lemma 4.9. *Assume* (4.39) *in Lemma 4.8. Then, we have*

$$\omega(r) \le C\Big[\big(\tfrac{r}{R_0}\big)^{n+2+2\alpha} \omega(R_0) + \sigma(r_N)\Big],$$

for $0 < r \le R_0$*, where* $R_0 < \frac{1}{2}d(Y) - \frac{1+2\sqrt{n}}{N}$*,* $r_N = r + \frac{1+2\sqrt{n}}{N}$ *and* $C = C(n, c_\pm)$*.*

Proof. We may check that $\sigma(r_N)$ satisfies the condition (3.50) in place of $\sigma(r)$, that is,

$$r^{-\delta}\sigma(r_N) \le s^{-\delta}\sigma(s_N) \qquad \text{if } 0 < s \le r \le R_0, \tag{4.43}$$

where $\delta = n+2+2\alpha$, $\bar{\alpha} \in (\delta, \beta)$, $\beta = n+4$ (note $\alpha \le 1$), $\sigma(r) = \widehat{C}(n, c_\pm, T) r^\delta$ from (4.40) and $r_N = r + \frac{c}{N}$. Indeed,

$$r^{-\delta}\sigma(r_N) = r^{-\delta}(r + \tfrac{c}{N})^\delta = (1 + \tfrac{c}{Nr})^\delta$$

is decreasing in r.

Thus, the conclusion follows by applying Lemma 3.7 and the comment after it: In our situation, $(4.15)'$ of [41] holds with $\beta = n+4$, that is $\omega(\tau r) \le C\tau^\beta \omega(r) + \sigma(r_N)$ for all small τ and $C = C(n, c_\pm)$. Then, (3.51) in Lemma 3.7 holds for any $\delta < \bar{\alpha} < \beta$, choosing $\tau \in (0,1)$ such that $C\tau^\beta < \tau^{\bar{\alpha}}$.

Note that the increasing property of $\sigma(r_N)$ is clear, while that of $\omega(r)$ follows by showing $\omega'(r) \ge 0$. Indeed, ω is increasing in its integral domain $Q(r)$ and another term coming from the derivative of $\{\nabla_e^N \tilde{u}\}_r$ in r vanishes: $\int_{Q(r)} \big(\nabla_e^N \tilde{u} - \{\nabla_e^N \tilde{u}\}_r\big) dX \cdot \partial_r \{\nabla_e^N \tilde{u}\}_r = 0$. $\square$

We are at the position to give the estimate on $U_{1+\alpha}$. It is important that the coefficient δ of $\mathcal{U}$ in the estimate (4.44) can be made arbitrary small. This is because our estimate is shown in terms of $\sqrt{\mathcal{U}}$.

Proposition 4.10. *We have*

$$U_{1+\alpha} = [\tilde{u}^N]^*_{1+\alpha} \le C[(A+1)U_1 + G_\infty] + \delta\mathcal{U} \tag{4.44}$$

for all small $\delta > 0$ with some $C = C(n, c_\pm, T, \delta)$.

Proof. We apply Lemma 4.9 to get

$$\omega(r) \le C\big(\tfrac{r}{R_0}\big)^{n+2+2\alpha}\omega(R_0) + C[A^2U_1^2 + G_\infty\mathcal{U}] r_N^{n+2+2\alpha} d(Y)^{-(2+2\alpha)}, \tag{4.45}$$

for $0 < r \le R_0$, taking $R_0 = \frac{1}{3}d(Y)$, if $\frac{1}{3}d(Y) < \frac{1}{2}d(Y) - \frac{1+2\sqrt{n}}{N}$, that is, if $d(Y) > \frac{6(1+2\sqrt{n})}{N}$ holds. Recall $r_N = r + \frac{c}{N}$ with $c = 1 + 2\sqrt{n}$ (later in the proof we may take c larger). Here, $C = C(n, c_\pm, T)$.

However, $\omega(R_0)$ is bounded as

$$\begin{aligned} \omega(R_0) &\le 2\int_{Q(R_0)} |\nabla^N \tilde{u}|^2 dX + 2|Q(R_0)|\, \{\nabla^N \tilde{u}\}^2_{R_0} \\ &\le 4\int_{Q(R_0)} |\nabla^N \tilde{u}|^2 dX \\ &\le C(n)\, U_1^2 R_0^{n+2}\, d(Y)^{-2}. \end{aligned}$$

Here, the first line is from the definition of $\omega(R_0)$ in (4.40), the second is by Schwarz's inequality applied for the second term, and the third is from (4.36) taking $r = R_0$.

Thus, first by Schwarz's inequality and then recalling $R_0 = \frac{1}{3}d(Y)$ so that the first term in the right hand side of (4.45) is bounded by the second term with A^2 replaced by $A^2 + 1$, we have

$$\int_{Q(r)} |\nabla_e^N \tilde{u} - \{\nabla_e^N \tilde{u}\}_r| dX \le \sqrt{|Q(r)|}\sqrt{\omega(r)} \tag{4.46}$$

$$\le C\sqrt{(A^2+1)U_1^2 + G_\infty \mathcal{U}}\, d(Y)^{-(1+\alpha)} r_N^{n+2+\alpha}$$
$$\le \left\{C'[(A+1)U_1 + G_\infty] + \delta\mathcal{U}\right\} d(Y)^{-(1+\alpha)} r_N^{n+2+\alpha},$$

for every $\delta > 0$ with some $C' = C'(n, c_\pm, T, \delta) > 0$, for $0 < r \le \frac{1}{3}d(Y)$, if $d(Y) > \frac{6(1+2\sqrt{n})}{N}$. We have estimated $C\sqrt{G_\infty \mathcal{U}} \le \delta\mathcal{U} + \frac{C^2}{\delta}G_\infty$ as usual.

In the case that $d(Y) \le \frac{6(1+2\sqrt{n})}{N}$, we can apply (4.51) in Lemma 4.11 stated below by making $c > 0$ in r_N larger if necessary. Indeed, by (4.51), we can estimate for $0 < r \le \frac{1}{3}d(Y)$

$$\text{(4.47)} \quad \int_{Q(r)} |\nabla_e^N \tilde{u} - \{\nabla_e^N \tilde{u}\}_r| dX \le \int_{Q(r)} \frac{dX}{|Q(r)|} \int_{Q(r)} |\nabla_e^N \tilde{u}(X) - \nabla_e^N \tilde{u}(Z)| dZ$$
$$\le C(n, c_+) r^{n+2} \cdot r^\alpha \big(U_{1+\alpha} + U_1 A + G_\infty\big) d(Y)^{-(1+\alpha)}.$$

Here, note that $|X - Z| \le 2r \le \frac{c_1}{N}$ with $c_1 = 4(1+2\sqrt{n})$ and also $d(X), d(Z) \ge \frac{\sqrt{8}}{3}d(Y)$ for $X, Z \in Q(r)$. However, from $r \le \frac{1}{3}d(Y) \le \frac{c_2}{N}$ with $c_2 = 2(1+2\sqrt{n})$, one can obtain

$$\text{(4.48)} \qquad r^{n+2+\alpha} \le \left(\tfrac{c_2}{c_2+c}\right)^{2+n+\alpha} r_N^{2+n+\alpha} \le \tfrac{\delta}{C}\, r_N^{2+n+\alpha},$$

for every $\delta > 0$, since $\frac{c_2}{c_2+c}$ can be small by making c large enough.

From (4.46)–(4.48), recalling $U_{1+\alpha} \le \mathcal{U}$, we have shown (without any restriction on $d(Y)$)

$$\text{(4.49)} \quad \int_{Q(Y,r)} |\nabla_e^N \tilde{u} - \{\nabla_e^N \tilde{u}\}_r| dX \le \left\{C[(A+1)U_1 + G_\infty] + \delta\mathcal{U}\right\} d(Y)^{-(1+\alpha)} r_N^{n+2+\alpha},$$

for $0 < r \le \frac{1}{3}d(Y)$ and any $Y = (t_1, y) \in \Omega$, where $C = C(n, c_\pm, T, \delta)$.

Now we apply Lemma 3.6. We fix any $X_0 = (t_0, z_0) \in \Omega$ and take $R = \frac{1}{4}d(X_0) = \frac{1}{4}\sqrt{t_0}$. Then, (4.49) shows that the condition (3.47) in Lemma 3.6 holds for any $Y \in Q(X_0, R)$, $r \in (0, R)$ with

$$F = \nabla_e^N \tilde{u},\ U_F = U_{1+\alpha},\ G(Y, r) = \{\nabla_e^N \tilde{u}\}_r$$
$$\text{and}\ \ H = \{C[(A+1)U_1 + G_\infty] + \delta\mathcal{U}\}\, d(Y)^{-(1+\alpha)}.$$

Indeed, for $Y = (t_1, y) \in Q(X_0, R)$, we have $0 < t_0 - t_1 < R^2 = \frac{1}{16}t_0$ so that $\sqrt{t_1} > \frac{\sqrt{15}}{4}\sqrt{t_0}$ and, combined with $r < R = \frac{1}{4}\sqrt{t_0}$, this shows $r < \frac{1}{4}\frac{4}{\sqrt{15}}\sqrt{t_1} = \frac{1}{\sqrt{15}}\sqrt{t_1} < \frac{1}{3}d(Y)$ so that (4.49) is applicable.

Therefore, by Lemma 3.6, we see that, for every $\delta > 0$ and $M > 0$, there exists $C = C_{\delta,M}(n, c_\pm, T) > 0$ such that

$$\text{(4.50)} \quad |\nabla_e^N \tilde{u}(Y) - \nabla_e^N \tilde{u}(Y_1)| \le \Big(C[(A+1)U_1 + G_\infty] + \delta\mathcal{U} + \delta U_{1+\alpha}\Big)$$
$$\times \big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)} |Y - Y_1|^\alpha,$$

holds if $Y, Y_1 \in Q(X_0, R)$ satisfy $|Y - Y_1| \ge \frac{1}{MN}$.

The case $|Y - Y_1| \le \frac{1}{MN}$ is covered by (4.52) in Lemma 4.11 stated below, and we obtain (4.50) for any $Y, Y_1 \in Q(X_0, R)$.

Thus, moving X_0, we have shown (4.50) for any $Y, Y_1 \in \Omega$. This implies the concluding estimate (4.44) on $U_{1+\alpha}$ by recalling the definition of $U_{1+\alpha}$ in (4.2), $U_{1+\alpha} \le \mathcal{U}$ and by changing $\delta > 0$ to $\frac{1}{2}\delta$. $\square$

Remark 4.1. *At this point, we comment that estimation in terms of the solution v^N to the discrete heat equation* (3.2) *was an important device to derive the power $n+4$ in the bounds in Lemma 4.8 and the bound* (4.44)*, leading to the desired α-Hölder continuity of $\nabla_e^N \tilde{u}$.*

The following Hölder estimate (4.52) for $\nabla_e^N \tilde{u}$ in the short distance regime $|Y - Y_1| \le \frac{1}{MN}$ was used in the proof of Proposition 4.10. This complements the Hölder estimate for $|Y - Y_1| \ge \frac{1}{MN}$ obtained by applying Lemma 3.6. It is essential that the front factor (especially that of $U_{1+\alpha}$) can be taken arbitrary small by choosing $M \ge 1$ large enough. By the polylinearity, at least in spatial directions, the function is Lipschitz continuous and therefore, in view of the Hölder estimate, the front factor can be made small in the short distance regime.

Lemma 4.11. (1) *If $|Y - Y_1| \le \frac{c_1}{N}$, we have*

$$|\nabla_e^N \tilde{u}(Y) - \nabla_e^N \tilde{u}(Y_1)| \le C\big(U_{1+\alpha} + U_1 A + G_\infty\big)\big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)} |Y - Y_1|^\alpha, \tag{4.51}$$

for some $C = C(n, c_+, c_1) > 0$.
(2) *Furthermore, for every $\delta > 0$, there exists $M = M(n, c_+, \alpha) \ge 1$ such that*

$$|\nabla_e^N \tilde{u}(Y) - \nabla_e^N \tilde{u}(Y_1)| \le \delta\big(U_{1+\alpha} + U_1 A + G_\infty\big)\big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)} |Y - Y_1|^\alpha, \tag{4.52}$$

holds if $|Y - Y_1| \le \frac{1}{MN}$.

Proof. We now show (4.52). For (4.51), we may take $\frac{1}{M} = c_1$ in the proof.

Step 1. (spatial direction) Let $Y = (t, z^{(1)})$ and $Y_1 = (t, z^{(2)})$ have the same t coordinate. We will assume first that $z^{(1)} = (z_i^{(1)})_{i=1}^n$ and $z^{(2)} = (z_i^{(2)})_{i=1}^n$ belong to the same $\frac{1}{N}$-box, so that $\frac{[Nz^{(1)}]}{N} = \frac{[Nz^{(2)}]}{N} =: \bar{z} = (\bar{z}_i)_{i=1}^n \in \frac{1}{N}\mathbb{T}_N^n$ are common, and also that $z_j^{(1)} = z_j^{(2)} =: z_j$ except $j = i$. Recall $\nabla_e^N \tilde{u}(Y) = \widetilde{\nabla_e^N u}(Y)$ as in (3.46) so that

$$\nabla_e^N \tilde{u}(z) = \sum_{v \in \{0,1\}^n} \vartheta^N(v, z) \nabla_e^N u\big(\bar{z} + \tfrac{v}{N}\big),$$

from (3.11), where we have now only specified the spatial variable. Then,

$$\begin{aligned} &\nabla_e^N \tilde{u}(z^{(1)}) - \nabla_e^N \tilde{u}(z^{(2)}) \\ &\quad = \sum_{v \in \{0,1\}^n} \big(\vartheta^N(v_i, z_i^{(1)}) - \vartheta^N(v_i, z_i^{(2)})\big) \prod_{j \ne i} \vartheta^N(v_j, z_j) \nabla_e^N u\big(\bar{z} + \tfrac{v}{N}\big). \end{aligned}$$

However,

$$\sum_{v_i = 0,1} \vartheta^N(v_i, z_i^{(1)}) \nabla_e^N u\big(\bar{z} + \tfrac{v}{N}\big)$$

$$= N(z_i^{(1)} - \bar z_i)\nabla_e^N u\big(\bar z + \tfrac{e_i+\hat v_i}{N}\big) + (1 - N(z_i^{(1)} - \bar z_i))\nabla_e^N u\big(\bar z + \tfrac{\hat v_i}{N}\big),$$

where $\hat v_i$ is defined below (3.12) from v. Therefore, noting $|\vartheta^N(v_j, z_j)| \le 1$, we have

$$\begin{aligned}
&\big|\nabla_e^N \tilde u(z^{(1)}) - \nabla_e^N \tilde u(z^{(2)})\big| \\
&\quad \le \sum_{v\in\{0,1\}^{n-1}} \Big| N(z_i^{(1)} - z_i^{(2)})\nabla_e^N u\big(\bar z + \tfrac{e_i+\hat v_i}{N}\big) - N(z_i^{(1)} - z_i^{(2)})\nabla_e^N u\big(\bar z + \tfrac{\hat v_i}{N}\big)\Big| \\
&\quad = \sum_{v\in\{0,1\}^{n-1}} N|z^{(1)} - z^{(2)}| \Big|\nabla_e^N u\big(\bar z + \tfrac{e_i+\hat v_i}{N}\big) - \nabla_e^N u\big(\bar z + \tfrac{\hat v_i}{N}\big)\Big| \\
&\quad \le 2^{n-1} N|z^{(1)} - z^{(2)}| \cdot U_{1+\alpha}(\tfrac{1}{N})^\alpha d(Y)^{-(1+\alpha)}.
\end{aligned}$$

Here, recalling $|Y - Y_1| \le \frac{1}{MN}$, we estimate

$$|z^{(1)} - z^{(2)}| = |z^{(1)} - z^{(2)}|^\alpha \cdot |z^{(1)} - z^{(2)}|^{1-\alpha} \le |z^{(1)} - z^{(2)}|^\alpha (\tfrac{1}{MN})^{1-\alpha},$$

and obtain

$$\big|\nabla_e^N \tilde u(z^{(1)}) - \nabla_e^N \tilde u(z^{(2)})\big| \le |z^{(1)} - z^{(2)}|^\alpha (\tfrac{1}{M})^{1-\alpha} \cdot 2^{n-1} U_{1+\alpha} d(Y)^{-(1+\alpha)}. \tag{4.53}$$

When $z^{(1)}$ and $z^{(2)}$ belong to different $\frac{1}{N}$-boxes, we consider the segment connecting these two points, divided into pieces belonging to the same $\frac{1}{N}$-boxes and apply the above result. Thus, the desired Hölder estimate (4.52) in the spatial direction is shown by taking M large, as $1 - \alpha > 0$.

Step 2. (temporal direction) Let $Y = (t, z)$ and $Y_1 = (s, z)$; we may assume $z = \frac{x}{N} \in \frac{1}{N}\mathbb{T}_N^n$. By the equation (1.6) or equivalently (4.1), for $X = (t, z)$, we have

$$\partial_t \nabla_e^N u(X) = -\tfrac12 \sum_{e':|e'|=1} \nabla_e^N \nabla_{e'}^{N,*}\big(a_{x,e'}(t)\nabla_{e'}^N u\big)(X) + \nabla_e^N g(X).$$

We apply the following rough estimates to the right hand side: By (2.9) applied for $\nabla_{e'}^{N,*}$ and (A.1)–(A.3),

$$\begin{aligned}
|\nabla_e^N \nabla_{e'}^{N,*}\big(a_{x,e'}(t)\nabla_{e'}^N u\big)(X)| &\le 2N^2 \Big\{ c_+ \sup_{X'} |\nabla_{e'}^N u(X' - \tfrac{e'}{N}) - \nabla_{e'}^N u(X')| \\
&\qquad + U_1 d(X)^{-1} \sup_{X'=(t,x')} |a_{x'-e',e'}(t) - a_{x',e'}(t)| \Big\} \\
&\le 2N^2 \Big\{ c_+ U_{1+\alpha} + U_1 A \Big\} (\tfrac{1}{N})^\alpha d(X)^{-(1+\alpha)},
\end{aligned}$$

and $|\nabla_e^N g(X)| \le 2NG_\infty$. Therefore, we have

$$\begin{aligned}
|\nabla_e^N u(t,z) - \nabla_e^N u(s,z)| \le& 2nN^{2-\alpha}|t-s| \Big\{ c_+ U_{1+\alpha} + U_1 A \Big\} \big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)} \\
&+ 2N|t-s|G_\infty.
\end{aligned}$$

Here, we estimate

$$|t-s| = |t-s|^{\frac{\alpha}{2}} |t-s|^{1-\frac{a}{2}} \le |t-s|^{\frac{\alpha}{2}} (\tfrac{1}{MN})^{2-\alpha},$$

by noting $|t-s| \le (\frac{1}{MN})^2$ from $|Y - Y_1| \le \frac{1}{MN}$. Thus, we obtain

$$|\nabla_e^N u(t,z) - \nabla_e^N u(s,z)|$$
$$\le |t-s|^{\frac{\alpha}{2}} (\tfrac{1}{M})^{2-\alpha} \Big[2n \Big\{ c_+ U_{1+\alpha} + U_1 A \Big\} \big(d(Y) \wedge d(Y_1) \big)^{-(1+\alpha)} + 2G_\infty \Big].$$

The desired Hölder estimate (4.52) in the temporal direction is shown by taking M large, as $2 - \alpha > 0$.

The proof of the lemma is completed by combining the results obtained in Steps 1 and 2. □

4.5 Estimate on $\langle u^N \rangle^{*,N}_{1+\alpha}$

The purpose of this subsection is to provide an estimate on $\langle u^N \rangle^{*,N}_{1+\alpha}$ used in the proof of Theorem 4.1. The argument, although different in our discrete setting, is inspired by that of Theorem 4.8 (p. 58) of [41]. Recall (4.2) for $U_1, U_{1+\alpha}$ and the assumptions (A.1)–(A.3) for $a(t)$ and $g(t)$, especially for the constants A and G_∞.

Proposition 4.12. *We have*

$$\langle u^N \rangle^{*,N}_{1+\alpha} \le C[AU_1 + U_{1+\alpha} + G_\infty]$$

where $C = C(n, c_+, T, \alpha)$.

Proof. Step 1. Let $X = (s, y/N)$ and $Y = (t, y/N)$ be given with $y \in \mathbb{T}_N^n$ and $0 \le t < s \le T$ such that $s - t = r^2$, $0 < r < \frac{1}{2} d(Y)$. Let

$$S_N(r,y) = \big\{ x \in \mathbb{T}_N^n : |\tfrac{x}{N} - \tfrac{y}{N}|_{L^\infty} \le r \big\}$$

be a square with center $\frac{y}{N}$ with width r. Note that $|S_N(r,y)| = (2[Nr]+1)^n$, recall that $[Nr]$ denotes the integer part of Nr. Define, for $y \in \mathbb{T}_N^n$ and $t \in [0,T]$,

$$U(t) = \frac{1}{|S_N(r,y)|} \sum_{x \in S_N(r,y)} u^N(t, \tfrac{x}{N}),$$

and divide

$$|u^N(Y) - u^N(X)| \le |u^N(t, \tfrac{y}{N}) - U(t)| + |U(t) - U(s)| + |U(s) - u^N(s, \tfrac{y}{N})|. \tag{4.54}$$

We will develop bounds for each of these terms.

Step 2. First, we rewrite the second term of (4.54) as

$$\begin{aligned} U(s) - U(t) &= \frac{1}{|S_N(r,y)|} \sum_{x \in S_N(r,y)} \int_t^s \partial_q u^N(q, \tfrac{x}{N}) dq \\ &= \frac{1}{|S_N(r,y)|} \sum_{x \in S_N(r,y)} \int_t^s \Big\{ - \sum_{|e|=1, e>0} \nabla_e^{N,*} \big(a_{x,e}(q) \nabla_e^N u^N \big) \big(q, \tfrac{x}{N}\big) + g\big(q, \tfrac{x}{N}\big) \Big\} dq \end{aligned} \tag{4.55}$$

$$
\begin{aligned}
&= \frac{N}{|S_N(r,y)|} \sum_{|e|=1, e>0} \int_t^s \Big\{ \sum_{x\in S_N(r,y)} a_{x,e}(q) \nabla_e^N u^N\big(q, \tfrac{x}{N}\big) \\
&\qquad\qquad - \sum_{x\in S_N(r,y-e)} a_{x,e}(q) \nabla_e^N u^N\big(q, \tfrac{x}{N}\big) \Big\} dq \\
&\quad + \frac{1}{|S_N(r,y)|} \sum_{x\in S_N(r,y)} \int_t^s g\big(q, \tfrac{x}{N}\big) dq \\
&=: \sum_{|e|=1, e>0} M_e^1 + M^2,
\end{aligned}
$$

where we use the equation (4.1) for the second line and then recall $\nabla_e^{N,*} = \nabla_{-e}^N$ for the third line.

One can write $a_{x,e}(q) = a_{y,e}(q) + \big(a_{x,e}(q) - a_{y,e}(q)\big)$ and insert so that

$$
\begin{aligned}
M_e^1 &= \frac{N}{|S_N(r,y)|} \int_t^s \Big\{ \sum_{x\in S_N(r,y)} \big(a_{x,e}(q) - a_{y,e}(q)\big) \nabla_e^N u^N\big(q, \tfrac{x}{N}\big) \\
&\qquad - \sum_{x\in S_N(r,y-e)} \big(a_{x,e}(q) - a_{y,e}(q)\big) \nabla_e^N u^N\big(q, \tfrac{x}{N}\big) \Big\} dq \\
&\quad + \frac{N}{|S_N(r,y)|} \int_t^s a_{y,e}(q) \Big\{ \sum_{x\in S_N(r,y)} \nabla_e^N u^N\big(q, \tfrac{x}{N}\big) - \sum_{x\in S_N(r,y-e)} \nabla_e^N u^N\big(q, \tfrac{x}{N}\big) \Big\} dq \\
&=: J_{1,e} + J_{2,e}.
\end{aligned}
$$

Further, as the difference of sums of $\nabla_e^N u^N\big(q, \frac{y}{N}\big)$, which does not depend on x, over $x \in S_N(r, y-e)$ and $x \in S_N(r,y)$ vanishes, we may rewrite $J_{2,e}$ as

$$
\begin{aligned}
J_{2,e} = \frac{N}{|S_N(r,y)|} \int_t^s a_{y,e}(q) \Big\{ &\sum_{x\in S_N(r,y)} \big[\nabla_e^N u^N\big(q, \tfrac{x}{N}\big) - \nabla_e^N u^N\big(q, \tfrac{y}{N}\big)\big] \\
- &\sum_{x\in S_N(r,y-e)} \big[\nabla_e^N u^N\big(q, \tfrac{x}{N}\big) - \nabla_e^N u^N\big(q, \tfrac{y}{N}\big)\big] \Big\} dq.
\end{aligned}
$$

Step 3. Note that the size of the symmetric difference satisfies

$$
|S_N(r,y) \Delta S_N(r, y-e)| = 2\big(2[Nr]+1\big)^{n-1}.
$$

We now see that, recalling $s - t = r^2$,

$$
|J_{1,e}| \le \tfrac{2}{2[Nr]+1} A U_1 N r^2 \big(\sqrt{n}(r + \tfrac{1}{N})\big)^\alpha d(Y)^{-(1+\alpha)},
$$

since, for $x \in S_N(r, y-e) \cup S_N(r,y)$, we have $|x-y|_{L^\infty} \le r + \frac{1}{N}$ so that $|a_{x,e}(q) - a_{y,e}(q)| \le A\big(\sqrt{n}(r + \frac{1}{N})\big)^\alpha d(Y)^{-\alpha}$ noting that $d((q, \frac{x}{N})) \ge d(Y)$ for $q \in [t,s]$, and also

$$
|\nabla_e^N u^N\big(q, \tfrac{x}{N}\big)| \le |\nabla_e^N u^N|_0^{(1),N} d(Y)^{-1} = U_1 d(Y)^{-1},
$$

by (3.43) and (3.46) in Lemma 3.5.

On the other hand, for $J_{2,e}$, since $0 < a_{y,e}(q) \le c_+$ and also

$$\big|\nabla_e^N u^N\big(q, \tfrac{x}{N}\big) - \nabla_e^N u^N\big(q, \tfrac{y}{N}\big)\big| \le U_{1+\alpha} d(Y)^{-(1+\alpha)} |\tfrac{x}{N} - \tfrac{y}{N}|^\alpha,$$

by (3.44) in Lemma 3.5, we observe that

$$|J_{2,e}| \le \tfrac{2}{2[Nr]+1} c_+ U_{1+\alpha} N r^2 \big(\sqrt{n}(r + \tfrac{1}{N})\big)^\alpha d(Y)^{-(1+\alpha)}.$$

Hence, if $r > \frac{1}{N}$,

$$|M_e^1| = |J_{1,e} + J_{2,e}| \le 2\big(AU_1 + c_+ U_{1+\alpha}\big) \frac{Nr^2 d(Y)^{-(1+\alpha)}}{[Nr]+1} (2\sqrt{n} r)^\alpha.$$

Since $[Nr] + 1 \ge Nr$, further

$$|M_e^1| \le 2(2\sqrt{n})^\alpha \big(AU_1 + c_+ U_{1+\alpha}\big) r^{1+\alpha} d(Y)^{-(1+\alpha)}. \tag{4.56}$$

However, if $r \le \frac{1}{N}$, as $[Nr] + 1 \ge 1$ and $0 < \alpha < 1$, then

$$\frac{N\big(\sqrt{n}(r + \frac{1}{N})\big)^\alpha r^2}{[Nr]+1} \le (2\sqrt{n})^\alpha N^{1-\alpha} r^2 \le (2\sqrt{n})^\alpha r^{1+\alpha}.$$

Hence, in this case also we recover the same estimate on M_e^1 in (4.56).

Moreover, as $\alpha < 1$, $d(Y) \le \sqrt{T}$ and $r^2 \le T$ so that $r^{1-\alpha} \le \sqrt{T}^{1-\alpha}$, we bound the term M^2 by

$$\begin{aligned} |M^2| &\le \frac{1}{|S_N(r,y)|} \sum_{x \in S_N(r,y)} \Big| \int_t^s g\big(q, \tfrac{x}{N}\big) dq \Big| \\ &\le G_\infty r^2 \ \le \ G_\infty r^{1+\alpha} d(Y)^{-(1+\alpha)} T^{1+\alpha}. \end{aligned} \tag{4.57}$$

Hence, from (4.55), (4.56) and (4.57), we obtain the bound

$$|U(t) - U(s)| \le C(AU_1 + c_+ U_{1+\alpha} + G_\infty) \big(\tfrac{r}{d(Y)}\big)^{1+\alpha} \tag{4.58}$$

where $C = C(n, T, \alpha)$.

Step 4. For the first and third terms of (4.54), noting that $\sum_{x \in S_N(r,y)} (\frac{y}{N} - \frac{x}{N}) = 0$, write

$$\begin{aligned} &|u^N(t, \tfrac{y}{N}) - U(t)| \\ &\le \Big| \frac{1}{|S_N(r,y)|} \sum_{x \in S_N(r,y)} \big| u^N(t, \tfrac{y}{N}) - u^N(t, \tfrac{x}{N}) - \nabla^N u^N(t, \tfrac{y}{N}) \cdot (\tfrac{y}{N} - \tfrac{x}{N}) \big|. \end{aligned}$$

When $r < \frac{1}{N}$, as there is a single point in $S_N(r,y)$, the above display vanishes.

When $r \ge \frac{1}{N}$ and there are multiple points in $S_N(r,y)$, as in the proof of (4.37) in Lemma 4.7, consider a path $\{(t, \frac{z_j}{N})\} \subset \{t\} \times \frac{1}{N} S_N(r,y)$ which moves from $(t, \frac{y}{N})$ to $(t, \frac{x}{N})$ in nearest-neighbor steps in $\{t\} \times S_N(r,y)$ with at most $\sqrt{n}|x-y|$ terms. Then,

$$|u^N(t, \tfrac{y}{N}) - u^N(t, \tfrac{x}{N}) - \nabla^N u^N(t, \tfrac{y}{N}) \cdot (\tfrac{y}{N} - \tfrac{x}{N})| \le \tfrac{1}{N} \sum_j |\nabla^N u^N(t, \tfrac{z_j}{N}) - \nabla^N u^N(t, \tfrac{y}{N})|$$

and, as $\frac{1}{N}|x-y| \le r$, we have by (3.44) in Lemma 3.5 that

$$|u^N(t, \tfrac{y}{N}) - U(t)| \le C(n)[\tilde{u}^N]^*_{1+\alpha} r^\alpha \cdot r d(Y)^{-(1+\alpha)} = C(n) U_{1+\alpha} (\tfrac{r}{d(Y)})^{1+\alpha}. \tag{4.59}$$

Step 5. Finally, combining (4.58) and (4.59), inputting back into (4.54), we have the concluding estimate on $\langle u^N \rangle^{*,N}_{1+\alpha}$ noting that $d(X) \wedge d(Y) = d(Y)$. □

5 Schauder estimate for the second discrete derivatives

The goal of this section is to derive uniform L^∞ bounds and Hölder estimates for the second discrete derivatives $\nabla^N_{e_1} \nabla^N_{e_2} u^N(t, \frac{x}{N})$ of the solution of the equation (1.1). To derive such estimates, as in the proof of Theorem 4.9 of [41], it is natural to consider the equation for $\nabla^N_e u^N(t, \frac{x}{N})$ and repeat the same argument for proving Theorem 4.1 for this equation. However, it turns out to be more convenient in our situation to consider ψ^N, instead of u^N, defined by the nonlinear transformation

$$\psi^N(t, \tfrac{x}{N}) := \varphi(u^N(t, \tfrac{x}{N})) \tag{5.1}$$

and the equation satisfied by its discrete derivatives; see (5.4) below.

In Section 5.1, we study the system of linear discrete PDEs (5.4), which is obtained as above and has a different form from (1.6), but as we will see, a similar application of the energy inequality works well for this equation. In Theorem 5.5, we give Schauder bounds for (5.4) which suit our applications; see Remark 5.6 for an alternative. In Section 5.2, we return to the original setting and formulate corresponding Schauder estimates there; see Corollary 5.8. When the initial value is smooth enough, we also give in Corollary 5.10 an improved regularity estimate.

5.1 Schauder bounds for the associated linear discrete PDE (5.4)

From (1.1), ψ^N defined by (5.1) satisfies

$$\partial_t \psi^N = \bar{a} \partial_t u^N = \bar{a}\{\Delta^N \psi^N + K f(u^N)\}, \tag{5.2}$$

where the last term can also be written as $K f(\varphi^{-1}(\psi^N))$ in terms of ψ^N and

$$\bar{a} \equiv \bar{a}(t, \tfrac{x}{N}) := \varphi'(u^N(t, \tfrac{x}{N})). \tag{5.3}$$

Note that the coefficient $\bar{a}$ is a site function, while $a_{x,e}(u)$ in (1.8), which appears in (1.1) by (1.5) and (1.7), depends also on the edges or directions e. In the continuous setting, these two functions are the same.

For $e \in \mathbb{Z}^n : |e| = 1, e > 0$, consider the discrete derivative of ψ^N in the direction e:

$$\xi_e \equiv \xi^N_e(t, \tfrac{x}{N}) := \nabla^N_e \psi^N(t, \tfrac{x}{N}).$$

By acting ∇^N_e on the equation (5.2) and recalling (2.3) for Δ^N, $\{\xi_e\}_e$ satisfies the system of equations

$$\partial_t \xi_e = -\nabla^N_e \Big\{ \bar{a}(X) \sum_{|e'|=1, e'>0} \nabla^{N,*}_{e'} \xi_{e'} \Big\} + \nabla^N_e g, \tag{5.4}$$

where $X = (t, \frac{x}{N})$ and

$$g(X) \equiv g^N(t, \tfrac{x}{N}) := K\bar{a}(t, \tfrac{x}{N}) f(u^N(t, \tfrac{x}{N})). \tag{5.5}$$

One may view the system (5.4) as a perturbation of a closed 'diagonal' or 'decoupled' system over $\{e\}$: Noting (2.9), and (2.3) and (2.4) to write the term with Δ^N, we have

$$\partial_t \xi_e = \bar{a}(X)\Delta^N \xi_e^N - \nabla_e^N \bar{a}(X) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*} \xi_{e'}(X + \tfrac{e}{N}) + \nabla_e^N g, \tag{5.6}$$

where $X + \frac{e}{N} = (t, \frac{x+e}{N})$ as in Lemma 4.6.

In the following, apart from (5.2) and similarly to Section 4.1, we will study the equation (5.4), where ξ_e is in the form of a gradient of abstract function ψ, that is $\xi_e = \nabla_e^N \psi$, with functions $\bar{a}$ and g satisfying the following abstract assumptions:

(B.1) (nondegeneracy, boundedness) $\quad c_- \le \bar{a} \le c_+$,

(B.2) (Hölder continuity) $\quad [\bar{a}]_\alpha^{*,N} \le \bar{A} < \infty$, $\alpha \in (0,1)$,

(B.3) (boundedness of $\nabla^N g$) $\quad |\nabla^N g|_0^{(1),N} \equiv \max_{e:|e|=1} |\nabla_e^N g|_0^{(1),N} \le G_0^1 < \infty$.

We always assume the continuity of $g(t)$ in t. Note that $\bar{a}$ defined by (5.3) from the solution u^N of (1.1) with (1.2) satisfies the conditions (B.1) and (B.2) with $\alpha = \sigma$ noting (2.16) in Corollary 2.3 and $\varphi \in C^2([u_-, u_+])$ is strictly increasing. Also, we will observe that g satisfying (5.5) satisfies (B.3) as a consequence of the Schauder estimate proved for u^N in Theorem 4.1. See Section 5.2 for these computations.

One might think that an L^∞ bound on $\nabla^N g$ could be substituted instead of (B.3). However, without further regularity, if one only assumes an L^∞ bound on the initial data $u^N(0, \cdot)$, then (B.3) is natural in view of the bound of $\nabla^N u^N$ in Theorem 4.1.

To begin, we need to rewrite Lemma 4.6 to adjust in the present setting; see Lemma 5.1 below. Take $Y = (t_1, y) \in \Omega = [0,T] \times \mathbb{T}^n$ and $r : 0 < r < \frac{1}{2} d(Y) = \frac{1}{2} t_1^{\frac{1}{2}}$. Recall that $Q_N(Y, r) = (t_1 - r^2, t_1) \times D_N(y, r)$ with the discrete interior $D_N(y, r)$ of the minimal cover of $D(y, r)$ by $\frac{1}{N}$-boxes, and $\mathcal{P}_N^+ Q_N(Y, r)$ is the parabolic outer boundary of $Q_N(Y, r)$.

Take the closest point $\tilde{y} \in \frac{1}{N}\mathbb{T}_N^n$ to y, in particular, $|y - \tilde{y}| \le \frac{\sqrt{n}}{2N}$ and set $\tilde{Y} := (t_1, \tilde{y}) \in \Omega_N$. For each $e : |e| = 1, e > 0$, let $\zeta_e = \zeta_e^N = \zeta_e^{N, Q(r)}(t, \frac{x}{N})$ be the solution of the discrete heat equation (with direction-independent coefficient) on $Q_N(r) = Q_N(Y, r)$:

$$\partial_t \zeta_e(t, \tfrac{x}{N}) = \bar{a}(\tilde{Y}) \Delta^N \zeta_e(t, \tfrac{x}{N}), \tag{5.7}$$

satisfying $\zeta_e = \xi_e$ at $\mathcal{P}_N^+ Q_N(r)$, and set

$$W_e \equiv W_e^N := \xi_e - \zeta_e \quad \text{on } \overline{Q_N(r)}.$$

Recall (4.28) and (4.29) for $\mathcal{P}_N^+ Q_N(r)$ and $\overline{Q_N(r)}$, respectively. Note, as before, by (2.3) and (2.4), that

$$-\nabla_e^N \Big\{ \bar{a}(\tilde{Y}) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*} \xi_{e'}^N \Big\} = -\nabla_e^N \Big\{ \bar{a}(\tilde{Y}) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*} \nabla_{e'}^N \psi^N \Big\} \tag{5.8}$$

$$= \bar{a}(\tilde{Y})\Delta^N \nabla_e^N \psi^N = \bar{a}(\tilde{Y})\Delta^N \xi_e^N.$$

We have the following discrete energy inequality for W_e.

Lemma 5.1. *Assume $0 < r < \frac{1}{2}d(Y) = \frac{1}{2}t_1^{\frac{1}{2}}$ and $Q_N(r) \neq \emptyset$. Then, for each e, we have*

$$\begin{aligned}(5.9) \qquad \int_{t_1-r^2}^{t_1} N^{-n} &\sum_{\frac{x}{N} \text{ or } \frac{x+e'}{N} \in D_N(y,r); |e'|=1, e'>0} |\nabla_{e'}^N W_e|^2(X)dt \\ &\le C\bar{A}^2\Big(\frac{r+\frac{1+\sqrt{n}}{N}}{d(Y)}\Big)^{2\alpha}(r+\tfrac{1}{N})^n r^2 \sup_{X \text{ or } X+\frac{e}{N} \in Q_N(r); |e'|=1, e'>0} |\nabla_{e'}^N \xi_{e'}^N|^2(X) \\ &\quad + CG_0^1(r+\tfrac{1}{N})^n r^2 d(Y)^{-1} \sup_{X \in Q_N(r)} |W_e(X)|,\end{aligned}$$

where $X = (t, \frac{x}{N})$ and $X + \frac{e}{N} = (t, \frac{x+e}{N})$. Here, the constants $C = C(n, c_-)$.

Proof. The proof is similar to that of Lemma 4.6. Rewriting as $\bar{a}(X) = \bar{a}(\tilde{Y}) + (\bar{a}(X) - \bar{a}(\tilde{Y}))$ in (5.4), noting (5.8), and from (5.7), we have

$$\begin{aligned}\partial_t W_e(t, \tfrac{x}{N}) =& \bar{a}(\tilde{Y})\Delta^N W_e(t, \tfrac{x}{N}) \\ &- \nabla_e^N \Big\{(\bar{a}(X) - \bar{a}(\tilde{Y})) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\xi_{e'}\Big\}(t, \tfrac{x}{N}) + \nabla_e^N g(t, \tfrac{x}{N}).\end{aligned}$$

Noting that $W_e(t, \cdot) = 0$ at the boundary $\partial_N^+ D_N(y, r)$, by (3.54) in Lemma 3.8, we have

$$\begin{aligned}&\tfrac{1}{2}\partial_t\Big(N^{-n} \sum_{\frac{x}{N} \in D_N(y,r)} W_e^2(X)\Big) \\ &= -N^{-n}\bar{a}(\tilde{Y}) \sum_{\frac{x}{N} \in D_N(y,r)} W_e(X) \sum_{|e'|=1, e'>0} \big(\nabla_{e'}^{N,*}\nabla_{e'}^N W_e\big)(X) \\ &\quad - N^{-n} \sum_{\frac{x}{N} \in D_N(y,r)} W_e(X)\nabla_e^N\Big\{(\bar{a}(X) - \bar{a}(\tilde{Y})) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\xi_{e'}^N\Big\}(X) \\ &\quad + N^{-n} \sum_{\frac{x}{N} \in D_N(y,r)} W_e(X)\nabla_e^N g(X) \\ &= -N^{-n}\bar{a}(\tilde{Y}) \sum_{*} |\nabla_{e'}^N W_e|^2(X) \\ &\quad - N^{-n} \sum_{**} (\bar{a}(X) - \bar{a}(\tilde{Y}))\nabla_e^{N,*} W_e(X) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\xi_{e'}^N(X) \\ &\quad + N^{-n} \sum_{\frac{x}{N} \in D_N(y,r)} W_e(X)\nabla_e^N g(X) \\ &=: -I_1 + I_2 + I_3,\end{aligned}$$

where $X = (t, \frac{x}{N})$, $\sum_{*}$ means the sum over (x, e') such that $\frac{x}{N}$ or $\frac{x+e'}{N} \in D_N(y,r)$ and $|e'| = 1, e' > 0$, while $\sum_{**}$ means the sum over x such that $\frac{x}{N}$ or $\frac{x-e}{N} \in D_N(y,r)$. Note that $I_1 \geq 0$.

Integrate both sides in $t \in [t_1 - r^2, t_1]$. Since $W_e(t_1 - r^2, \cdot) = 0$, the left hand side is

$$\tfrac{1}{2} N^{-n} \sum_{\frac{x}{N} \in D_N(y,r)} W_e^2(t_1, \tfrac{x}{N}) \geq 0.$$

On the other hand, for every $\varepsilon > 0$,

$$\begin{aligned}\int_{t_1 - r^2}^{t_1} |I_2| dt \leq \varepsilon \int_{t_1-r^2}^{t_1} N^{-n} \sum_{**} |\nabla_e^{N,*} W_e|^2(X) dt \\ + \frac{n}{\varepsilon} \int_{t_1-r^2}^{t_1} N^{-n} \sum_{**} |\bar{a}(X) - \bar{a}(\tilde{Y})|^2 \sum_{|e'|=1, e'>0} |\nabla_{e'}^{N,*} \xi_{e'}^N|^2(X) dt.\end{aligned}$$

Here, for the first term, since $\nabla_e^{N,*} = \nabla_{-e}^N$ implies $\nabla_e^{N,*} W_e(X) = -\nabla_e^N W_e(X - \frac{e}{N})$, we have

$$\begin{aligned}\sum_{**} |\nabla_e^{N,*} W_e|^2(X) &= \sum_{**} |\nabla_e^N W_e|^2(X - \tfrac{e}{N}) \\ &= \sum_{\frac{x}{N} \text{ or } \frac{x+e}{N} \in D_N(y,r)} |\nabla_e^N W_e|^2(X),\end{aligned}$$

by replacing $X - \frac{e}{N}$ by X. For the second term, by a similar replacement, we estimate

$$|\nabla_{e'}^{N,*} \xi_{e'}^N|^2(X) \leq J_{N,r} := \sup_{X \text{ or } X + \frac{e}{N} \in Q_N(r); |e'|=1, e'>0} |\nabla_{e'}^N \xi_{e'}^N|^2(X),$$

and by $[\bar{a}]_\alpha^{*,N} \leq \bar{A}$,

$$|\bar{a}(X) - \bar{a}(\tilde{Y})| \leq \bar{A}(d(X) \wedge d(\tilde{Y}))^{-\alpha} |X - \tilde{Y}|^\alpha \leq \bar{A} 2^\alpha d(Y)^{-\alpha} (r + \tfrac{1+\sqrt{n}}{N})^\alpha,$$

since $|X - \tilde{Y}| \leq r + \frac{1+\sqrt{n}}{N}$ for $X \in \overline{Q_N(r)}$ and $r < \frac{1}{2} d(Y) = \frac{1}{2} d(\tilde{Y})$ implies $d(X) \geq \frac{1}{2} d(Y)$ as in (4.33) so that $d(X) \wedge d(\tilde{Y}) \geq \frac{1}{2} d(Y)$. Moreover,

$$\int_{t_1-r^2}^{t_1} N^{-n} \sum_{\frac{x}{N} \text{ or } \frac{x-e}{N} \in D_N(y,r)} 1 dt \leq C(n)(r + \tfrac{1}{N})^n r^2.$$

Therefore, recalling $\bar{a}(\tilde{Y}) \geq c_- > 0$, we obtain

$$\begin{aligned}\int_{t_1-r^2}^{t_1} |I_2| dt \leq \frac{\varepsilon}{c_-} \int_{t_1-r^2}^{t_1} I_1 dt \\ + \frac{n^2}{\varepsilon} C(n) 2^{2\alpha} \bar{A}^2 \Big(\tfrac{r + \frac{1+\sqrt{n}}{N}}{d(Y)}\Big)^{2\alpha} (r + \tfrac{1}{N})^n r^2 J_{N,r}.\end{aligned}$$

For I_3, since $|Q_N(r)| \le C(n)N^n(r+\frac{1}{N})^n r^2$, we obtain by $|\nabla_e^N g|_0^{(1),N} \le G_0^1$,

$$\int_{t_1-r^2}^{t_1} |I_3|dt \le G_0^1 C(n)(r+\tfrac{1}{N})^n r^2 d(Y)^{-1} \sup_{Q_N(r)} |W_e^N|.$$

Summarizing these estimates, we have

$$\begin{aligned}(1-\tfrac{\varepsilon}{c_-})\int_{t_1-r^2}^{t_1} I_1 dt \le & \frac{n^2}{\varepsilon} c 2^{2\alpha} \bar{A}^2 (\tfrac{r+\frac{1+\sqrt{n}}{N}}{d(Y)})^{2\alpha}(r+\tfrac{1}{N})^n r^2 J_{N,r} \\ & + C(n)G_0^1(r+\tfrac{1}{N})^n r^2 d(Y)^{-1} \sup_{Q_N(r)} |W_e^N|.\end{aligned}$$

Choosing $\varepsilon > 0$ small and noting $\bar{a}(\tilde{Y}) \ge c_- > 0$ and $\alpha < 1$, we have shown (5.9) in terms of a $C = C(n, c_-)$. □

We now rewrite the discrete estimate obtained in Lemma 5.1 into a continuous one via polylinear interpolation. We will take $\zeta_e^N = \zeta_e^{N,Q(r_N^1)}$ as the solution of the discrete heat equation (5.7) on the wider domain $Q_N(Y, r_N^1)$ with boundary condition $\zeta_e^N = \xi_e^N$ on $\mathcal{P}_N^+ Q_N(Y, r_N^1)$, where $r_N^1 = r + \frac{\sqrt{n}+1}{N}$. As before, $W_e^N = \zeta_e^N - \xi_e^N$ on $\overline{Q_N(Y, r_N^1)}$. Set

$$\begin{aligned} & U_{2+\alpha,e} := [\tilde{\xi}_e^N]_{1+\alpha}^{(1)}, \\ (5.10)\qquad & U_2 := \max_{e_1,e_2} |\nabla_{e_1}^N \tilde{\xi}_{e_2}^N|_0^{(2)} = \max_{e_1,e_2} \sup_{X\in\Omega} d^2(X)|\nabla_{e_1}^N \tilde{\xi}_{e_2}^N(X)|, \\ & \mathcal{U}_e^1 := [\tilde{\xi}_e^N]_{1+\alpha}^{(1)} + \langle \xi_e^N \rangle_{1+\alpha}^{(1),N}, \end{aligned}$$

and also

$$(5.11)\qquad U_{2+\alpha} := \max_e U_{2+\alpha,e}, \qquad \mathcal{U}^1 := \max_e \mathcal{U}_e^1.$$

Recall (3.37) and (3.41) for $[\tilde{\xi}_e^N]_{1+\alpha}^{(1)}$ and $\langle \xi_e^N \rangle_{1+\alpha}^{(1),N}$, respectively. We have the following estimate extending (4.41) in the present setting.

Lemma 5.2. *We have*

$$(5.12)\qquad \int_{Q(r)} |\widetilde{\nabla^N W_e}|^2(X)dX \le C[\bar{A}^2 U_2^2 + G_0^1 \mathcal{U}_e^1] r_N^n (\tfrac{r_N}{d(Y)})^{2+2\alpha} d(Y)^{-2},$$

for every $r > 0$ such that $r_N = r + \frac{1+2\sqrt{n}}{N} < \frac{1}{2}d(Y)$, where $C = C(n, c_-, T, \alpha)$. Recall here $Q(r) = Q(Y, r)$.

Proof. By the same argument given for (4.37), except that we multiply and divide by one more factor of $d(Y) \le \sqrt{T}$, we have

$$(5.13)\qquad \sup_{X\in Q_N(r)} |W_e(X)| \le C(n,T)\mathcal{U}_e^1 (\tfrac{r}{d(Y)})^{1+\alpha} d(Y)^{-1},$$

for $0 < r < \frac{1}{2}d(Y)$.

The main link now is given by Lemma 4.7. In fact, by (4.35) for $\nabla^N W_e$ in place of $\nabla^N w$, then by Lemma 5.1 (with r replaced by $r+\frac{\sqrt{n}}{N}$) and (4.36) in Lemma 4.7 and also (5.13) (with r replaced by $r+\frac{\sqrt{n}}{N}$), we obtain similarly to (4.41) that

$$\begin{aligned}\int_{Q(r)} |\widetilde{\nabla^N_{e'} W_e}|^2(X)dX &\le C\big[\bar{A}^2 U_2^2 d(Y)^{-2} + G_0^1 \mathcal{U}_e^1 (\tfrac{r_N}{d(Y)})^{1-\alpha}\big] r_N^n (\tfrac{r_N}{d(Y)})^{2+2\alpha} \\ &\le C[\bar{A}^2 U_2^2 + G_0^1 \mathcal{U}_e^1] r_N^n (\tfrac{r_N}{d(Y)})^{2+2\alpha} d(Y)^{-2}\end{aligned} \tag{5.14}$$

for every e, e': $|e| = |e'| = 1, e > 0, e' > 0$ and all $r > 0$ if it satisfies $r_N < \frac{1}{2}d(Y)$. For the second line, note that $1-\alpha > 0$ and $2r_N/d(Y) \le 1 \le Td(Y)^{-2}$. Here, $C = C(n, c_-, T)$ changed line to line. □

Since $\nabla^N_{e'}\zeta_e$ solves the discrete heat equation, analogous to the derivation of (4.42), we obtain

$$\int_{Q(\rho)} |\widetilde{\nabla^N_{e'}\zeta_e} - \{\widetilde{\nabla^N_{e'}\zeta_e}\}_\rho|^2 dX \le C\big(\tfrac{\rho}{r}\big)^{n+4} \int_{Q(r)} |\widetilde{\nabla^N_{e'}\zeta_e} - \{\widetilde{\nabla^N_{e'}\zeta_e}\}_r|^2 dX, \tag{5.15}$$

for $\rho \in (0, r)$ and $C = C(n, c_\pm)$. Indeed, in the present setting, the coefficient $\bar{a}(\tilde{Y})$ is direction-independent and one can directly apply (3.25) in Proposition 3.4.

Therefore, combined with (5.12) in Lemma 5.2, we have as in Lemma 4.8,

$$\omega(\rho) \le \bar{C}\big(\tfrac{\rho}{r}\big)^{n+4}\omega(r) + \sigma(r_N), \tag{5.16}$$

for $\rho \in (0, r)$ if $r_N = r + \frac{1+\sqrt{n}}{N} < \frac{1}{2}d(Y)$ is satisfied, where

$$\begin{aligned}\omega(r) \equiv \omega_e(r) &= \int_{Q(r)} |\widetilde{\nabla^N \xi_e} - \{\widetilde{\nabla^N \xi_e}\}_r|^2 dX, \\ \sigma(r) \equiv \sigma_e(r) &= \widehat{C}[\bar{A}^2 U_2^2 + G_0^1 \mathcal{U}_e^1] r^{n+2+2\alpha} d(Y)^{-(4+2\alpha)}.\end{aligned} \tag{5.17}$$

Here, $\bar{C} = \bar{C}(n, c_\pm)$ and $\widehat{C} = \widehat{C}(n, c_\pm, T)$.

To derive the bound on $U_{2+\alpha}$ analogous to Proposition 4.10, we need the following Hölder estimates for $\nabla^N_{e'}\tilde{\xi}_e$ in a short distance regime as in Lemma 4.11.

Lemma 5.3. (1) *If* $|Y - Y_1| \le \frac{c_1}{N}$, *we have*

$$|\nabla^N_{e'}\tilde{\xi}_e(Y) - \nabla^N_{e'}\tilde{\xi}_e(Y_1)| \le C\big(U_{2+\alpha} + U_2\bar{A} + G_0^1\big)\big(d(Y)\wedge d(Y_1)\big)^{-(2+\alpha)}|Y - Y_1|^\alpha, \tag{5.18}$$

for some $C = C(n, c_+, c_1) > 0$.
(2) *Furthermore, for every* $\delta > 0$, *there exists* $M = M(n, c_+) \ge 1$ *such that*

$$|\nabla^N_{e'}\tilde{\xi}_e(Y) - \nabla^N_{e'}\tilde{\xi}_e(Y_1)| \le \delta\big(U_{2+\alpha} + U_2\bar{A} + G_0^1\big)\big(d(Y)\wedge d(Y_1)\big)^{-(2+\alpha)}|Y - Y_1|^\alpha, \tag{5.19}$$

holds if $|Y - Y_1| \le \frac{1}{MN}$.

Proof. We only outline the proof following that of Lemma 4.11.

For the spatial direction, we give the estimate in terms of $U_{2+\alpha}$ instead of $U_{1+\alpha}$. This yields an extra factor of $d(Y)^{-1}$ and, instead of (4.53), we obtain

$$\left|\nabla_{e'}^N \tilde{\xi}_e(z^{(1)}) - \nabla_{e'}^N \tilde{\xi}_e(z^{(2)})\right| \le |z^{(1)} - z^{(2)}|^\alpha (\tfrac{1}{M})^{1-\alpha} \cdot 2^{n-1} U_{2+\alpha,e} d(Y)^{-(2+\alpha)}.$$

For the temporal direction, by the equation (5.4), we have

$$\partial_t \nabla_{e'}^N \xi_e(X) = -\sum_{|e_1|=1, e_1>0} \nabla_{e'}^N \nabla_e^N \big(\bar{a}(X) \nabla_{e_1}^{N,*} \xi_{e_1}\big)(X) + \nabla_{e'}^N \nabla_e^N g(X).$$

We apply the following rough estimates to the right hand side: By (2.9) and (B.1)–(B.3),

$$\begin{aligned}|\nabla_{e'}^N \nabla_e^N \big(\bar{a}(X) \nabla_{e_1}^{N,*} \xi_{e_1}\big)(X)| &\le 2N^2 \Big\{ c_+ \sup_{X'} |\nabla_{e_1}^{N,*} \xi_{e_1}(X' + \tfrac{e}{N}) - \nabla_{e_1}^{N,*} \xi_{e_1}(X')| \\ &\qquad + U_2 d(X)^{-2} \sup_{X'} |\bar{a}(X' + \tfrac{e}{N}) - \bar{a}(X')| \Big\} \\ &\le 2N^2 \Big\{ c_+ U_{2+\alpha} + U_2 \bar{A} \Big\} (\tfrac{1}{N})^\alpha d(X)^{-(2+\alpha)},\end{aligned}$$

and $|\nabla_{e'}^N \nabla_e^N g(X)| \le 2N G_0^1 d(X)^{-1}$. Therefore, similarly to Step 2 in the proof of Lemma 4.11, we obtain

$$\begin{aligned}&|\nabla_e^N u(t,z) - \nabla_e^N u(s,z)| \\ &\quad \le C|t-s|^{\frac{\alpha}{2}} (\tfrac{1}{M})^{2-\alpha} \Big[\Big\{ U_{2+\alpha} + U_2 \bar{A} \Big\} \big(d(Y) \wedge d(Y_1)\big)^{-(1+\alpha)} + G_0^1 \Big] \big(d(Y) \wedge d(Y_1)\big)^{-1},\end{aligned}$$

where $Y = (t,z)$, $Y_1 = (s,z)$ and $C = C(n, c_+)$. This concludes the proof of the lemma. □

We are at the position to give the bound on $U_{2+\alpha}$, analogous to Proposition 4.10.

Proposition 5.4. *We have*

$$U_{2+\alpha} = \max_{|e|=1, e>0} [\tilde{\xi}_e^N]_{1+\alpha}^{(1)} \le C\big[(\bar{A}+1)U_2 + G_0^1\big] + \delta \mathcal{U}^1, \tag{5.20}$$

for every $\delta > 0$ with some $C = C(n, c_\pm, T, \delta)$.

Proof. We apply Lemma 4.9, combining with (5.16), to get

$$\omega(r) \le C\big(\tfrac{r}{R_0}\big)^{n+2+2\alpha} \omega(R_0) + C[\bar{A}^2 U_2^2 + G_0^1 \mathcal{U}_e^1] r_N^{n+2+2\alpha} d(Y)^{-(4+2\alpha)},$$

for $0 < r \le R_0 = \frac{1}{3} d(Y)$, if $d(Y) > \frac{6(1+2\sqrt{n})}{N}$. Here, $C = C(n, c_\pm, T)$ and recall $r_N = r + \frac{c}{N}$ with $c = 1 + 2\sqrt{n}$.

However, $\omega(R_0)$ is bounded as

$$\omega(R_0) \le 4 \int_{Q(R_0)} |\nabla^N \tilde{\xi}_e|^2 dX \le C(n)\, U_2^2 R_0^{n+2}\, d(Y)^{-4}.$$

Therefore, we have

$$\int_{Q(r)} |\nabla^N_{e'}\tilde{\xi}_e - \{\nabla^N_{e'}\tilde{\xi}_e\}_r| dX \le C\sqrt{(\bar{A}^2+1)U_2^2 + G_0^1\mathcal{U}_e^1}\, d(Y)^{-(2+\alpha)} r_N^{n+2+\alpha} \tag{5.21}$$
$$\le \{C'[(\bar{A}+1)U_2 + G_0^1] + \delta\mathcal{U}_e^1\}\, d(Y)^{-(2+\alpha)} r_N^{n+2+\alpha},$$

for every $\delta > 0$ with some $C' = C'(n, c_\pm, T, \delta) > 0$, for $0 < r \le \frac{1}{3}d(Y)$, if $d(Y) > \frac{6(1+2\sqrt{n})}{N}$.

In the case that $d(Y) \le \frac{6(1+2\sqrt{n})}{N}$, we can apply (5.18) in Lemma 5.3 by making $c > 0$ in r_N larger if necessary. Indeed, we obtain for $0 < r \le \frac{1}{3}d(Y)$

$$\int_{Q(r)} |\nabla^N_{e'}\tilde{\xi}_e - \{\nabla^N_{e'}\tilde{\xi}_e\}_r| dX \le C r^{n+2+\alpha}\big(U_{2+\alpha} + U_2\bar{A} + G_0^1\big) d(Y)^{-(2+\alpha)} \tag{5.22}$$

where $C = C(n, c_+)$.

From (5.21), (5.22), making $c > 0$ in r_N large as in the proof of Proposition 4.10 and recalling $U_{2+\alpha} \le \mathcal{U}^1$, we have shown

$$\int_{Q(Y,r)} |\nabla^N_{e'}\tilde{\xi}_e - \{\nabla^N_{e'}\tilde{\xi}_e\}_r| dX \le \{C[(\bar{A}+1)U_2 + G_0^1] + \delta\mathcal{U}^1\}\, d(Y)^{-(2+\alpha)} r_N^{n+2+\alpha}, \tag{5.23}$$

for $0 < r \le \frac{1}{3}d(Y)$ and any $Y = (t_1, y) \in \Omega$ where $C = C(n, c_\pm, T, \delta)$.

We now fix any $X_0 = (t_0, z_0) \in \Omega$ and take $R = \frac{1}{4}d(X_0) = \frac{1}{4}\sqrt{t_0}$. Then, by Lemma 3.6 (with a proper modification in $U_{F,1+\alpha}$), we see that, for every $\delta > 0$ and $M > 0$, there exists $C = C_{\delta,M}(n, c_\pm, T) > 0$ such that

$$|\nabla^N_{e'}\tilde{\xi}_e(Y) - \nabla^N_{e'}\tilde{\xi}_e(Y_1)| \le \Big(C[(\bar{A}+1)U_2 + G_0^1] + \delta\mathcal{U}^1 + \delta U_{2+\alpha}\Big) \tag{5.24}$$
$$\times \big(d(Y) \wedge d(Y_1)\big)^{-(2+\alpha)} |Y - Y_1|^\alpha,$$

holds if $Y, Y_1 \in Q(X_0, R)$ satisfy $|Y - Y_1| \ge \frac{1}{MN}$.

The case $|Y - Y_1| \le \frac{1}{MN}$ is covered by (5.19) in Lemma 5.3, and we obtain (5.24) for any $Y, Y_1 \in Q(X_0, R)$. Thus, moving X_0, we have shown (5.24) for any $Y, Y_1 \in \Omega$. This implies the concluding estimate (5.20) on $U_{2+\alpha}$. □

Now, by the same argument given for Proposition 4.12, using in place of $U_{1+\alpha}$, U_1 and G_∞ the quantities $U_{2+\alpha,e}$, U_2 and G_0^1 (which introduce an extra factor of $d(Y)^{-1}$), we have

$$\langle \xi_e^N \rangle_{1+\alpha}^{(1),N} \le C'\big[\bar{A}U_2 + U_{2+\alpha,e} + G_0^1\big] \tag{5.25}$$

and so

$$\mathcal{U}_e^1 = U_{2+\alpha,e} + \langle \xi_e^N \rangle_{1+\alpha}^{(1),N} \le U_{2+\alpha,e} + C'\big[\bar{A}U_2 + U_{2+\alpha,e} + G_0^1\big] \tag{5.26}$$

where $C' = C'(n, c_+, T, \alpha)$.

Hence, inserting (5.26), maximized over e, into (5.20) in Proposition 5.4, we obtain

$$U_{2+\alpha} \le C\Big[(\bar{A}+1)U_2 + G_0^1 + \delta\big\{U_{2+\alpha} + C'[\bar{A}U_2 + U_{2+\alpha} + G_0^1]\big\}\Big]$$

and by choosing $\delta > 0$ small that

$$U_{2+\alpha} \le C''\big[(\bar{A}+1)U_2 + G_0^1\big] \tag{5.27}$$

where $C'' = C''(n, c_\pm, T, \alpha)$.

However, by applying the interpolation inequality (4.24) in Lemma 4.5 to U_2,

$$U_2 \le 5 \max_{e'} (|\tilde{\xi}_{e'}^N|_0^{(1)})^{\frac{\alpha}{1+\alpha}} \big(|\tilde{\xi}_{e'}^N|_0^{(1)} + [\tilde{\xi}_{e'}^N]_{1+\alpha}^{(1)}\big)^{\frac{1}{1+\alpha}}. \tag{5.28}$$

Denote

$$|\tilde{\xi}_\cdot^N|_0^{(1)} := \max_{e'} |\tilde{\xi}_{e'}^N|_0^{(1)}.$$

Then, from (5.28), we have in view of (5.27) that

$$\begin{aligned} C''(\bar{A}+1)U_2 &\le \Big(2^{\frac{1}{\alpha}} \big(5C''(\bar{A}+1)\big)^{\frac{1+\alpha}{\alpha}} |\tilde{\xi}_\cdot^N|_0^{(1)}\Big)^{\frac{\alpha}{1+\alpha}} \Big(\tfrac{1}{2}\big(|\tilde{\xi}_\cdot^N|_0^{(1)} + U_{2+\alpha}\big)\Big)^{\frac{1}{1+\alpha}} \\ &\le 2^{\frac{1}{\alpha}} \big(5C''(\bar{A}+1)\big)^{\frac{1+\alpha}{\alpha}} |\tilde{\xi}_\cdot^N|_0^{(1)} + \tfrac{1}{2}\big(|\tilde{\xi}_\cdot^N|_0^{(1)} + U_{2+\alpha}\big), \end{aligned} \tag{5.29}$$

where we have used a trivial bound: $ab \,(\le \frac{a^p}{p} + \frac{b^q}{q}) \le a^p + b^q$ for $a, b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Inserting this into (5.27), we obtain

$$U_{2+\alpha} \le \bar{C}\Big[(\bar{A}+1)^{1+\frac{1}{\alpha}} |\tilde{\xi}_\cdot^N|_0^{(1)} + G_0^1\Big] \tag{5.30}$$

where $\bar{C} = \bar{C}(n, c_\pm, T, \alpha)$.

Reporting these estimates, as an extension of Theorem 4.1, we arrive at the following bound for U_2 and

$$|\tilde{\xi}_\cdot^N|_{1+\alpha}^{(1)} := U_2 + U_{2+\alpha} + \max_e \langle \xi_e^N \rangle_{1+\alpha}^{(1),N}.$$

Theorem 5.5. *Under the assumptions (B.1), (B.2) and (B.3), we have*

$$|\tilde{\xi}_\cdot^N|_{1+\alpha}^{(1)} \le C\Big[(\bar{A}+1)^{1+\frac{1}{\alpha}} |\tilde{\xi}_\cdot^N|_0^{(1)} + G_0^1\Big], \tag{5.31}$$

and

$$U_2 \le C\Big[(\bar{A}+1)^{\frac{1}{\alpha}} |\tilde{\xi}_\cdot^N|_0^{(1)} + (\bar{A}+1)^{-1} G_0^1\Big] \tag{5.32}$$

where $C = C(n, c_\pm, T, \alpha)$.

Proof. For (5.31), it is enough to bound each term in $|\tilde{\xi}_\cdot^N|_{1+\alpha}^{(1)}$ by the right hand side. For $U_{2+\alpha}$, the bound follows from (5.30). For U_2, we can use (5.29), where multiplying the right hand side of (5.30) (or (5.31)) by $(\bar{A}+1)^{-1}$ gives (5.32). Finally for $\langle \xi_e^N \rangle_{1+\alpha}^{(1),N}$, we may use (5.25) together with (5.29) and (5.30). $\square$

Theorem 5.5 shows a form of the first Schauder estimate for the linear discrete PDE (5.4). It is an estimate in terms of norms which weight more near the boundary $t = 0$. In fact, the estimate (5.32) on U_2 yields the singularity $\frac{1}{t}$ for $\nabla_{e_1}^N \xi_{e_2}^N$ near $t = 0$. However, we note here an analog of Theorem 4.1 and (4.8) for the equation (5.4), which exhibits a weaker singularity $\frac{1}{\sqrt{t}}$; see the second estimate in (5.33).

Corollary 5.6. *Under the assumptions (B.1), (B.2) and $\|\nabla^N g\|_\infty < \infty$ (replacing (B.3)), and also $\|\xi^N_\cdot\|_\infty < \infty$ (compare with* (1.2)*), we have*

$$
\begin{aligned}
|\tilde{\xi}^N_\cdot|^*_{1+\alpha} &\le C\big[(\bar{A}+1)^{1+\frac{1}{\alpha}}\|\xi^N_\cdot\|_\infty + \|\nabla^N g\|_\infty\big], \\
|\nabla^N \xi^N_\cdot|^{(1)}_0 &\le C\big[(\bar{A}+1)^{\frac{1}{\alpha}}\|\xi^N_\cdot\|_\infty + (\bar{A}+1)^{-1}\|\nabla^N g\|_\infty\big],
\end{aligned}
\tag{5.33}
$$

where $C = C(n, c_\pm, T, \alpha)$.

Proof. The argument follows the same proof as for Theorem 5.5. Observe that the seminorms $|\tilde{\xi}^N_\cdot|^{(1)}_{1+\alpha}$, $|\tilde{\xi}^N_\cdot|^{(1)}_0$, $|\nabla^N g|^{(1),N}_0$ all involve just one more weight $d(X)$ compared to $|\tilde{\xi}^N_\cdot|^*_{1+\alpha}$, $\|\xi^N_\cdot\|_\infty$, $\|\nabla^N g\|_\infty$, respectively; see also the proof of Corollary 5.9. $\square$

Now, as in Theorem 4.2, when the initial value of ψ and $\xi^N_e = \nabla^N_e \psi$ are smooth enough, we will obtain a more regular estimate at $t = 0$ without any singularity. We will suppose that ψ is C^4 and so ξ^N_e is C^3 in the sense that

$$
\sup_N \|\psi(0)\|_{C^4_N} \le \mathcal{C}_0 < \infty \quad \text{and so} \quad \sup_N \|\xi^N_e(0)\|_{C^3_N} \le \mathcal{C}_0 < \infty. \tag{5.34}
$$

We will need to following additional assumptions for the coefficient $\bar{a}$, similar to (4.12) and (4.13):

$$
[\bar{a}]^{(-\alpha),N}_\alpha \le \bar{B} < \infty : |\bar{a}(t_1, \tfrac{x_1}{N}) - a(t_2, \tfrac{x_2}{N})| \le \bar{B}\big\{|t_2 - t_1|^{\frac{\alpha}{2}} + \big|\tfrac{x_2}{N} - \tfrac{x_1}{N}\big|^\alpha\big\}, \tag{5.35}
$$

$$
\sup_N \|\bar{a}(0,\cdot)\|_{C^4_N} \le \mathcal{C}_1 < \infty. \tag{5.36}
$$

Also, we assume that $\nabla^N g$ satisfies

$$
\sup_{N,e} |\nabla^N_e g|_0 \le G^1_\infty < \infty. \tag{5.37}
$$

Theorem 5.7. *Consider the discrete PDE* (5.4)*, satisfying (B.1),* (5.35)*,* (5.36)*,* (5.37)*, such that*

$$
\sup_{N,e} |\xi^N_e|_0 \le W_\infty < \infty, \tag{5.38}
$$

and with initial value given in (5.34)*. Then, with respect to the unweighted norms,*

$$
|\tilde{\xi}^N_\cdot|_{1+\alpha} \le C\big[(\bar{B}+1)^{1+\frac{1}{\alpha}}\big(W_\infty + \mathcal{C}_6\big) + G^1_\infty + \mathcal{C}_7\big] \tag{5.39}
$$

$$
|\nabla^N \tilde{\xi}^N_\cdot|_0 \le C\big[(\bar{B}+1)^{\frac{1}{\alpha}}\big(W_\infty + \mathcal{C}_6\big) + (\bar{B}+1)^{-1}(G^1_\infty + \mathcal{C}_7)\big]. \tag{5.40}
$$

$$
[\nabla^N \tilde{\xi}^N_\cdot]_\alpha \le C\big[(\bar{B}+1)^{1+\frac{1}{\alpha}}\big(W_\infty + \mathcal{C}_6\big) + G^1_\infty + \mathcal{C}_7\big], \tag{5.41}
$$

where $C = C(n, c_\pm, T, \alpha)$ and $\mathcal{C}_6, \mathcal{C}_7$, specified in the proof, depend on $\bar{B}$, $\mathcal{C}_0$, $\mathcal{C}_1$ and α.

In particular, we have polynomial bounds in $\bar{B}$, $\mathcal{C}_0$ and $\mathcal{C}_1$, in terms of a universal constant C,

$$
\mathcal{C}_6 \le C(\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_0(1+\mathcal{C}_1^5) \quad \text{and} \quad \mathcal{C}_7 \le C(\bar{B}+1)^{\frac{2}{\alpha}}\mathcal{C}_0(1+\mathcal{C}_1^{11}).
$$

Proof. We will apply the same sort of scheme as given for Theorem 4.2, where now Theorem 5.5 will be invoked with respect to a time extended system. Recall equation (5.4) for $t > 0$,

$$\partial_t \xi_e^N = \nabla_e^N \big\{ -\bar{a}(X) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*} \xi_{e'}^N \big\} + \nabla_e^N g, \tag{5.42}$$

where $\{\xi_e^N = \nabla_e^N \psi\}_{|e|=1}$ is a gradient of the function ψ. As before, the idea now is to extend the evolution below time $t = 0$.

Consider $\bar{a}$ evaluated at time $t = 0$, and define $b(X) := \bar{a}\big(0, \frac{x}{N}\big)$. Note that b does not depend on time t or direction e, and satisfies $c_- \le b \le c_+$ from (B.1). We will need to consider the reciprocal $1/b$ in the following. Note by (5.36) that $\|(1/b)\|_{C_N^4} \le C(n, c_\pm)\mathcal{C}_1^4$.

For $t \ge 0$, define Z by

$$\partial_t Z = - \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big(b \nabla_{e'}^N Z\big) \tag{5.43}$$

with initial condition $Z(0) = \frac{1}{b}\psi(0)$. Note as $\sup_N \|(1/b)\|_{C_N^4} \le C(n, c_\pm)\mathcal{C}_1^4$ that we have $\sup_N \|Z(0)\|_{C_N^4} \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^4)$. Note also that equation (5.43) is at the level of (5.2) for $\psi = \psi^N$, but we divide it by b to make the equation in divergence form so that we can apply the gradient bounds obtained in Theorem 4.2.

By the maximum principle (Lemma 2.7), $\|Z\|_\infty \le \|Z(0)\|_\infty \le C(c_-)\mathcal{C}_0$. Moreover, we may apply Theorem 4.2 to bound gradients of Z: We have b satisfies (A.1) and also (4.12) and (4.13) with $B = [b]_\alpha^{(-\alpha),N} \le \bar{B}$ and $C_1 = \sup_{N,e} |\nabla_e^N b_0| \le \mathcal{C}_1$. Here, $G_\infty = 0$, $C_0 = \sup_N \|Z(0)\|_{C_N^2} \le C(n, c_\pm)\mathcal{C}_0(1+\mathcal{C}_1^4)$, and (4.4) holds with bound $\|Z\|_\infty \le C(c_-)\mathcal{C}_0$. Then, by (4.15),

$$\begin{aligned} \max_e |\nabla_e^N Z|_0 &\le C(n, c_\pm, T, \alpha)\big[(B+1)^{\frac{1}{\alpha}}|Z|_0 + (B+1)^{-1}\big\{G_\infty + C_2\big\}\big] \\ &\le C(n, c_\pm, T, \alpha)\mathcal{C}_3 \end{aligned} \tag{5.44}$$

where

$$\begin{aligned} C_2 &= n(C_1 C_0 + c_+ C_0) \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^5), \\ \mathcal{C}_3 &= (\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_0 + (\bar{B}+1)^{-1}\mathcal{C}_0(1+\mathcal{C}_1^5) \\ &\le C(\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_0(1+\mathcal{C}_1^5), \end{aligned} \tag{5.45}$$

in terms of a universal constant C.

Define now, and noting (2.9),

$$\begin{aligned} h_1 &:= \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big(b \nabla_{e'}^N Z\big) \\ &= -b\Delta^N Z + \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*} b \cdot \nabla_{e'}^N Z(X + \tfrac{e}{N}). \end{aligned} \tag{5.46}$$

Observe that h_1 satisfies (5.43). Then, by the maximum principle, and previous bounds $b \le c_+$, $\max_e |\nabla_e^N b|_0 \le \mathcal{C}_1$ in (5.36), and $\|Z(0)\|_{C_N^4} \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^4)$, we have

$$\|h_1\|_\infty \le \|h_1(0)\|_\infty \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^5). \tag{5.47}$$

Hence, noting (5.46), by (5.44) and triangle inequality, we also have

$$\|\Delta^N Z\|_\infty \le C(n, c_\pm, T, \alpha)\mathcal{C}_1\mathcal{C}_3 + C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^5). \tag{5.48}$$

Further, by Theorem 4.2, as in the set-up of (5.44), we obtain an estimate for the gradient of h_1: Indeed, as $Z = \frac{1}{b}\psi$ and $\|Z(0)\|_{C_N^4} \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^4)$, noting (5.36), we have $C_0 = \sup_N \|h_1(0)\|_{C_N^2} \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^5)$. Also, as before, $C_1 = \sup_{N,e} |\nabla_e^N b|_0 \le \mathcal{C}_1$, $B = [b]_\alpha^{(-\alpha),N} \le \bar{B}$ and $G_\infty = 0$. Hence, in this context (cf. (5.45)), $C_2 \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^6)$. By (5.47), $|h_1|_0 \le C\mathcal{C}_0(1 + \mathcal{C}_1^5)$. Then,

$$\begin{aligned} \max_e |\nabla_e^N h_1|_0 &\le C(n, c_\pm, T, \alpha)\big[(B+1)^{\frac{1}{\alpha}}|h_1|_0 + (B+1)^{-1}\big\{G_\infty + C_2\big\}\big] \\ &\le C(n, c_\pm, \alpha, T)\mathcal{C}_4 \end{aligned} \tag{5.49}$$

where, in terms of a universal constant C,

$$\begin{aligned} \mathcal{C}_4 &= (\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_0(1 + \mathcal{C}_1^5) + (\bar{B}+1)^{-1}\mathcal{C}_0(1 + \mathcal{C}_1^6) \\ &\le C(\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_0(1 + \mathcal{C}_1^6). \end{aligned}$$

By considering the equation for $Z_e := \nabla_e^N Z$,

$$\partial_t Z_e = -\sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big(b\nabla_{e'}^N Z_e\big) - \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big((\nabla_e^N b)Z_{e'}(X + \tfrac{e}{N})\big), \tag{5.50}$$

we may bound second gradients $\nabla_{e_1}^N \nabla_{e_2}^N Z$. Indeed, $\eta_e := \nabla_e^N(bZ)$ is more natural object that we consider below, which is parallel to ξ_e, but here we consider Z_e to deduce the second gradient bounds.

Observe here that $C_0 = \sup_N \|Z_e(0)\|_{C_N^2} \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^4)$. As before, in this context, $C_1 = \sup_{N,e} |\nabla_e^N b|_0 \le \mathcal{C}_1$, $B = [b]_\alpha^{(-\alpha),N} \le \bar{B}$ and $C_2 \le C(n, c_\pm)\mathcal{C}_0(1 + \mathcal{C}_1^5)$ (cf. (5.45)). However, here $|\nabla_{e_2}^N Z|_0 \le C(n, c_\pm, T, \alpha)\mathcal{C}_3$ and, by (5.44) and (5.48),

$$\begin{aligned} G_\infty &= \Big\| \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big((\nabla_e^N b)Z_{e'}(X + \tfrac{e}{N})\big)\Big\|_\infty \le C(n)\mathcal{C}_1 \max_e \|Z_e\|_\infty + \mathcal{C}_1\|\Delta^N Z\|_\infty \\ &= C(n, c_\pm, T, \alpha)\big\{\mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_1\big[\mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_0(1 + \mathcal{C}_1^5)\big]\big\}. \end{aligned}$$

Then, by Theorem 4.2 applied to (5.50), we have

$$\begin{aligned} \max_{e_1, e_2} |\nabla_{e_1}^N \nabla_{e_2}^N Z|_0 &\le C(n, c_\pm, T, \alpha)\big[(B+1)^{\frac{1}{\alpha}}|\nabla_{e_1}^N Z|_0 + (B+1)^{-1}\big\{G_\infty + C_2\big\}\big] \\ &\le C(n, c_\pm, T, \alpha)\mathcal{C}_5, \end{aligned} \tag{5.51}$$

where

$$\begin{aligned}\mathcal{C}_5 &= (\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_3 + (\bar{B}+1)^{-1}\big\{\mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_1\big[\mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_0(1+\mathcal{C}_1^5)\big] + \mathcal{C}_0(1+\mathcal{C}_1^5)\big\}\\ &\le C(\bar{B}+1)^{\frac{2}{\alpha}}\mathcal{C}_0(1+\mathcal{C}_1^7),\end{aligned}$$

incorporating the expression for C_2 and $\mathcal{C}_3$ in (5.45) in terms of a universal constant C.

Let now $\hat{Z}(t) = Z(1-t)$ and $\hat{h}_1(t) = h_1(1-t)$ for $0 \le t < 1$. Then,

$$\partial_t \hat{Z} = -\sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big(b\nabla_{e'}^N \hat{Z}\big) + 2\hat{h}_1.$$

Define $\hat{\Psi} = b\hat{Z}$ to be back to the level of $\psi = \psi^N$. Then, by using (2.9),

$$\partial_t \hat{\Psi} = b\Delta^N \hat{\Psi} - b\hat{h}_2 + 2b\hat{h}_1,$$

where

$$\hat{h}_2 = \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*}\big[b(\nabla_{e'}^N \tfrac{1}{b})\hat{\Psi}(X + \tfrac{e'}{N})\big].$$

We have by the previous bounds (5.34), (5.36), (5.44) and $\|(1/b)\|_{C_N^4} \le C(n, c_\pm)\mathcal{C}_1^4$, noting $\hat{\Psi} = b\hat{Z}$, that

$$\begin{aligned}\|\hat{h}_2\|_\infty &\le C(n,c_\pm)\max_{e'}\Big\{|\nabla_{e'}^{N,*}\big[b(\nabla_{e'}^N\tfrac{1}{b})(X)b(X+\tfrac{e'}{N})\big]|_0|Z|_0\\ &\qquad + |b(\nabla_{e'}^N\tfrac{1}{b})(X)b(X+\tfrac{e'}{N})|_0|\nabla_{e'}^{N,*}Z|_0\Big\}\\ &\le C(n,c_\pm,T,\alpha)\big\{\mathcal{C}_0\mathcal{C}_1^4(1+\mathcal{C}_1) + \mathcal{C}_1^4\mathcal{C}_3\big\}.\end{aligned}\tag{5.52}$$

Analogously, using also (5.51), we have

$$\begin{aligned}\|\nabla_e^N\hat{h}_2\|_\infty &\le C(n,c_\pm)\max_{e'}\Big\{|\nabla_e^N\nabla_{e'}^{N,*}\big[b(\nabla_{e'}^N\tfrac{1}{b})(X)b(X+\tfrac{e'}{N})\big]|_0|Z|_0\\ &\qquad + |b(\nabla_{e'}^N\tfrac{1}{b})(X)b(X+\tfrac{e'}{N})|_0|\nabla_e^N\nabla_{e'}^{N,*}Z|_0\\ &\qquad + |\nabla_{e'}^{N,*}\big[b(\nabla_{e'}^N\tfrac{1}{b})(X)b(X+\tfrac{e'}{N})\big]|_0|\nabla_e^N Z|_0\\ &\qquad + |\nabla_e^N\big[b(\nabla_{e'}^N\tfrac{1}{b})(X)b(X+\tfrac{e'}{N})\big]|_0|\nabla_{e'}^{N,*}Z|_0\Big\}\\ &\le C(n,c_\pm,T,\alpha)\big\{\mathcal{C}_0\mathcal{C}_1^4(1+\mathcal{C}_1^2) + \mathcal{C}_1^4\mathcal{C}_5 + \mathcal{C}_3\mathcal{C}_1^4(1+\mathcal{C}_1)\big\}.\end{aligned}\tag{5.53}$$

Now define $\hat{\eta}_e = \nabla_e^N\hat{\Psi}$ for $|e|=1$ to be at the level of ξ_e^N. Then,

$$\partial_t\hat{\eta}_e = \nabla_e^N\Big\{-b\sum_{|e'|=1, e'>0}\nabla_{e'}^{N,*}\hat{\eta}_{e'}\Big\} - \nabla_e^N(b\hat{h}_2) + 2\nabla_e^N(b\hat{h}_1).\tag{5.54}$$

Note that $\hat{\eta}_e(1) = \nabla_e^N(bZ(0)) = \nabla_e^N\psi(0) = \xi_e^N(0)$, matching the initial data to (5.42). Also, by (5.34), (5.36) and (5.44), noting $Z = \frac{1}{b}\psi$ and via (2.9) $\nabla_e^N(b\hat{Z}) = (\nabla_e^N b)\hat{Z} + b\nabla_e^N\hat{Z}(X+\frac{e}{N})$,

$$\|\hat{\eta}_e\|_\infty = \|\nabla_e^N(b\hat{Z})\|_\infty \le C(n,c_\pm,T,\alpha)\big[\mathcal{C}_0\mathcal{C}_1 + \mathcal{C}_3\big]\tag{5.55}$$

$$=: C(n, c_\pm, T, \alpha)\mathcal{C}_6$$

where, incorporating the expression for $\mathcal{C}_3$ in (5.45), in terms of a universal constant C,

$$\mathcal{C}_6 = \mathcal{C}_0\mathcal{C}_1 + \mathcal{C}_3 \le C(\bar{B}+1)^{\frac{1}{\alpha}}\mathcal{C}_0(1+\mathcal{C}_1^5).$$

Now, as in (4.18), define

$$\hat{a} = \begin{cases} \bar{a}(t-1) & \text{for } t \ge 1 \\ b & \text{for } 0 \le t < 1, \end{cases} \quad \text{and} \quad \hat{g} = \begin{cases} \nabla_e^N g & \text{for } t \ge 1 \\ 2\nabla_e^N(b\hat{h}_1) - \nabla_e^N(b\hat{h}_2) & \text{for } 0 \le t < 1. \end{cases}$$

Note that $\hat{a}$ satisfies condition (B.1), and also (B.2) with

$$[\hat{a}]_\alpha^{*,N} \le (T+1)^{\frac{\alpha}{2}}[\bar{a}]_\alpha^{(-\alpha),N} \le C(T)\bar{B} = \bar{A}$$

since $b = \bar{a}(0)$ does not depend on time t and $d(X) \le \sqrt{T+1}$.

Also, (B.3) holds, noting the bounds (5.36), (5.37), (5.47), (5.49), (5.52) (5.53), via (2.9), with

$$\begin{aligned} |\hat{g}|_0^{(1)} &\le (T+1)^{\frac{\alpha}{2}}\big\{G_\infty^1 + 2|\nabla_e^N(b\hat{h}_1)|_0 + |\nabla_e^N(b\hat{h}_2)|_0\big\} \\ &= C(n, c_\pm, T, \alpha)\big\{G_\infty^1 + \mathcal{C}_7\big\} = G_0^1 \end{aligned}$$

where

$$\begin{aligned} \mathcal{C}_7 &= |\nabla_e^N b|_0|\hat{h}_1|_0 + |\nabla_e^N \hat{h}_1|_0 + |\nabla_e^N b|_0|\hat{h}_2|_0 + |\nabla_e^N \hat{h}_2|_0 \\ &\le \mathcal{C}_1\big[\mathcal{C}_0(1+\mathcal{C}_1^5)\big] + \mathcal{C}_4 + \mathcal{C}_1\big[\mathcal{C}_0\mathcal{C}_1^4(1+\mathcal{C}_1) + \mathcal{C}_1^4\mathcal{C}_3\big] \\ &\quad + \mathcal{C}_0\mathcal{C}_1^4(1+\mathcal{C}_1^2) + \mathcal{C}_1^4\mathcal{C}_5 + \mathcal{C}_3\mathcal{C}_1^4(1+\mathcal{C}_1) \\ &\le C\Big\{\mathcal{C}_0\mathcal{C}_1(1+\mathcal{C}_1^5) + \mathcal{C}_3\mathcal{C}_1^4(1+\mathcal{C}_1) + \mathcal{C}_4 + \mathcal{C}_5\mathcal{C}_1^4\Big\} \\ &\le C(\bar{B}+1)^{\frac{2}{\alpha}}\mathcal{C}_0(1+\mathcal{C}_1^{11}). \end{aligned}$$

The last line follows by incorporating the expressions of $\mathcal{C}_3$ in (5.45), $\mathcal{C}_4$ below (5.49), and $\mathcal{C}_5$ below (5.51). Here, the constants C are universal.

We now formulate the extended system for $t \ge 0$, which corresponds to (5.54) when $0 \le t < 1$ and (5.42) when $t \ge 1$, as

$$\partial_t V_e = \nabla_e^N\Big\{ - \hat{a}(X) \sum_{|e'|=1, e'>0} \nabla_{e'}^{N,*} V_{e'}(X)\Big\} + \hat{g}(X).$$

Observe, by (5.38) and (5.55), that

$$|V_e|_0^{(1)} \le (T+1)^{\frac{\alpha}{2}}\big\{\max_e |\xi_e^N|_0 + \max_e |\eta_e|_0\big\} \le C(n, c_\pm, T, \alpha)\big(W_\infty + \mathcal{C}_6\big).$$

We may now apply Theorem 5.5 to the system $\{V_e\}_{|e|=1}$. As $\xi_e(t) = V_e(t+1)$ for $t \ge 0$, the desired statements (5.39), (5.40), and (5.41) follow. $\square$

5.2 Second Schauder estimate for (1.1)

We now specialize to the setting introduced at the beginning of this section and apply Theorem 5.5, Corollary 5.6, and Theorem 5.7 to the equation (1.1) satisfying (1.2). See, correspondingly, Corollaries 5.8, 5.9 and 5.10 below, in which we obtain estimates of the second discrete derivatives of $u^N(t, \frac{x}{N})$ exhibiting singularities $\frac{1}{t}, \frac{1}{\sqrt{t}}, 1$, respectively, near $t = 0$, without assuming or assuming some regularity conditions for the initial value $u^N(0)$. Recall the correspondences at the beginning of Section 5.1.

Corollary 5.8. *Consider the nonlinear discrete PDE* (1.1) *satisfying* (1.2). *Then, we have*

$$|\tilde{\xi}_e^N|_{1+\sigma}^{(1)} \le C\big(K^{1+\frac{2}{\sigma}} + 1\big), \tag{5.56}$$

for every e, and

$$|\xi_e^N(X)| \le \frac{C(K^{\frac{1}{\sigma}} + 1)}{\sqrt{t}}, \tag{5.57}$$

$$|\nabla_{e_1}^N \xi_{e_2}^N(X)| \le \frac{C(K^{\frac{2}{\sigma}} + 1)}{t}, \tag{5.58}$$

$$|\nabla_{e_1}^N \nabla_{e_2}^N u^N(X)| \le \frac{C(K^{\frac{2}{\sigma}} + 1)}{t}, \tag{5.59}$$

for every e, e_1, e_2 and $X = (t, \frac{x}{N}) \in \Omega_N$ where $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : i = 1, 2)$.

Also, (5.56) *implies a σ-Hölder estimate for $\nabla_{e_1}^N \nabla_{e_2}^N u^N(X)$, that is*

$$[\nabla_{e_1}^N \nabla_{e_2}^N u^N]_\sigma^{(3-\sigma)} \le C\big(K^{\frac{3}{\sigma}} + 1\big) \tag{5.60}$$

for every e_1, e_2, where $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : 1 \le i \le 3)$ and $\|h\|_\infty := \|h\|_{L^\infty([u_-, u_+])}$ for a function h.

The Hölder seminorm $[\,\cdot\,]_\sigma^{(3-\sigma)}$ in (5.60) is weaker than $[\,\cdot\,]_\sigma^{(2)}$ which is expected from (5.56). The reason is that a larger diverging factor appears from $\nabla_{e_1}^N \nabla_{e_2}^N a_{x,e}(t)$ in the Taylor expansion; see the estimates for J_1 in the proof. However, in the next two corollaries, the diverging factor matches better.

We remark that we can also get an estimate $\langle \nabla_e^N u^N \rangle_{1+\sigma}^{(1)} \le C(K^{1+\frac{2}{\sigma}} + 1)$ (and therefore for $\langle \nabla_e^N \tilde{u}^N \rangle_{1+\sigma}^{(1)}$ by Lemma 3.5), by elements of the proof of the bound (5.60), but this is not detailed here. Similar estimates would also hold in the next two corollaries.

Proof. To apply Theorem 5.5, we estimate the right hand sides of (5.31) and (5.32) in terms of K in the context of (1.1) or (5.2). First, by the Hölder continuity of u^N given in (2.16) of Corollary 2.3 with $\alpha = \sigma$, we have

$$[\bar{a}]_\sigma^{*,N} = [\varphi'(u^N)]_\sigma^{*,N} \le \|\varphi''\|_\infty [u^N]_\sigma^{*,N} \le C'(K + 1) \tag{5.61}$$

where $C' = C'(n, c_\pm, T, \|f\|_\infty, \|\varphi''\|_\infty, \|u^N(0)\|_\infty)$. Next for g in (5.5), by noting (2.9), we have

$$|\nabla^N g|_0^{(1),N} \le KC(\varphi, f)|\nabla^N u^N|_0^{(1),N}, \tag{5.62}$$

where $C(\varphi, f) := \|\varphi''\|_\infty \|f\|_\infty + \|\varphi'\|_\infty \|f'\|_\infty$. However, by Corollary 4.3,

$$|\nabla^N u^N|_0^{(1),N} \le C''(K^{\frac{1}{\sigma}} + 1) \tag{5.63}$$

where $C'' = C''(n, c_\pm, T, \sigma, \|f\|_\infty, \|\varphi''\|_\infty, u_\pm)$. Thus, one can take $\bar{A} = C'(K+1)$ and $G_0^1 = C(\varphi, f)C''(K^{1+\frac{1}{\sigma}} + K)$ in the assumptions (B.2) and (B.3), respectively.

Next, for $|\tilde{\xi}_e^N|_0^{(1)}$, seen in (5.31) and (5.32),

$$|\xi_e^N|_0^{(1),N} = |\nabla_e^N \varphi(u^N)|_0^{(1),N} \le \|\varphi'\|_\infty |\nabla_e^N u^N|_0^{(1),N}. \tag{5.64}$$

Thus, by Lemma 3.5 and (5.63), we have

$$|\tilde{\xi}_e^N|_0^{(1)} = |\xi_e^N|_0^{(1),N} \le C'''(K^{\frac{1}{\sigma}} + 1)$$

where $C''' = C'''(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi'\|_\infty, \|\varphi''\|_\infty, u_\pm)$. Therefore, we can apply Theorem 5.5 to obtain the desired bounds: (5.56) from (5.31), (5.57) from (5.64) and (5.58) from (5.32), respectively.

We now argue the bound (5.59). As φ has a strictly increasing C^2 inverse, (5.59) follows from (5.57) and (5.58). Indeed, set

$$a_{x,e}^0(t) = \big(a_{x,e}(t)\big)^{-1} \equiv \big(a_{x,e}(u^N(t))\big)^{-1}. \tag{5.65}$$

Note, by the Lipschitz property of φ, and also the proof of Lemma 2.4 and (5.64), that

$$\begin{aligned} &\tfrac{1}{c_+} \le a_{x,e}^0 \le \tfrac{1}{c_-} \quad \text{and} \\ &|\nabla_e^N a_{x,e'}^0(t)| = \Big|\frac{\nabla_e^N a_{x,e'}(t)}{a_{x+e,e'}(t)a_{x,e'}(t)}\Big| \le C(c_\pm) \max_z |\nabla_e^N \varphi(u^N(t, \tfrac{z}{N})| \\ &\qquad \le C(c_\pm, \|\varphi'\|_\infty) \max_z |\nabla_e^N u^N(t, \tfrac{z}{N})| \le \frac{C(c_\pm, \|\varphi'\|_\infty)C'''}{\sqrt{t}}(K^{\frac{1}{\sigma}} + 1). \end{aligned} \tag{5.66}$$

Calculate, noting $\nabla_{e_2}^N u^N(X) = \frac{\xi_{e_2}^N(X)}{a_{x,e}(u^N(t))} = a_{x,e}^0(t)\xi_{e_2}^N(X)$ and by (2.9),

$$\begin{aligned} \nabla_{e_1}^N \nabla_{e_2}^N u^N(X) &= \nabla_{e_1}^N\big(a_{x,e_2}^0(t)\xi_{e_2}^N(X)\big) \\ &= \nabla_{e_1}^N a_{x,e_2}^0(t) \cdot \xi_{e_2}^N(X + \tfrac{e_1}{N}) + a_{x,e_2}^0(t) \cdot \nabla_{e_1}^N \xi_{e_2}^N(X). \end{aligned} \tag{5.67}$$

Now, the desired estimate (5.59) holds, by inputting (5.66), (5.57) and (5.58) into (5.67), with constant $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : i = 1, 2)$.

Finally, to show the Hölder estimate in (5.60), by (5.67), write

(5.68)

$$\nabla_{e_1}^N \nabla_{e_2}^N u^N(X) - \nabla_{e_1}^N \nabla_{e_2}^N u^N(Y)$$

$$= \big(\nabla^N_{e_1} a^0_{x,e_2}(t) - \nabla^N_{e_1} a^0_{y,e_2}(s)\big)\xi^N_{e_2}(X + \tfrac{e_1}{N}) + \nabla^N_{e_1} a^0_{y,e_2}(s)\big(\xi^N_{e_2}(X+\tfrac{e_1}{N}) - \xi^N_{e_2}(Y+\tfrac{e_1}{N})\big)$$
$$+ \big(a^0_{x,e_2}(t) - a^0_{y,e_2}(s)\big)\nabla^N_{e_1}\xi^N_{e_2}(X) + a^0_{y,e_2}(s)\big(\nabla^N_{e_1}\xi^N_{e_2}(X) - \nabla^N_{e_1}\xi^N_{e_2}(Y)\big)$$
$$=: J_1 + J_2 + J_3 + J_4.$$

Recall, in the following, $|X-Y|^\sigma = \max\big\{|\frac{x}{N} - \frac{y}{N}|^\sigma, |t-s|^{\frac{\sigma}{2}}\big\}$.

The term J_4, as $|a^0|_0 \le c_-^{-1}$, is bounded via (5.56):

$$|J_4| = a^0_{y,e_2}(s)\big|\nabla^N_{e_1}\xi^N_{e_2}(X) - \nabla^N_{e_1}\xi^N_{e_2}(Y)\big| \le \frac{C}{(t\wedge s)^{1+\frac{\sigma}{2}}}\big(K^{1+\frac{2}{\sigma}}+1\big)|X-Y|^\sigma. \tag{5.69}$$

To bound J_3, note by the argument for Corollary 2.3 via Lemma 2.4, that $[a^0]^{*,N}_\sigma \le C(n, c_\pm, T, \|f\|_\infty, \|\varphi''\|_\infty)(K+1)$. Hence,

$$\big|a^0_{x,e_2}(t) - a^0_{y,e_2}(s)\big| \le \frac{C}{(t\wedge s)^{\frac{\sigma}{2}}}(K+1)|X-Y|^\sigma. \tag{5.70}$$

Then, by (5.70) and also the gradient bound (5.58), we obtain

$$|J_3| = \big|a^0_{x,e_2}(t) - a^0_{y,e_2}(s)\big||\nabla^N_{e_1}\xi^N_{e_2}(X)| \le \frac{C}{(t\wedge s)^{1+\frac{\sigma}{2}}}(K+1)(K^{\frac{2}{\sigma}}+1)|X-Y|^\sigma. \tag{5.71}$$

Recall the definition of $a_{x,e}(t)$ (cf. (1.8)). To deal with the terms J_1 and J_2, it will be helpful to observe $\nabla^N_e\varphi(u^N(t,\frac{x}{N})) = a_{x,e}(t)\nabla^N_e u^N(t,\frac{x}{N})$ and so, by the Lipschitz property of φ, that

$$\begin{aligned} |\xi^N_e(X) - \xi^N_e(Y)| &= |\nabla^N_e\varphi(u^N(t,\tfrac{x_1}{N})) - \nabla^N_e\varphi(u^N(s,\tfrac{x_2}{N}))| \\ &\le |a_{x_1,e}(t) - a_{x_2,e}(s)||\nabla^N_e u^N(t,\tfrac{x_1}{N})| + a_{x_2,e}(s)|\nabla^N_e u^N(t,\tfrac{x_1}{N}) - \nabla^N_e u^N(s,\tfrac{x_2}{N})|. \end{aligned} \tag{5.72}$$

Moreover, by Corollary 2.3, we have $|a_{x_1,e}(t) - a_{x_2,e}(s)| \le C(K+1)(t\wedge s)^{-\frac{\sigma}{2}}|X-Y|^\sigma$ and, by the Schauder estimate and gradient bound in Corollary 4.3, we have $|\nabla^N_e u^N(X) - \nabla^N_e(Y)| \le C(t\wedge s)^{-\frac{1+\sigma}{2}}(K^{1+\frac{1}{\sigma}}+1)|X-Y|^\sigma$ and $|\nabla^N_e u^N(X)| \le Ct^{-\frac{1}{2}}(K^{\frac{1}{\sigma}}+1)$.

Then, equation (5.72) is bounded

$$\begin{aligned} |\xi^N_e(X) - \xi^N_e(Y)| \le{}& \frac{C}{(t\wedge s)^{\frac{\sigma}{2}}}(K+1)|X-Y|^\sigma\cdot\frac{C}{\sqrt{t}}(K^{\frac{1}{\sigma}}+1) \\ &+ \frac{C}{(t\wedge s)^{\frac{1+\sigma}{2}}}(K^{1+\frac{1}{\sigma}}+1)|X-Y|^\sigma. \end{aligned} \tag{5.73}$$

We now bound J_2. By the bound $|\nabla^N_{e_1}a^0_{y,e_2}(s)| \le C\max_z|\nabla^N_{e_1}u^N(s,\frac{z}{N})| \le \frac{C}{\sqrt{s}}(K^{\frac{1}{\sigma}}+1)$ in (5.66), and (5.72), (5.73), we have

$$\begin{aligned} |J_2| &= |\nabla^N_{e_1}a^0_{y,e_2}(x)|\big|\xi^N_{e_2}(X+\tfrac{e_1}{N}) - \xi^N_{e_2}(Y+\tfrac{e_1}{N})\big| \\ &\le |\nabla^N_{e_1}a^0_{y,e_2}(x)|\Big[|a_{x_1,e}(t) - a_{x_2,e}(s)||\nabla^N_e u^N(t,\tfrac{x_1}{N})| \\ &\qquad + a_{x_2,e}(s)|\nabla^N_e u^N(t,\tfrac{x_1}{N}) - \nabla^N_e u^N(s,\tfrac{x_2}{N})\Big] \end{aligned} \tag{5.74}$$

$$\le \frac{C}{(t\wedge s)^{1+\frac{\sigma}{2}}}(K^{1+\frac{2}{\sigma}}+1)|X-Y|^\sigma.$$

We now address the term J_1. We rewrite the quantity $\nabla^N_{e_1}a^0_{x,e_2}(t)-\nabla^N_{e_1}a^0_{y,e_2}(s)$ into two terms, one where time t is fixed, and the other where space y is fixed:

$$\begin{aligned}\nabla^N_{e_1}a^0_{x,e_2}(t)-\nabla^N_{e_1}a^0_{y,e_2}(s) &= \big(\nabla^N_{e_1}a^0_{x,e_2}(t)-\nabla^N_{e_1}a^0_{y,e_2}(t)\big)\\ &\quad+\big(\nabla^N_{e_1}a^0_{y,e_2}(t)-\nabla^N_{e_1}a^0_{y,e_2}(s)\big) \ =: \ J_{1,1}+J_{1,2}.\end{aligned}$$

To bound $J_{1,1}$, we observe that it has the same form as $\nabla^N_{e_1}a_{x,e_2}(t)-\nabla^N_{e_1}a_{y,e_2}(t)$ except now the dependent variable is $\varphi(u^N)$ in the composition $\varphi^{-1}(\varphi(u^N))$, instead of simply u^N. Hence, the same proof of the Hölder estimate (6.18) via Lemma 6.5 and Remark 6.3, applied to a^0, gives

$$\begin{aligned}|J_{1,1}| \le C(\|\varphi''\|_\infty,\|\varphi'''\|_\infty)\Big[&\max_{e,z}|\nabla^N_e\varphi\big(u^N(t,\tfrac{x+z}{N})\big)-\nabla^N_e\varphi\big(u^N(t,\tfrac{y+z}{N})\big)|\\ &+\max_{e,e',z,z'}|\nabla^N_e\varphi\big(u^N(t,\tfrac{z}{N})\big)||\nabla^N_{e'}\varphi\big(u^N(t,\tfrac{z'}{N})\big)||X-Y|^\sigma\Big].\end{aligned} \tag{5.75}$$

Since $\xi^N_e=\nabla^N_e u^N$, we may now bound the first term on the right-hand side of (5.75) by (5.72) and (5.73). The second term on the right-hand side of (5.75), by the Lipschitz property of φ, is bounded by $C\max_{e,e',z,z'}|\nabla^N_e u^N(t,\frac{z}{N})||\nabla^N_{e'}u^N(t,\frac{z'}{N})||X-Y|^\sigma$ and further bounded using the gradient bound for $|\nabla^N_e u^N(X)|\le Ct^{-\frac12}(K^{\frac1\sigma}+1)$ in Corollary 4.3.

Then, together, we have

(5.76)

$$\begin{aligned}|J_{1,1}| &\le C\Big[\max_{e,z}|a_{x,e}(t)-a_{y,e}(t)||\nabla^N_e u^N(t,\tfrac{z}{N})|\\ &\quad+\max_{e,z}|\nabla^N_e u^N(t,\tfrac{x+z}{N})-\nabla^N_e u^N(t,\tfrac{y+z}{N})|\\ &\quad+\max_{e,e',z,z'}|\nabla^N_e u^N(t,\tfrac{z}{N})||\nabla^N_{e'}u^N(t,\tfrac{z'}{N})||X-Y|^\sigma\Big]\\ &\le \frac{C}{t^{\frac{1+\sigma}{2}}}(K^{1+\frac1\sigma}+1)|X-Y|^\sigma+\frac{C}{t}(K^{\frac2\sigma}+1)|X-Y|^\sigma \le \frac{C}{t}(K^{\frac2\sigma}+1)|X-Y|^\sigma.\end{aligned}$$

When $|J_{1,1}|$ is multiplied by $|\xi^N_{e_2}(X+\frac{e_1}{N})|$, since

$$|\xi^N_{e_2}(X)|\le\|\varphi'\|_\infty|\nabla^N_{e_2}u^N(X)|\le\frac{C}{\sqrt t}(K^{\frac1\sigma}+1), \tag{5.77}$$

by the gradient bound in Corollary 4.3, we have

$$|J_{1,1}|\cdot|\xi^N_{e_2}(X+\tfrac{e_1}{N})|\le\frac{C}{t^{\frac32}}(K^{\frac3\sigma}+1)|X-Y|^\sigma. \tag{5.78}$$

The argument with respect to $J_{1,2}$ is analogous. By the proof of (6.19) discussed in Remark 6.4, we obtain

$$|J_{1,2}|\le C(\|\varphi''\|_\infty,\|\varphi'''\|_\infty)\Big[\max_{e,z}|\nabla^N_e\varphi(u^N(t,\tfrac{z}{N}))-\nabla^N_e\varphi(u^N(s,\tfrac{z}{N}))|$$

$$+\max_z \frac{|\varphi(u^N(t,\frac{z}{N}))-\varphi(u^N(s,\frac{z}{N}))|}{\sqrt{|t-s|}}$$
$$+\max_{e,e',z,z'}\max_{r\geq t\wedge s}|\nabla_e^N\varphi(u^N(r,\tfrac{z}{N}))||\nabla_{e'}^N\varphi(u^N(r,\tfrac{z'}{N}))||X-Y|^\sigma\Big].$$

The first term on the right-hand side is bounded via (5.72) and (5.73) again. By the Lipschitz property of φ, the other two terms can be bounded in terms of u^N and the Hölder estimate $|u^N(t,\frac{z}{N})-u^N(s,\frac{z}{N})|\leq C(t\wedge s)^{-\frac{1+\sigma}{2}}(K^{1+\frac{1}{\sigma}}+1)|t-s|^{\frac{1+\sigma}{2}}$ and gradient bound $|\nabla_e^N u^N(X)|\leq Ct^{-\frac12}(K^{\frac{1}{\sigma}}+1)$ in Corollary 4.3.

Then,

$$\begin{aligned}(5.79)\qquad |J_{1,2}| &\leq C\Big[\max_{e,z}|a_{x,e}(t)-a_{y,e}(t)||\nabla_e^N u^N(t,\tfrac{z}{N})| \\ &\quad+\max_{e,z}|\nabla_e^N u^N(t,\tfrac{x+z}{N})-\nabla_e^N u^N(t,\tfrac{y+z}{N})| \\ &\quad+\max_z\frac{|u^N(t,\frac{z}{N})-u^N(s,\frac{z}{N})|}{\sqrt{|t-s|}} \\ &\quad+\max_{e,e',z,z'}\max_{r\geq t\wedge s}|\nabla_e^N u^N(r,\tfrac{z}{N})||\nabla_{e'}^N u^N(r,\tfrac{z'}{N})||X-Y|^\sigma\Big] \\ &\leq \frac{C}{(t\wedge s)^{\frac{1+\sigma}{2}}}(K^{1+\frac{1}{\sigma}}+1)|t-s|^{\frac{\sigma}{2}}+\frac{C}{t\wedge s}(K^{\frac{2}{\sigma}}+1)|X-Y|^\sigma.\end{aligned}$$

Multiplying the estimate of $J_{1,2}$ in (5.79) by $|\xi_{e_2}^N(X+\frac{e_1}{N})|$, noting the bound (5.77), we obtain

$$(5.80)\qquad |J_{1,2}|\cdot|\xi_{e_2}^N(X+\tfrac{e_1}{N})|\leq\frac{C}{(t\wedge s)^{\frac32}}(K^{\frac{3}{\sigma}}+1)|X-Y|^\sigma.$$

Hence, by combining the bounds (5.78) and (5.80), we have

$$|J_1|\leq\frac{C}{(t\wedge s)^{\frac32}}(K^{\frac{3}{\sigma}}+1)|X-Y|^\sigma=\frac{C}{(t\wedge s)^{\frac{3-\sigma+\sigma}{2}}}(K^{\frac{3}{\sigma}}+1)|X-Y|^\sigma.$$

Collecting the bounds for the terms J_1,J_2,J_3,J_4 gives (5.60) with respect to a constant $C=C(n,c_\pm,T,\sigma,\|f\|_\infty,\|f'\|_\infty,\|\varphi'\|_\infty,\|\varphi''\|_\infty,\|\varphi'''\|_\infty,u_\pm)$. In fact, we remark that the estimate for J_1 is the dominant one. □

A better estimate can be derived when the initial value $u^N(0,\cdot)$ has some smoothness, namely when condition (4.11) holds:

$$\sup_N\|u^N(0)\|_{C_N^2}\leq C_0.$$

A sufficient condition for (4.11) is when the initial data is between u_- and u_+ such that $u^N(0,\cdot)=u_0(\cdot)$ for $u_0\in C^2(\mathbb{T}^n)$.

Consider the weighted norm, for fixed e,

$$|\tilde\xi_e^N|_{1+\sigma}^*:=[\tilde\xi_e^N]_{1+\sigma}^*+\langle\xi_e^N\rangle_{1+\sigma}^*+|\nabla^N\tilde\xi_e^N|_0^{(1)},$$

analogous to that given in (4.3).

Corollary 5.9. *Consider the solution of the discrete PDE* (1.1) *satisfying* (1.2). *When* $u^N(0,\cdot)$ *satisfies the condition* (4.11), *we have*

$$|\tilde{\xi}_e^N|_{1+\sigma}^* \le CK_0^{1+\frac{2}{\sigma}},$$
$$|\xi_e^N(X)| \le CK_0^{\frac{1}{\sigma}},$$

for every $X = (t, \frac{x}{N}) \in \Omega_N$ *and* e, *and*

$$|\nabla_{e_1}^N \xi_{e_2}^N(X)| \le \frac{C}{\sqrt{t}} K_0^{\frac{2}{\sigma}},$$
$$|\nabla_{e_1}^N \nabla_{e_2}^N u^N(X)| \le \frac{C}{\sqrt{t}} K_0^{\frac{2}{\sigma}},$$

for every e_1, e_2, *where* $K_0 = K + C_0 + 1$ *and* $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : i = 1, 2)$.

Moreover, we have the Hölder estimate:

$$[\nabla_{e_1}^N \nabla_{e_1}^N u^N]_\sigma^{(1)} \le CK_0^{\frac{3}{\sigma}}, \tag{5.81}$$

for every e_1, e_2 *where* $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : 1 \le i \le 3)$.

Proof. When the condition (4.11) is satisfied, by Corollary 4.4, (5.63) is improved as

$$\|\nabla^N u^N\|_\infty = |\nabla^N \tilde{u}^N|_0^* \le CK_0^{\frac{1}{\sigma}}. \tag{5.82}$$

Then, the desired bound for $\xi_\cdot^N$ follows,

$$\|\xi_\cdot^N\|_\infty \le \|\varphi'\|_\infty \|\nabla^N u^N\|_\infty \le CK_0^{\frac{1}{\sigma}} \tag{5.83}$$

and also (5.62) is improved to

$$\|\nabla^N g\|_\infty \le C(\varphi, f) K \|\nabla^N u^N\|_\infty \le CKK_0^{\frac{1}{\sigma}}. \tag{5.84}$$

In these bounds, $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi'\|_\infty, \|\varphi''\|_\infty, u_\pm)$.

We have $\bar{A} = C(n, c_\pm, T, \|f\|_\infty, \|\varphi''\|_\infty, u_\pm) K_0$ (cf. (5.61)) in the estimate of $[\bar{a}]_\sigma^{*,N}$ by Corollary 2.6.

By Corollary 5.6, and the bound (5.83) shown for ξ_e^N above, we obtain the desired estimates for $|\tilde{\xi}_e^N|_{1+\sigma}^*$ and $|\nabla_{e_1}^N \xi_{e_2}^N|_0^*$. Indeed, by (5.33), we have

$$|\tilde{\xi}_e^N|_{1+\sigma}^* \le CK_0^{1+\frac{2}{\sigma}}$$

and

$$|\nabla_{e_1}^N \xi_{e_2}^N|_0^{(1)} \le CK_0^{\frac{2}{\sigma}},$$

with $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi'\|_\infty, \|\varphi''\|_\infty, u_\pm)$.

The desired bound on $\nabla^N_{e_1}\nabla^N_{e_2}u^N$ now follows as in the proof of Corollary 5.8. Indeed, in the present context, by (5.67), we need to bound

$$\nabla^N_{e_1}a^0_{x,e_2}(t)\cdot\xi^N_{e_2}(X+\tfrac{e_1}{N})+a^0_{x,e_2}(t)\cdot\nabla^N_{e_1}\xi^N_{e_2}(X).$$

The first term, noting (5.66), is bounded $|\nabla^N_{e_1}a^0_{x,e_2}\cdot\xi^N_{e_2}|\le C(c_\pm)\max_e\|\xi^N_e\|^2_\infty$. The second term, also noting (5.66), is bounded $|a^0_{x,e_2}\nabla^N_{e_1}\xi^N_{e_2}|\le C(c_-)|\nabla_{e_1}\xi^N_{e_2}|$. The estimate follows now by inserting the already proven bounds for ξ^N_e and $\nabla^N_{e_1}\xi^N_{e_2}$, with respect to a constant $C=C(n,c_\pm,T,\sigma,\|f\|_\infty,\|f'\|_\infty,\|\varphi'\|_\infty,\|\varphi''\|_\infty,u_\pm)$.

Finally, the Hölder bound (5.81) follows by scheme given in Corollary 5.8, using the proven bounds of $|\tilde\xi^N_e|^*_{1+\sigma}$ and $|\nabla^N_{e_1}\xi^N_{e_2}|^*_0$ and now Corollaries 4.4 and 2.6 instead of Corollaries 4.3 and 2.3. Indeed, consider the decomposition (5.68) of the difference $\nabla^N_{e_1}\nabla^N_{e_2}u^N(X)-\nabla^N_{e_1}\nabla^N_{e_2}(Y)=J_1+J_2+J_3+J_4$. By (5.69), (5.71), and (5.74), we may bound J_4, J_3 and J_2 respectively. By adding (5.76) to (5.79), multiplied by $\max_{e'}|\xi^N_{e'}|\le C(\|\varphi'\|_\infty)\max_{e'}|\nabla^N_{e'}u^N|_0$, we may bound J_1.

In particular, noting (5.66) and (5.70), with respect to (5.69) and (5.71), we have

$$|J_4|+|J_3|\le C\Big[|\nabla^N_{e_1}\xi^N_{e_2}(X)-\nabla^N_{e_1}\xi^N_{e_2}(Y)| + \max_z|u^N(t,\tfrac{z}{N})-u^N(s,\tfrac{z}{N})|\cdot|\nabla^N_{e_1}\xi^N_{e_2}(X)|\Big]. \tag{5.85}$$

By Corollary 4.4, we may bound $|u^N(t,\frac{z}{N})-u^N(s,\frac{z}{N})|\le CK_0|t-s|^{\frac{\sigma}{2}}$. Then, noting the already proven bounds for $[\nabla^N_{e_1}\xi^N_{e_2}]^{(1)}_\sigma\le|\tilde\xi^N_\cdot|^*_{1+\sigma}$ and for $|\nabla^N_{e_1}\xi^N_{e_2}(X)|$, we have the further bound

$$|J_4|+|J_3|\le\frac{C}{(t\wedge s)^{\frac{1+\sigma}{2}}}K_0^{\frac{3}{\sigma}}|X-Y|^\sigma.$$

Also, note the estimate $|\nabla^N_e a^0_{x,e'}(t)|\le C\max_z|\nabla^N_e u^N(t,\frac{z}{N})|$ in (5.66), $a_{x,e}\le c_+$, and $|a_{x,e}(t)-a_{y,e}(s)|\le CK_0|X-Y|^\sigma$ in Corollary 2.6. Then, by the uniform bounds for u^N in Corollary 4.4, we have, with respect to (5.74), that

$$\begin{aligned}|J_2|&\le C|\nabla^N_{e_1}u^N|_0\Big[|a_{x_1,e}(t)-a_{x_2,e}(s)||\nabla^N_e u^N(t,\tfrac{x_1}{N})| \\ &\qquad + a_{x_2,e}(s)|\nabla^N_e u^N(t,\tfrac{x_1}{N})-\nabla^N_e u^N(s,\tfrac{x_2}{N})|\Big] \\ &\le CK_0^{\frac{3}{\sigma}}|X-Y|^\sigma.\end{aligned} \tag{5.86}$$

Finally, from adding (5.76) to (5.79) and then multiplying by $\max_{e'}|\nabla^N_{e'}u^N|_0$, we have

$$\begin{aligned}|J_1|&\le C\max_{e'}|\nabla^N_{e'}u^N|\big(J_{1,1}+J_{1,2}\big) \\ &\le C\max_{e'}|\nabla^N_{e'}u^N|\Big[\max_{e,z}|a_{x,e}(t)-a_{y,e}(t)||\nabla^N_e u^N(t,\tfrac{z}{N})| \\ &\quad+\max_{e,z}|\nabla^N_e u^N(t,\tfrac{x+z}{N})-\nabla^N_e u^N(t,\tfrac{y+z}{N})| \\ &\quad+\max_z\frac{|u^N(t,\frac{z}{N})-u^N(s,\frac{z}{N})|}{\sqrt{|t-s|}}\end{aligned}$$

$$+ \max_{e,e',z,z'} \max_{r\geq t\wedge s} |\nabla_e^N u^N(r,\tfrac{z}{N})||\nabla_{e'}^N u^N(r,\tfrac{z'}{N})||X-Y|^\sigma\Big].$$

By the bounds for u^N in Corollary 4.4, we obtain

$$|J_1| \leq CK_0^{\frac{3}{\sigma}}|X-Y|^\sigma. \tag{5.87}$$

Note that the constant $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : 1 \leq i \leq 3)$, with respect to J_1, whereas with respect to J_2, J_3, J_4 the constant does not depend on $\|\varphi'''\|_\infty$. Therefore, combining the bounds for J_1, J_2, J_3, J_4, we arrive at the desired Hölder estimate for $\nabla_{e_1}^N\nabla_{e_2}^N u^N$, with constant $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : 1 \leq i \leq 3)$. □

Still more can be said with a more regular initial condition at the level of Theorem 5.7. Consider the initial value

$$\xi_e^N(0,\tfrac{x}{N}) = \nabla_e^N\varphi(u^N(0,\tfrac{x}{N})), \tag{5.88}$$

where

$$\sup_N \|u^N(0)\|_{C_N^4} \leq \bar{C}_0 < \infty. \tag{5.89}$$

Recall (1.10) for the discrete C^4-norm. A sufficient condition, when $\varphi \in C^4$, is that $u^N(0,\frac{x}{N}) = u_0(\frac{x}{N})$ for $x \in \mathbb{T}_N^n$ and $u_0 \in C^4(\mathbb{T}^n)$.

We now observe that $\sup_N \|\varphi(u^N(0,\cdot))\|_{C_N^4} \leq \mathcal{C}_0$ and so $\sup_N \|\xi_e^N(0)\|_{C_N^3} \leq \mathcal{C}_0$ where

$$\mathcal{C}_0 = C(\|\varphi^{(i)}\|_\infty : 1 \leq i \leq 4)\big[\bar{C}_0 + \bar{C}_0^2 + \bar{C}_0^3 + \bar{C}_0^4\big]. \tag{5.90}$$

Indeed, recalling (1.8) and denoting $a_{x,e_3}(0) := a_{x,e_3}(u^N(0))$, by (2.9), the second derivative of $\xi_{e_3}^N(0)$ is written as

$$\begin{aligned}
&\nabla_{e_1}^N\nabla_{e_2}^N\xi_{e_3}^N(0,\tfrac{x}{N})\\
&\quad= \nabla_{e_1}^N\nabla_{e_2}^N\nabla_{e_3}^N\varphi(u^N(0,\tfrac{x}{N}))\\
&\quad= \nabla_{e_1}^N\nabla_{e_2}^N\big[a_{x,e_3}(0)\nabla_{e_3}^N u^N(0,\tfrac{x}{N})\big]\\
&\quad= [\nabla_{e_1}^N\nabla_{e_2}^N a_{x,e_3}(0)]\nabla_{e_3}^N u^N(0,\tfrac{x+e_2+e_1}{N}) + [\nabla_{e_2}^N a_{x,e_3}(0)]\nabla_{e_1}^N\nabla_{e_3}^N u^N(0,\tfrac{x+e_2}{N})\\
&\qquad+ [\nabla_{e_1}^N a_{x,e_3}(0)]\nabla_{e_2}^N\nabla_{e_3}^N u^N(0,\tfrac{x+e_1}{N}) + a_{x,e_3}(0)\nabla_{e_1}^N\nabla_{e_2}^N\nabla_{e_3}^N u^N(0,\tfrac{x}{N}).
\end{aligned}$$

An analogous but longer expression can be written for $\nabla_{e_1}^N\nabla_{e_2}^N\nabla_{e_3}^N\xi_{e_4}^N(0,\frac{x}{N})$.

Uniform bounds over e_1, e_2, e_3 of $\nabla_{e_1}^N a_{x,e_4}(0)$, $\nabla_{e_1}^N\nabla_{e_2}^N a_{x,e_4}(0)$ and $\nabla_{e_1}^N\nabla_{e_2}^N\nabla_{e_3}^N a_{x,e_4}(0)$, in terms of φ and $\bar{C}_0$, follow now from 'mean-value' formulas (6.23) (or Lemma 2.4) and (6.27), as well as Remark 6.5 in the next section.

We remark that we will use that $\varphi \in C^5$ in the following corollary. This is the only place where the property that φ is *five* times continuously differentiable is used.

Corollary 5.10. *Consider the solution of the discrete PDE* (1.1) *satisfying* (1.2). *When* $u^N(0,\cdot)$ *satisfies the condition* (5.89), *we have, for every* e_1, e_2, *that*

$$|\tilde{\xi}_{e_1}^N|_{1+\sigma} \leq C\big[\bar{K}_0^{1+\frac{2}{\sigma}}(1+\bar{C}_0^{24}) + \bar{K}_0^{\frac{2}{\sigma}}\bar{C}_0^{48}\big], \tag{5.91}$$

$$(5.92)\qquad |\nabla_{e_1}^N \tilde{\xi}_{e_2}^N|_0 \le C\big[\bar{K}_0^{\frac{2}{\sigma}}(1+\bar{C}_0^{24}) + \bar{K}_0^{\frac{2}{\sigma}-1}\bar{C}_0^{48}\big],$$

$$(5.93)\qquad |\nabla_{e_1}^N \nabla_{e_2}^N u^N(X)| \le C\big[\bar{K}_0^{\frac{2}{\sigma}}(1+\bar{C}_0^{24}) + \bar{K}_0^{\frac{2}{\sigma}-1}\bar{C}_0^{48}\big],$$

$$(5.94)\qquad [\nabla_{e_1}^N \nabla_{e_2}^N u^N]_\sigma \le C\big[\bar{K}_0^{1+\frac{2}{\sigma}}(1+\bar{C}_0^{24}) + \bar{K}_0^{\frac{2}{\sigma}}\bar{C}_0^{48} + \bar{K}_0^{\frac{3}{\sigma}}\big].$$

Here, $\bar{K}_0 = K + \bar{C}_0 + 1$ and $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi^{(i)}\|_\infty, u_\pm : 1 \le i \le 5)$.

Proof. We apply Theorem 5.7 with $\bar{a} = \varphi'(u^N)$ and $g = K\bar{a}(t, \frac{x}{N})f(u^N(t, \frac{x}{N}))$ as in (5.3) and (5.5). First, we see the condition (5.35) for this $\bar{a}$ as

$$[\bar{a}]_\sigma^{(-\sigma),N} \le \bar{B} = C\big(K + \bar{C}_0 + 1\big) =: C\bar{K}_0$$

by the proof of Corollary 2.6, where $C = C(n, c_\pm, T, \|f\|_\infty, \|\varphi''\|_\infty, \|u^N(0)\|_\infty)$.

Next, let us check (5.36). We have $|\nabla_e^N \bar{a}(0)| \le \|\varphi''\|_\infty |\nabla_e^N u^N(0)|$. Also, we have $|\nabla_{e'}^N \nabla_e^N \bar{a}(0)| = |\nabla_{e'}^N\big(a_{x,e}^1(0)\nabla_e^N u^N(0)\big)|$, $|\nabla_{e''}^N \nabla_{e'}^N \nabla_e^N \bar{a}(0)| = |\nabla_{e''}^N \nabla_{e'}^N\big(a_{x,e}^1(0)\nabla_e^N u^N(0)\big)|$ and in addition $|\nabla_{e'''}^N \nabla_{e''}^N \nabla_{e'}^N \nabla_e^N \bar{a}(0)| = |\nabla_{e'''}^N \nabla_{e''}^N \nabla_{e'}^N\big(a_{x,e}^1(0)\nabla_e^N u^N(0)\big)|$, where

$$(5.95)\qquad a_{x,e}^1 = \begin{cases} \frac{\varphi'(u^N(X+\frac{e}{N})) - \varphi'(u^N(X))}{u^N(X+\frac{e}{N}) - u^N(X)} & \text{when } u^N(X+\frac{e}{N}) \ne u^N(X) \\ \varphi''(u^N(X)) & \text{when } u^N(X+\frac{e}{N}) = u^N(X). \end{cases}$$

Note $\|a_{x,e}^1\|_\infty \le \|\varphi''\|_\infty$ and $\|\nabla_{e'}^N a_{x,e}^1(0)\|_\infty \le C(\|\varphi'''\|_\infty)\|\nabla_{e'}^N u^N(0)\|_\infty$ by the proof of Lemma 2.4. Moreover, $|\nabla_{e''}^N \nabla_{e'}^N a_{x,e}^1(0)| \le C(\|\varphi'''\|_\infty, \|\varphi^{(4)}\|_\infty)\sum_{j=1}^2 \|u^N(0)\|_{C_N^2}^j$ by the proof of Lemma 6.5. Also, we have $|\nabla_{e'''}^N \nabla_{e''}^N \nabla_{e'}^N a_{x,e}^1(0)| \le C(\|\varphi^{(i)}\|_\infty : 2 \le i \le 5)\sum_{j=1}^3 \|u^N(0)\|_{C_N^3}^j$, applying Remark 6.5 to a^1. Hence, by (5.89), we have (5.36) in the form

$$(5.96)\qquad \begin{aligned} &\|\nabla_e^N \bar{a}(0)\|_\infty + \|\nabla_{e'}^N \nabla_e^N \bar{a}(0)\|_\infty + \|\nabla_{e''}^N \nabla_{e'}^N \nabla_e^N \bar{a}(0)\|_\infty + \|\nabla_{e'''}^N \nabla_{e''}^N \nabla_{e'}^N \nabla_e^N \bar{a}(0)\|_\infty \\ &\le C(\|\varphi^{(i)}\|_\infty : 2 \le i \le 5)\big(\bar{C}_0 + \bar{C}_0^2 + \bar{C}_0^3 + \bar{C}_0^4\big) \\ &= \mathcal{C}_1 \le C(\varphi)\mathcal{C}_0. \end{aligned}$$

Also, by (5.84), we see the condition (5.37) for g:

$$\|\nabla^N g\|_\infty \le C(\varphi, f)K\|\nabla^N u^N\|_\infty \le CK\bar{K}_0^{\frac{1}{\sigma}} = G_\infty^1$$

and, by (5.83), we have the condition (5.38):

$$\|\xi_e^N\|_\infty \le \|\varphi'\|_\infty \|\nabla_e^N u^N\|_\infty \le C\bar{K}_0^{\frac{1}{\sigma}} = W_\infty.$$

Here, $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|f'\|_\infty, \|\varphi'\|_\infty, \|\varphi''\|_\infty, u_\pm)$.

Now, by Theorem 5.7, recalling $\bar{K}_0$, noting $\alpha = \sigma$ and $\mathcal{C}_1 \le C(\varphi)\mathcal{C}_0$, we have

$$(5.97)\qquad |\tilde{\xi}_{e_1}^N|_{1+\sigma} \le C\big[(\bar{K}_0+1)^{1+\frac{1}{\sigma}}\big(W_\infty + \mathcal{D}_6\big) + G_\infty^1 + \mathcal{D}_7\big],$$

$$(5.98)\qquad |\nabla_{e_1}^N \tilde{\xi}_{e_2}^N|_0 \le C\big[(\bar{K}_0+1)^{\frac{1}{\sigma}}\big(W_\infty + \mathcal{D}_6\big) + (\bar{K}_0+1)^{-1}(G_\infty^1 + \mathcal{D}_7)\big],$$

with $\mathcal{D}_6 \le (\bar K_0+1)^{\frac{1}{\sigma}}\mathcal{C}_0(1+\mathcal{C}_0^5)$ and $\mathcal{D}_7 \le (\bar K_0+1)^{\frac{2}{\sigma}}\mathcal{C}_0(1+\mathcal{C}_0^{11})$. Therefore, bounds (5.91) and (5.92) follow, noting $\bar K_0 + 1 \le 2\bar K_0$ and the forms of G^1_∞ and W_∞, and $\mathcal{C}_0(1+\mathcal{C}_0^5) \le C(\bar C_0 + \bar C_0^{24})$, $\mathcal{C}_0(1+\mathcal{C}_0^{11}) \le C(\bar C_0 + \bar C_0^{48})$.

To derive (5.93), as in the proof of Corollaries 5.8 and 5.9, we have

$$\nabla^N_{e_1}\nabla^N_{e_2}u^N(X) = \nabla^N_{e_1}a^0_{x,e_2}(t)\cdot \xi^N_{e_2}(X+\tfrac{e_1}{N}) + a^0_{x,e_2}(t)\cdot \nabla^N_{e_1}\xi^N_{e_2}(X),$$

where $|\nabla^N_{e_1}a^0_{x,e_2}\cdot\xi^N_{e_2}| \le C(c_-,\|\varphi''\|_\infty)\max_e\|\xi^N_e\|^2_\infty$ and $|a^0_{x,e_2}\cdot\nabla^N_{e_1}\xi^N_{e_2}| \le C(c_-)|\nabla^N_{e_1}\xi^N_{e_2}|_0$. In particular, by (5.98), we have

$$|\nabla^N_{e_1}\nabla^N_{e_2}u^N(X)| \le C\big[W^2_\infty + (\bar K_0+1)^{\frac{1}{\sigma}}\big(W_\infty+\mathcal{D}_6\big) + (\bar K_0+1)^{-1}(G^1_\infty+\mathcal{D}_7)\big]. \tag{5.99}$$

The bound (5.93) now follows from the forms of G^1_∞ and W_∞ above.

Finally, the Hölder estimate (5.94) follows as in Corollary 5.9 (through the method given for Corollary 5.8), using now (5.97) and (5.98) to bound J_3 and J_4. Indeed, J_1, J_2 have the same estimates (5.87) and (5.86) as in the proof of Corollary 5.9.

To bound J_3 and J_4, by (5.85) and the estimate $|u^N(t,\cdot)-u^N(s,\cdot)| \le C\bar K_0|t-s|^{\frac{\sigma}{2}}$ by Corollary 4.4, we have

$$|J_4|+|J_3| \le C\Big[|\nabla^N_{e_1}\xi^N_{e_2}(X) - \nabla^N_{e_1}\xi^N_{e_2}(Y)| + \bar K_0|\nabla^N_{e_1}\xi^N_{e_2}(X)||X-Y|^\sigma\Big].$$

We now input the Hölder bound for $\nabla^N_e\xi^N_{e'}$ in (5.97) and the gradient bound in (5.98), and note that the term with the Hölder bound is dominant:

$$\bar K_0\big[\bar K_0^{\frac{1}{\sigma}}(W_\infty+\mathcal{D}_6) + \bar K_0^{-1}(G^1_\infty+\mathcal{D}_7)\big].$$

Hence,

$$|J_4|+|J_3| \le C\Big[\bar K_0^{1+\frac{1}{\sigma}}(W_\infty+\mathcal{D}_6) + G^1_\infty + \mathcal{D}_7\Big]|X-Y|^\sigma.$$

The following bound (5.100) now follows by adding the (equal) bounds (5.87) and (5.86) for J_1 and J_2 to the bound for $|J_4|+|J_5|$:

$$[\nabla^N_{e_1}\nabla^N_{e_2}u^N]_\sigma \le C\Big\{\bar K_0^{1+\frac{1}{\sigma}}\big(W_\infty+\mathcal{D}_6\big) + G^1_\infty + \mathcal{D}_7 + (\bar K_0+1)^{\frac{3}{\sigma}}\Big\} \tag{5.100}$$

and this implies the bound (5.94) similarly as above. □

6 Construction and estimates of fundamental solutions

This section presents a different approach to the discrete Schauder estimates. We rely on the fundamental solutions constructed by the parametrix method of E.E. Levi. This route is also well-known in PDE theory to derive the Schauder estimates; see Friedman [18] Chapter 1, Il'in, Kalashnikov and Oleinik [32], Ladyženskaja, Solonnikov and Ural'ceva [39] p.356– and Eidel'man [14]. Cannizzaro and Matetski [8] used Schauder estimates shown in this direction to study discrete KPZ equation.

We first construct the fundamental solutions of the linear difference operators of parabolic type and derive the estimate on its spatial discrete derivatives in Section 6.1. Then, we apply it to the quasilinear discrete PDE (1.1) in Section 6.2.

6.1 Fundamental solutions of linear parabolic difference operators

Let $L \equiv L_{t,x}$ be a linear (elliptic) discrete difference operator of second order with coefficients depending on $(t, \frac{x}{N})$ and let L^* be its dual operator with respect to the inner product $\langle f, g\rangle = \frac{1}{N^n}\sum_{x\in\mathbb{T}_N^n} f(\frac{x}{N})g(\frac{x}{N})$.

Definition 6.1. (i) *We call $p = p(s, \frac{y}{N}; t, \frac{x}{N})$, $s \le t$, a fundamental solution (in the forward sense) of $L - \partial_t$, if $(L_{t,x} - \partial_t)p = 0$ (L is acting on the variable $\frac{x}{N}$) and $p(s, \frac{y}{N}; s, \frac{x}{N}) = N^n\delta_{xy}$, where δ_{xy} is the Kronecker's δ.*
(ii) *We call $p^* = p^*(s, \frac{y}{N}; t, \frac{x}{N})$, $s \le t$, a fundamental solution (in the backward sense) of $L^* + \partial_s$, if $(L^*_{s,y} + \partial_s)p^* = 0$ and $p^*(t, \frac{y}{N}; t, \frac{x}{N}) = N^n\delta_{xy}$.*

We recall the following identity from [18], Chapter 1, Theorem 15.

Lemma 6.1. *$p(s, \frac{y}{N}; t, \frac{x}{N}) = p^*(s, \frac{y}{N}; t, \frac{x}{N})$.*

Proof. For $s \le r \le t$, set

$$q(r) := \frac{1}{N^n}\sum_z p(s, \tfrac{y}{N}; r, \tfrac{z}{N})p^*(r, \tfrac{z}{N}; t, \tfrac{x}{N}).$$

Then, its derivative in r vanishes:

$$\partial_r q(r) = \frac{1}{N^n}\sum_z \Big\{L_{r,z}p(s, \tfrac{y}{N}; r, \tfrac{z}{N})p^*(r, \tfrac{z}{N}; t, \tfrac{x}{N}) - p(s, \tfrac{y}{N}; r, \tfrac{z}{N})L^*_{r,z}p^*(r, \tfrac{z}{N}; t, \tfrac{x}{N})\Big\} = 0.$$

Thus, we have $p^*(s, \frac{y}{N}; t, \frac{x}{N}) = q(s) = q(t) = p(s, \frac{y}{N}; t, \frac{x}{N})$. $\square$

We specifically consider as $L = L_t$ the linear difference operator of the following form:

$$L_t u(\tfrac{x}{N}) = -\frac{1}{2}\sum_{|e|=1} a_e(t, \tfrac{x}{N})\nabla_e^{N,*}\nabla_e^N u(\tfrac{x}{N}) + \sum_{|e|=1} b_e(t, \tfrac{x}{N})\nabla_e^N u(\tfrac{x}{N}) + c_e(t, \tfrac{x}{N})u(\tfrac{x}{N}). \tag{6.1}$$

The coefficients are continuous in t and satisfy the following conditions:

$$\begin{aligned} &0 < c_- \le a_e(t, \tfrac{x}{N}) \le c_+ < \infty, \qquad a_e(t, \tfrac{x}{N}) = a_{-e}(t, \tfrac{x}{N}),\\ &|b_e(t, \tfrac{x}{N})|, |c_e(t, \tfrac{x}{N})| \le D_0, \end{aligned} \tag{6.2}$$

and Hölder continuity:

$$\begin{aligned} &|a_e(t, \tfrac{x}{N}) - a_e(s, \tfrac{y}{N})| \le C_H(|t-s|^{\alpha/2} + |\tfrac{x}{N} - \tfrac{y}{N}|^\alpha),\\ &|b_e(t, \tfrac{x}{N}) - b_e(t, \tfrac{y}{N})| \le C_H|\tfrac{x}{N} - \tfrac{y}{N}|^\alpha,\\ &|c(t, \tfrac{x}{N}) - c(t, \tfrac{y}{N})| \le C_H|\tfrac{x}{N} - \tfrac{y}{N}|^\alpha, \end{aligned} \tag{6.3}$$

for some $\alpha \in (0,1)$ and $D_0, C_H > 0$, recall that $|\cdot|$ denotes the distance in $\mathbb{T}^n$ and defined by (2.13). Writing C_H for $C_H \vee D_0$, we assume $D_0 = C_H$ for simplicity. The symmetry of a_e in e is assumed without loss of generality, since we have

$$\sum_{|e|=1} a_e\nabla_e^{N,*}\nabla_e^N = \sum_{|e|=1} a_e\nabla_{-e}^N\nabla_e^N = \sum_{|e|=1} a_e\nabla_e^N\nabla_{-e}^N = \sum_{|e|=1} a_{-e}\nabla_e^{N,*}\nabla_e^N$$

so that we may take a symmetrization $\frac{1}{2}(a_e + a_{-e})$ for a_e.

Applying the classical parametrix method due to Levi, we can construct the fundamental solution $p(s, \frac{y}{N}; t, \frac{x}{N})$, $0 \le s < t$, (in the forward sense) of the operator $\partial_t - L_t$ and obtain the following estimate on its (discrete) derivative in x.

Proposition 6.2. *The fundamental solution exists and we have the estimate*

$$\left|\nabla^N_{e,x} p(s, \tfrac{y}{N}; t, \tfrac{x}{N})\right| \le \frac{C}{\sqrt{t-s}}\, e^{CC_H^{2/\alpha}}\, g(c(t-s), \tfrac{x-y}{N}), \quad 0 \le s < t \le T, \tag{6.4}$$

for some $C = C(n, c_\pm, \alpha, T)$, where $g(t, \frac{x}{N}) := t^{-n/2} e^{-|\frac{x}{N}|^2/t}$ with $|\frac{x}{N}|$ defined in (2.13).

Proof. For each fixed $(s, \frac{y}{N})$, let $Z(s, \frac{y}{N}; t, \frac{x}{N}), t > s$, be the fundamental solution of $\mathcal{L}_{a(s,\frac{y}{N})} := \Delta^N_{a(s,\frac{y}{N})} - \partial_t$, where $\Delta^N_{a(s,\frac{y}{N})}$ is the discrete Laplacian with the constant coefficient $a(s, \frac{y}{N}) = (a_e(s, \frac{y}{N}))_{|e|=1}$ defined by (2.8). Recall that Δ^N_a is defined for $a = (a_e)_{|e|=1} \in \mathbb{R}^{2n}$ such that $c_- \le a_e \le c_+$ and $a_e = a_{-e}$. Namely, $\mathcal{L}_{a(s,\frac{y}{N})} Z(s, \frac{y}{N}; \cdot, \cdot) = 0$ for $t > s$, and $Z(s, \frac{y}{N}; s, \cdot) = N^n \delta_{\frac{y}{N}}(\cdot)$. Since $\mathbb{T}^n_N$ is spatially homogeneous, let us define $\bar{p}$ by $\bar{p}(t-s, \frac{x-y}{N}) = Z(s, \frac{y}{N}; t, \frac{x}{N})$. Note that $\bar{p}(t, \frac{x}{N}) = N^n \sum_{z \in \mathbb{T}^n_N} p^*(N^2 t, x+z)$ with p^* in [12]. By (1.9) of [12] for $p^*(= p_b)$, we have Aronson estimate:

$$\bar{p}(t, \tfrac{x}{N}) \le c g(kt, \tfrac{x}{N}).$$

Moreover, Gaussian bounds are known on the first and second discrete derivatives of Z:

$$|\nabla^N_{e,x} Z(s, \tfrac{y}{N}; t, \tfrac{x}{N})| \le c_1 \frac{g(k_1(t-s), \frac{x-y}{N})}{\sqrt{t-s}}, \tag{6.5}$$

$$|\nabla^N_{e',x} \nabla^N_{e,x} Z(s, \tfrac{y}{N}; t, \tfrac{x}{N})| \le c_2 \frac{g(k_2(t-s), \frac{x-y}{N})}{t-s}, \tag{6.6}$$

which follow from (1.4) and (1.5) of [12]; we take $N^2(t-s)$ for $t-s$ and estimate $1 \vee N^2(t-s) \ge N^2(t-s)$. Note that here we are in a very simple situation with the discrete Laplacian Δ^N_a with constant coefficient a, and without a random environment as in [12]. The condition $a \in \mathcal{A}(c_-, c_+, n, \Gamma)$ in [12] is satisfied due to the symmetry assumption $a_e = a_{-e}$ in our case. The constants c, c_1, c_2, k, k_1, k_2 depend only on n and $c_\pm$.

Let $\Phi(s, \frac{y}{N}; t, \frac{x}{N})$ be the solution of the integral equation

$$\Phi(s, \tfrac{y}{N}; t, \tfrac{x}{N}) = \mathcal{L} Z(s, \tfrac{y}{N}; t, \tfrac{x}{N}) + \int_s^t \tfrac{1}{N^n} \sum_z \mathcal{L} Z(r, \tfrac{z}{N}; t, \tfrac{x}{N}) \cdot \Phi(s, \tfrac{y}{N}; r, \tfrac{z}{N}) dr, \tag{6.7}$$

where $\mathcal{L} = L_t - \partial_t$ acts on the variable $(t, \frac{x}{N})$ (cf. (4.1) of [18]). Then, from (2.8) of [18] (or (11.13) of [39]),

$$p(s, \tfrac{y}{N}; t, \tfrac{x}{N}) := Z(s, \tfrac{y}{N}; t, \tfrac{x}{N}) + \int_s^t \tfrac{1}{N^n} \sum_z Z(r, \tfrac{z}{N}; t, \tfrac{x}{N}) \cdot \Phi(s, \tfrac{y}{N}; r, \tfrac{z}{N}) dr \tag{6.8}$$

is the fundamental solution of $\mathcal{L} = L_t - \partial_t$.

The equation (6.7) can be solved as

$$\Phi(s, \tfrac{y}{N}; t, \tfrac{x}{N}) = \sum_{k=1}^{\infty} (\mathcal{L}Z)_k(s, \tfrac{y}{N}; t, \tfrac{x}{N}), \tag{6.9}$$

where $(\mathcal{L}Z)_1 = \mathcal{L}Z$ and

$$(\mathcal{L}Z)_{k+1}(s, \tfrac{y}{N}; t, \tfrac{x}{N}) = \int_s^t \frac{1}{N^n} \sum_z \mathcal{L}Z(r, \tfrac{z}{N}; t, \tfrac{x}{N}) \cdot (\mathcal{L}Z)_k(s, \tfrac{y}{N}; r, \tfrac{z}{N}) dr; \tag{6.10}$$

see (4.4) and (4.5) of [18] (or (11.23) and (11.24) of [39]). Then, noting that $\mathcal{L}Z = L_t Z - \Delta^N_{a(s,\frac{y}{N})} Z$, from the bounds with $c_\pm, D_0 (= C_H)$ and Hölder continuity (6.3) of the coefficients of L_t in (6.1) and using (6.6), we have

$$|\mathcal{L}Z(s, \tfrac{y}{N}; t, \tfrac{x}{N})| \le C(n, c_\pm)(1 + C_H) \Big\{ \big|t - s\big|^{\frac{\alpha}{2}} + \big|\tfrac{x-y}{N}\big|^{\alpha} \Big\} \frac{g(k_2(t-s), \frac{x-y}{N})}{t-s}. \tag{6.11}$$

However, one can estimate g as

$$g(k_2 t, z) = (k_2 t)^{-\frac{n}{2}} e^{-\frac{|z|^2}{k_2 t}} \le C_a t^{-\frac{n}{2}} \Big(\tfrac{|z|^2}{t}\Big)^{-a}$$

for every $a > 0$. First, taking $a = \frac{1}{2}(d + 2 - 2\mu - \alpha)$, in terms of a parameter μ,

$$\begin{aligned} \big|t - s\big|^{\frac{\alpha}{2}} \frac{g(k_2(t-s), \frac{x-y}{N})}{t-s} &\le C_a (t-s)^{\frac{\alpha}{2} - 1 - \frac{n}{2}} \Big(\tfrac{|\frac{x-y}{N}|^2}{t-s}\Big)^{-\frac{1}{2}(n+2-2\mu-\alpha)} \\ &= C_a \frac{1}{(t-s)^{\mu}} \frac{1}{|\frac{x-y}{N}|^{n+2-2\mu-\alpha}}. \end{aligned} \tag{6.12}$$

Next, taking $a = \frac{1}{2}(d + 2 - 2\mu)$,

$$\begin{aligned} \big|\tfrac{x-y}{N}\big|^{\alpha} \frac{g(k_2(t-s), \frac{x-y}{N})}{t-s} &\le C_a \big|\tfrac{x-y}{N}\big|^{\alpha} (t-s)^{-1-\frac{n}{2}} \Big(\tfrac{|\frac{x-y}{N}|^2}{t-s}\Big)^{-\frac{1}{2}(n+2-2\mu)} \\ &= C_a \frac{1}{(t-s)^{\mu}} \frac{1}{|\frac{x-y}{N}|^{n+2-2\mu-\alpha}}. \end{aligned} \tag{6.13}$$

Thus, from the estimates (6.11), (6.12) and (6.13), we have shown

$$|\mathcal{L}Z(s, \tfrac{y}{N}; t, \tfrac{x}{N})| \le \frac{C(n, c_\pm, \mu)(1 + C_H)}{(t-s)^{\mu}} \frac{1}{|\frac{x}{N} - \frac{y}{N}|^{n+2-2\mu-\alpha}} \qquad (1 - \tfrac{\alpha}{2} < \mu < 1), \tag{6.14}$$

which corresponds to (4.3) of [18].

Then, by induction using (6.10), one can get for some $\lambda_0^*, C_1, C_2 > 0$, depending on $n, c_\pm, \alpha$:

$$|(\mathcal{L}Z)_m(s, \tfrac{y}{N}; t, \tfrac{x}{N})| \le C_1 \frac{(C_2 C_H)^m}{\Gamma(\frac{m\alpha}{2})} (t-s)^{\frac{m\alpha}{2} - \frac{n}{2} - 1} \exp\Big\{ -\frac{\lambda_0^* |\frac{x}{N} - \frac{y}{N}|^2}{4(t-s)} \Big\};$$

see (11.25) of [39] p.362 and (4.14) of [18]. Indeed, we replace the integral estimates such as Lemmas 2 and 3 in [18] by those on Riemann sums uniformly in N. Also the estimates on $\frac{\partial Z}{\partial x_i}, \frac{\partial^2 Z}{\partial x_i \partial x_j}$ in (4.10), (4.11) of [18] are replaced by those on discrete derivatives $\nabla^N_{e_i} Z, \nabla^N_{e_i}\nabla^N_{e_j} Z$ given in (6.5), (6.6). We note that two $m\alpha$ in (4.14) of [18] should be $\frac{m\alpha}{2}$ as in [39]. In fact, the estimate on $(\mathcal{L}Z)_2$ above (4.14) is shown with $\frac{2\alpha}{2}$. In our computation, we are especially concerned with the dependence of the estimate on the constant C_H, which should be C_H^m as above from (6.10). Therefore, from (6.9), we obtain (cf. (4.15) of [18]), in terms of C_3, C_4 depending on $n, c_\pm, \alpha, T$,

$$(6.15) \qquad |\Phi(s, \tfrac{y}{N}; t, \tfrac{x}{N})| \le \frac{C_3 e^{C_4 C_H^{2/\alpha}}}{(t-s)^{(n+2-\alpha)/2}} \exp\Big\{ -\frac{\lambda_0^* |\frac{x}{N} - \frac{y}{N}|^2}{4(t-s)} \Big\},$$

by estimating $(t-s)^{\frac{m\alpha}{2}} \le C_T^m (t-s)^{\frac{\alpha}{2}}$ for $m \ge 1$ and $0 \le t-s \le T$, and

$$\sum_{m=1}^\infty \frac{(C_2 C_H)^m}{\Gamma(\frac{m\alpha}{2})} \le C \sum_{n=1}^\infty \frac{(C_2 C_H)^{\frac{2}{\alpha} n}}{n!} \le C e^{(C_2 C_H)^{2/\alpha}}.$$

Further estimates are similar to [18]. We only note that the estimates on the Riemann sums are uniform in N as noted above, and by (6.8), we get the estimate (6.4) on $\nabla^N_{e,x} p$. $\square$

Remark 6.1. *A weaker estimate than* (6.4) *under the average in x (i.e., under $N^{-n}\sum_x$) might be available due to the result of [12]. Indeed, such an estimate can be shown at least if the coefficient a is independent of t so that it is temporally homogeneous, because the spatial shift acts ergodically.*

Remark 6.2. *If the coefficients have a singularity $t^{-\sigma/2}$ at $t = 0$ in* (6.3) *like in* (2.17)*, we face the following difficulty. Estimates* (6.11) *and* (6.14) *can be shown with an additional factor $s^{-\sigma/2}$ in the right hand side. Thus, from* (6.7)*, $\Phi(s, \frac{y}{N}; r, \frac{z}{N})$ has a singularity $s^{-\sigma/2}$ near $s = 0$. This is inherited by p^N so that the estimates diverge at $s = 0$.*

6.2 Application to the nonlinear problem

We now apply Proposition 6.2 to show the Schauder estimate for the solution $u^N = u^N(t, \frac{x}{N}), x \in \mathbb{T}^n_N$ of the quasilinear discrete PDE (1.1). One can rewrite the operator $L^N_{a(u^N(t))}$ in (1.5), determined from $\Delta^N \varphi(u^N(t))$ as in (1.7), into the form of (6.1). Indeed, recalling $\nabla^{N,*}_e = \nabla^N_{-e}$ and then using (2.9), we have

$$(6.16) \qquad \begin{aligned} L^N_{a(t)} u(\tfrac{x}{N}) &= -\tfrac{1}{2} \sum_{|e|=1} \nabla^N_{-e}\big(a_{x,e}(t) \nabla^N_e u\big)(\tfrac{x}{N}) \\ &= -\tfrac{1}{2} \sum_{|e|=1} \Big\{ a_{x-e,e}(t) \nabla^N_{-e}\nabla^N_e u(\tfrac{x}{N}) + \nabla^N_{-e} a_{x,e}(t) \cdot \nabla^N_e u(\tfrac{x}{N}) \Big\}. \end{aligned}$$

Therefore, we may take $a_e(t, \frac{x}{N}) = \frac{1}{4}(a_{x-e,e}(t) + a_{x+e,-e}(t))$, $b_e(t, \frac{x}{N}) = \frac{1}{2}\nabla^N_{-e} a_{x,e}(t)$ and $c_e(t, \frac{x}{N}) = 0$ in (6.1). Unfortunately, as b_e is given as a discrete derivative of $a_{x,e}(t)$, we need to use the first Schauder estimate for $u^N(t)$ to estimate it.

In the following, assuming the first Schauder estimate, we illustrate how one can derive the second Schauder estimate based on the estimates for the fundamental solutions. Constants C may change line to line, but these are referenced to earlier results.

We will assume the initial value $u_0 \in C^2(\mathbb{T}^n)$. Then, $\|u^N(0)\|_{C^2_N}$ is uniformly bounded in N and K. By (2.25) in Corollary 2.6, $a_e(X)$ satisfies

$$|a_e(t_1, \tfrac{x_1}{N}) - a_e(t_2, \tfrac{x_2}{N})| \le CK\Big\{\big|t_1 - t_2\big|^{\frac{\sigma}{2}} + \big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big|^{\sigma}\Big\}, \quad t_1, t_2 \ge 0, \tag{6.17}$$

where we simply write K instead of $K+1$ by assuming $K \ge 1$. Furthermore, by Corollary 4.4, under the assumption that $u_0 \in C^2(\mathbb{T}^n)$, we have $[\tilde{u}^N]_{1+\sigma} (\le |\tilde{u}^N|_{1+\sigma}) \le CK^{1+\frac{1}{\sigma}}$ and $|\nabla_e^N u^N| \le CK^{\frac{1}{\sigma}}$. From this, as we will indicate in Remarks 6.3 and 6.4, b_e satisfies the Hölder continuity

$$|b_e(t, \tfrac{x_1}{N}) - b_e(t, \tfrac{x_2}{N})| \le CK^{\frac{2}{\sigma}}\big|\tfrac{x_1}{N} - \tfrac{x_2}{N}\big|^{\sigma}, \quad t \ge 0, \tag{6.18}$$

$$|b_e(t, \tfrac{x}{N}) - b_e(s, \tfrac{x}{N})| \le CK^{\frac{2}{\sigma}}|t-s|^{\frac{\sigma}{2}}, \quad x \in \mathbb{T}_N^n. \tag{6.19}$$

The constant D_0 in (6.2) can be taken as $D_0 = CK^{\frac{1}{\sigma}}$ by (6.23) below and Corollary 4.4. Note that (6.19) is unnecessary when we apply Proposition 6.2. It is used to estimate $J_{1,2}$ in the proof of Corollary 5.8.

Thus, a_e and b_e satisfy the conditions (6.2) and (6.3) with $C_H = CK^{\frac{2}{\sigma}}$ and $\alpha = \sigma$. Therefore, by Proposition 6.2 and also by Lemma 6.1 noting that $L^N_{a(u^N(t))}$ is symmetric, we have the estimates on the discrete derivatives both in x and y of the fundamental solution $p(s, \frac{y}{N}; t, \frac{x}{N})$ of $L^N_{a(u^N(t))} - \partial_t$:

$$\big|\nabla^N_{e,x} p(s, \tfrac{y}{N}; t, \tfrac{x}{N})\big|, \big|\nabla^N_{e,y} p(s, \tfrac{y}{N}; t, \tfrac{x}{N})\big| \le \frac{C}{\sqrt{t-s}}\, e^{CK^{4/\sigma^2}}\, g(c(t-s), \tfrac{x-y}{N}), \tag{6.20}$$

for $0 \le s < t \le T$.

With the gradient estimate on p in y in hand, we can derive the estimates on $\nabla^N_{e_1}\nabla^N_{e_2} u^N$ for the solution u^N of (1.1) based on Duhamel's formula and computations on commutators of difference operators. The gradient estimate of p in x would be useful to show the first Schauder estimate, but here we have assumed it.

Our result is the following. Note that this bound is much worse compared to that obtained in Corollary 5.10, though we assume only $u_0 \in C^2$ not in C^5 here.

Proposition 6.3. *Suppose the condition* (1.2) *holds at $t = 0$ and $u_0 \in C^2(\mathbb{T}^n)$. Then, we have*

$$|\nabla^N_{e_1}\nabla^N_{e_2} u^N(t, \tfrac{x}{N})| \le C(\|\nabla^N\nabla^N u^N(0)\|_{L^\infty} + 1)\exp\{Ce^{CK^{4/\sigma^2}}\}, \tag{6.21}$$

for all $t \in [0, T]$, $x \in \mathbb{T}_N^n$, and unit vectors e_1, e_2.
Here, the constant $C = C(n, c_\pm, T, \sigma, \|f\|_\infty, \|\varphi''\|_\infty, \|\varphi'''\|_\infty, u_\pm) > 0$.

We defer the proof of Proposition 6.3 until after we develop some estimates. We now compute the commutator $[\nabla_e^N, L_a^N]$ and see that it has a gradient form.

Lemma 6.4. *We have*

$$\nabla_e^N L_a^N u(\tfrac{x}{N}) = L_a^N \nabla_e^N u(\tfrac{x}{N}) + \sum_{|e'|=1, e'>0} \nabla_{e'}^N \left\{ \nabla_e^N a_{x,e'}(u) \tau_{e-e'} \nabla_e^N u(\tfrac{x}{N}) \right\}, \tag{6.22}$$

and the bound

$$|\nabla_e^N a_{x,e'}(u)| \le C\{|\nabla_e^N u(\tfrac{x}{N})| + |\nabla_e^N u(\tfrac{x-e'}{N})|\}, \tag{6.23}$$

where $C = \frac{1}{2}\|\varphi''\|_{L^\infty([u_-,u_+])}$.

Proof. The identity (6.22) is shown as

$$\begin{aligned}
&\nabla_e^N L_a^N u(\tfrac{x}{N}) = \sum_{|e'|=1, e'>0} \nabla_e^N \nabla_{e'}^N \left\{ a_{x,e'}(u) \tau_{-e'} \nabla_{e'}^N u(\tfrac{x}{N}) \right\} \\
&= \sum_{|e'|=1, e'>0} \nabla_{e'}^N \nabla_e^N \left\{ a_{x,e'}(u) \tau_{-e'} \nabla_{e'}^N u(\tfrac{x}{N}) \right\} \\
&= \sum_{|e'|=1, e'>0} \nabla_{e'}^N \left\{ a_{x,e'}(u) \nabla_e^N \tau_{-e'} \nabla_{e'}^N u(\tfrac{x}{N}) + \tau_e \tau_{-e'} \nabla_{e'}^N u(\tfrac{x}{N}) \cdot \nabla_e^N a_{x,e'}(u) \right\} \\
&= L_a^N \nabla_e^N u(\tfrac{x}{N}) + \sum_{|e'|=1, e'>0} \nabla_{e'}^N \{ \tau_{e-e'} \nabla_{e'}^N u(\tfrac{x}{N}) \cdot \nabla_e^N a_{x,e'}(u) \}.
\end{aligned}$$

Next, to show the bound (6.23), we write

$$\nabla_e^N a_{x,e'}(u) = N \left\{ \frac{\varphi(u(x+e)) - \varphi(u(x+e-e'))}{u(x+e) - u(x+e-e')} - \frac{\varphi(u(x)) - \varphi(u(x-e'))}{u(x) - u(x-e')} \right\}, \tag{6.24}$$

where we omit $\frac{1}{N}$ for the variables of u in the right hand side. Then, applying Lemma 2.4 with $a = u(x+e)$, $b = u(x+e-e')$, $c = u(x)$, $d = u(x-e')$, we obtain the bound (6.23). $\square$

For the solution $u^N = u^N(t, \frac{x}{N})$ of the discrete PDE (1.1), using (6.22) with $e = e_2$, we have

$$\begin{aligned}
\partial_t \nabla_{e_1}^N \nabla_{e_2}^N u^N &= \nabla_{e_1}^N \nabla_{e_2}^N L_a^N u^N + K \nabla_{e_1}^N \nabla_{e_2}^N f(u^N) \\
&= \nabla_{e_1}^N L_a^N \nabla_{e_2}^N u^N + \sum_{e'} \nabla_{e_1}^N \nabla_{e'}^N \{ \nabla_{e_2}^N a_{x,e'}(u^N) \tau_{e_2-e'} \nabla_{e_2}^N u^N \} \\
&\quad + K \nabla_{e_1}^N \{ g_{x,e_2}(u^N) \nabla_{e_2}^N u^N \}.
\end{aligned} \tag{6.25}$$

Here, $g_{x,e}(u)$ is defined from $f(u)$ in a similar way to $a_{x,e}(u)$ in (1.8). For the second term, by (2.9), we expand

$$\begin{aligned}
&\nabla_{e_1}^N \{ \nabla_{e_2}^N a_{x,e'}(u^N) \tau_{e_2-e'} \nabla_{e_2}^N u^N \} \\
&= \nabla_{e_1}^N \nabla_{e_2}^N a_{x,e'}(u^N) \tau_{e_2-e'+e_1} \nabla_{e_2}^N u^N + \nabla_{e_2}^N a_{x,e'}(u^N) \tau_{e_2-e'} \nabla_{e_1}^N \nabla_{e_2}^N u^N.
\end{aligned} \tag{6.26}$$

We now state the next lemma, in which $e_1 \in \mathbb{Z}^d$ does not need to have norm 1.

Lemma 6.5. *We have*

$$\nabla^N_{e_1}\nabla^N_{e_2} a_{x,e'}(u) = \tfrac{1}{2}\varphi''(u(\tfrac{x}{N}))\big(\nabla^N_{e_1}\nabla^N_{e_2}u(\tfrac{x}{N}) - \nabla^N_{e_1}\nabla^N_{e_2}u(\tfrac{x-e'}{N})\big) + \varphi''(u(\tfrac{x+e_1}{N}))\nabla^N_{e_1}\nabla^N_{e_2}u(\tfrac{x}{N}) + R^N(x). \tag{6.27}$$

where $R^N(x)$ is a sum of quadratic functions of $\{\nabla^N_i u^N(\frac{z}{N}); i = e_1, e_2, e_1+e_2, -e', z = x, x+e_1, x+e_2, x+e_1+e_2\}$, which is explicitly given in the proof. In particular, in $R^N(x)$, the squares of $\nabla^N_{e_1}u^N(\frac{z}{N})$ and $\nabla^N_{e_1+e_2}u^N(\frac{z}{N})$ do not appear.

Proof. Omitting $\frac{1}{N}$ in the variables again, the left hand side of (6.27) is rewritten as

$$\begin{aligned}
N^2\Big\{&\frac{\varphi(u(x+e_1+e_2)) - \varphi(u(x+e_1+e_2-e'))}{u(x+e_1+e_2) - u(x+e_1+e_2-e')} - \frac{\varphi(u(x+e_2)) - \varphi(u(x+e_2-e'))}{u(x+e_2) - u(x+e_2-e')} \\
&- \frac{\varphi(u(x+e_1)) - \varphi(u(x+e_1-e'))}{u(x+e_1) - u(x+e_1-e')} + \frac{\varphi(u(x)) - \varphi(u(x-e'))}{u(x) - u(x-e')}\Big\} \\
= N^2\Big\{&\varphi'(u(x+e_1+e_2)) \\
&+ \tfrac{1}{2}\varphi''(u(x+e_1+e_2))(u(x+e_1+e_2) - u(x+e_1+e_2-e')) \\
&+ \tfrac{1}{6}\varphi'''(u_1^*)(u(x+e_1+e_2) - u(x+e_1+e_2-e'))^2 \\
&- \varphi'(u(x+e_2)) - \tfrac{1}{2}\varphi''(u(x+e_2))(u(x+e_2) - u(x+e_2-e')) \\
&- \tfrac{1}{6}\varphi'''(u_2^*)(u(x+e_2) - u(x+e_2-e'))^2 \\
&- \varphi'(u(x+e_1)) - \tfrac{1}{2}\varphi''(u(x+e_1))(u(x+e_1) - u(x+e_1-e')) \\
&- \tfrac{1}{6}\varphi'''(u_3^*)(u(x+e_1) - u(x+e_1-e'))^2 \\
&+ \varphi'(u(x)) + \tfrac{1}{2}\varphi''(u(x))(u(x) - u(x-e')) + \tfrac{1}{6}\varphi'''(u_4^*)(u(x) - u(x-e'))^2\Big\},
\end{aligned}$$

by the mean value theorem for some $u_1^*, .., u_4^* \in \mathbb{R}$.

The sum of the terms containing $\frac{1}{6}\varphi'''$, written as $R_1^N(x)$, is given by

$$\begin{aligned}
R_1^N(x) =& \tfrac{1}{6}\varphi'''(u_1^*)(\nabla^N_{-e'}u(x+e_1+e_2))^2 - \tfrac{1}{6}\varphi'''(u_2^*)(\nabla^N_{-e'}u(x+e_2))^2 \\
&- \tfrac{1}{6}\varphi'''(u_3^*)(\nabla^N_{-e'}u(x+e_1))^2 + \tfrac{1}{6}\varphi'''(u_4^*)(\nabla^N_{-e'}u(x))^2.
\end{aligned}$$

The terms containing φ' are summarized as

$$\begin{aligned}
&N^2\{\varphi'(u(x+e_1+e_2)) - \varphi'(u(x+e_1)) - \varphi'(u(x+e_2)) + \varphi'(u(x))\} \\
&= N^2\Big\{\varphi''(u(x+e_1))(u(x+e_1+e_2) - u(x+e_1)) \\
&\qquad + \tfrac{1}{2}\varphi'''(u_5^*)(u(x+e_1+e_2) - u(x+e_1))^2 \\
&\qquad - \varphi''(u(x))(u(x+e_2) - u(x)) - \tfrac{1}{2}\varphi'''(u_6^*)(u(x+e_2) - u(x))^2\Big\} \\
&= N\Big\{\varphi''(u(x+e_1))\nabla^N_{e_2}u(x+e_1) - \varphi''(u(x))\nabla^N_{e_2}u(x)\Big\} \\
&\qquad + \tfrac{1}{2}\varphi'''(u_5^*)(\nabla^N_{e_2}u(x+e_1))^2 - \tfrac{1}{2}\varphi'''(u_6^*)(\nabla^N_{e_2}u(x))^2.
\end{aligned}$$

However, the first term is rewritten as

$$\varphi''(u(x+e_1))\nabla^N_{e_1}\nabla^N_{e_2}u(x) + N\big(\varphi''(u(x+e_1)) - \varphi''(u(x))\big)\nabla^N_{e_2}u(x)$$

$$= \varphi''(u(x+e_1))\nabla^N_{e_1}\nabla^N_{e_2}u(x) + \varphi'''(u_7^*)\nabla^N_{e_1}u(x)\nabla^N_{e_2}u(x).$$

Thus, the terms containing φ' are rewritten as

$$\varphi''(u(x+e_1))\nabla^N_{e_1}\nabla^N_{e_2}u(x) + R^N_2(x),$$

where

$$R^N_2(x) = \tfrac{1}{2}\varphi'''(u_5^*)(\nabla^N_{e_2}u(x+e_1))^2 - \tfrac{1}{2}\varphi'''(u_6^*)(\nabla^N_{e_2}u(x))^2 + \varphi'''(u_7^*)\nabla^N_{e_1}u(x)\cdot\nabla^N_{e_2}u(x).$$

Finally, the terms containing $\frac{1}{2}\varphi''$ are summarized as

$$\begin{aligned}
&\tfrac{1}{2}N^2\Big\{\varphi''(u(x+e_1+e_2))(u(x+e_1+e_2)-u(x+e_1+e_2-e')) \\
&\qquad -\varphi''(u(x+e_2))(u(x+e_2)-u(x+e_2-e')) \\
&\qquad -\varphi''(u(x+e_1))(u(x+e_1)-u(x+e_1-e')) + \varphi''(u(x))(u(x)-u(x-e'))\Big\} \\
&= \tfrac{1}{2}N^2\varphi''(u(x))\Big\{(u(x+e_1+e_2)-u(x+e_1+e_2-e')) \\
&\qquad\qquad -(u(x+e_2)-u(x+e_2-e')) \\
&\qquad\qquad -(u(x+e_1)-u(x+e_1-e')) + (u(x)-u(x-e'))\Big\} + R^N_3(x) \\
&= \tfrac{1}{2}\varphi''(u(x))\big\{\nabla^N_{e_1}\nabla^N_{e_2}u(x) - \nabla^N_{e_1}\nabla^N_{e_2}u(x-e')\big\} + R^N_3(x),
\end{aligned}$$

where $R^N_3(x)$ is given and rewritten as

$$\begin{aligned}
R^N_3(x) = &-\tfrac{1}{2}N\big\{\varphi''(u(x+e_1+e_2)) - \varphi''(u(x))\big\}\nabla^N_{-e'}u(x+e_1+e_2) \\
&+\tfrac{1}{2}N\big\{\varphi''(u(x+e_2)) - \varphi''(u(x))\big\}\nabla^N_{-e'}u(x+e_2) \\
&+\tfrac{1}{2}N\big\{\varphi''(u(x+e_1)) - \varphi''(u(x))\big\}\nabla^N_{-e'}u(x+e_1) \\
= &-\tfrac{1}{2}\varphi'''(u_8^*)\nabla^N_{e_1+e_2}u(x)\nabla^N_{-e'}u(x+e_1+e_2) \\
&+\tfrac{1}{2}\varphi'''(u_9^*)\nabla^N_{e_2}u(x)\nabla^N_{-e'}u(x+e_2) \\
&+\tfrac{1}{2}\varphi'''(u_{10}^*)\nabla^N_{e_1}u(x)\nabla^N_{-e'}u(x+e_1).
\end{aligned}$$

This completes the proof of (6.27) with $R^N(x) = R^N_1(x) + R^N_2(x) + R^N_3(x)$. □

Remark 6.3. First recall that (6.27) holds for any $e_1 \in \mathbb{Z}^d$, i.e., not necessarily $|e_1| = 1$. Taking $e_1 = x_2 - x_1$, $e_2 = -e$, $e' = e$, $x = x_1$ and multiplying by $\frac{1}{N}$, (6.27) implies that

$$\begin{aligned}
b_e(t,\tfrac{x_2}{N}) - b_e(t,\tfrac{x_1}{N}) = &\tfrac{1}{4}\varphi''(u(\tfrac{x_1}{N}))\big\{\nabla^N_{-e}u(\tfrac{x_2}{N}) - \nabla^N_{-e}u(\tfrac{x_1}{N})\big\} \\
&-\tfrac{1}{4}\varphi''(u(\tfrac{x_1}{N}))\big\{\nabla^N_{-e}u(\tfrac{x_2-e}{N}) - \nabla^N_{-e}u(\tfrac{x_1-e}{N})\big\} \\
&+\tfrac{1}{2}\varphi''(u(\tfrac{x_2}{N}))\big\{\nabla^N_{-e}u(\tfrac{x_2}{N}) - \nabla^N_{-e}u(\tfrac{x_1}{N})\big\} + \tfrac{1}{2N}R^N(\tfrac{x_1}{N}).
\end{aligned}$$

For the first three terms, we recall (1.2). Then, we can apply the first Schauder estimate $[\tilde{u}^N]_{1+\sigma} (\le |\tilde{u}^N|_{1+\sigma}) \le CK^{1+\frac{1}{\sigma}}$ shown in Corollary 4.4 under the assumption that $u_0 \in C^2(\mathbb{T}^n)$, and see that these three terms are bounded by $CK^{1+\frac{1}{\sigma}}\big|\frac{x_1}{N} - \frac{x_2}{N}\big|^\sigma$. The last term is a sum of quadratic functions of $\{\nabla_i^N u^N(\frac{z}{N})\}$ as indicated in Lemma 6.5. The term with $i = e_1 = x_2 - x_1$ is bounded as

$$|\nabla_{e_1}^N u^N(\tfrac{z}{N})| = \Big| \sum_{k=1}^{|x_2-x_1|} \nabla_{\tilde{e}_k} u^N(\tfrac{z_k}{N}) \Big| \le CK^{\frac{1}{\sigma}} |x_2 - x_1|,$$

by Corollary 4.4 by choosing $\tilde{e}_k : |\tilde{e}_k| = 1$ and z_k properly, where $|x_1 - x_2| = |x_1 - x_2|_{L^\infty}$. Similar for the term with $i = e_1 + e_2$. These terms do not appear in squares as we noted. From these observation, for $x_1 \neq x_2$,

$$\tfrac{1}{N}|R^N(\tfrac{x_1}{N})| \le \tfrac{1}{N} C\{K^{\frac{1}{\sigma}}|x_1 - x_2| \cdot K^{\frac{1}{\sigma}} + (K^{\frac{1}{\sigma}})^2\} \le 2CK^{\frac{2}{\sigma}} |\tfrac{x_1}{N} - \tfrac{x_2}{N}|.$$

Noting $\sigma < 1$ so that $K^{1+\frac{1}{\sigma}} \le K^{\frac{2}{\sigma}}$, we obtain (6.18) when the time variable is fixed.

Remark 6.4. We comment, by the same style of proof given for Lemma 2.5, we obtain

$$\begin{aligned}(6.28)\qquad \big|\nabla_{e_2}^N a_{x,e_1}^N(t) - \nabla_{e_2}^N a_{x,e_1}^N(s)\big| &\le C(\|\varphi''\|_\infty) \max_{e,y} \big|\nabla_e^N u^N(t, \tfrac{y}{N}) - \nabla_e^N u^N(s, \tfrac{y}{N})\big| \\ &\quad + \frac{C(\|\varphi'''\|_\infty)}{N} \max_{e,e',z,z'} \max_{r \ge t\wedge s} \big|\nabla_e^N u^N(r, \tfrac{z}{N})\big|\big|\nabla_{e'}^N u^N(r, \tfrac{z'}{N})\big|.\end{aligned}$$

When $\frac{1}{N} \le |t-s|^{\frac{\sigma}{2}}$, (6.28) and Corollary 4.4 give the formula (6.19) when the space variable is fixed. On the ther hand, when $\frac{1}{N} \ge |t-s|^{\frac{\sigma}{2}}$, we reduce, by the estimate on $\langle u^N \rangle_{1+\sigma}$ in Corollary 4.4 and also Lemma 2.4, that

$$\begin{aligned}(6.29)\qquad &\big|\nabla_{e_2}^N a_{x,e_1}^N(t) - \nabla_{e_2}^N a_{x,e_1}^N(s)\big| \\ &\quad \le C(\|\varphi'\|_\infty) N\big[|u^N(t, \tfrac{x+e_1}{N}) - u^N(s, \tfrac{x+e_1}{N})| + |u^N(t, \tfrac{x}{N}) - u^N(s, \tfrac{x}{N})|\big] \\ &\quad \le C(K^{1+\frac{1}{\sigma}} + 1) \cdot N|t-s|^{\frac{1+\sigma}{2}} \le C(K^{1+\frac{1}{\sigma}} + 1)|t-s|^{\frac{1}{2}} \\ &\quad \le C(K^{1+\frac{1}{\sigma}} + 1)|t-s|^{\frac{\sigma}{2}},\end{aligned}$$

yielding (6.19) in this case also.

Remark 6.5. Moreover, recalling $\|\varphi^{(i)}\|_\infty = \|\varphi^{(i)}\|_{L^\infty([u_-,u_+])}$ as in Corollary 2.3, we claim that

$$(6.30)\qquad |\nabla_{e_1}^N \nabla_{e_2}^N \nabla_{e_3}^N a_{x,e_4}(t)| \le C(\|\varphi^{(i)}\|_\infty : 1 \le i \le 4) \sum_{j=1}^3 \|u^N(t,\cdot)\|_{C_N^3}^j.$$

Indeed, omitting $\frac{1}{N}$ in the variables again, write

$$\begin{aligned} a_{x,e_4}(t) &= \varphi'(u^N(t,x)) + \frac{1}{2N}\varphi''(u^N(t,x))\nabla_{e_4}^N u^N(t,x) \\ &\quad + \frac{1}{3!N^2}\varphi'''(u^N(t,x))\big[\nabla_{e_4}^N u^N(t,x)\big]^2 + \frac{1}{4!N^3}\varphi^{(4)}(z^N(x;t,e_4))\big[\nabla_{e_4}^N u^N(t,x)\big]^3 \\ &= I_1 + I_2 + I_3 + I_4 \end{aligned}$$

where $z^N(x;t,e_4) : \mathbb{T}_N^n \to \mathbb{R}$ is the value between $u^N(t,x)$ and $u^N(t,x+e_4)$ (closest to $u^N(t,x)$ if more than one) with respect to the mean-value theorem. Then, we need to bound $\nabla_{e_1}^N \nabla_{e_2}^N \nabla_{e_3}^N I_i$ for $1 \le i \le 4$.

Recall the notation for a^1 in (5.95). Note, by (2.9),

$$
\begin{aligned}
\nabla^N_{e_1}\nabla^N_{e_2}\nabla^N_{e_3} I_1 &= \nabla^N_{e_1}\nabla^N_{e_2}(a^1_{x,e_3}\nabla^N_{e_3}u^N(t,x)) \\
&= \nabla^N_{e_1}\big[\nabla^N_{e_2}a^1_{x,e_3}\cdot\nabla^N_{e_3}u^N(t,x+e_2)+a^1_{x,e_3}\nabla^N_{e_2}\nabla^N_{e_3}u^N(t,x)\big].
\end{aligned}
$$

By the proof of Lemma 6.5, $|\nabla^N_{e_1}\nabla^N_{e_2}a^1_{x,e_3}| \le C\sum_{j=1}^2 \|u^N(t,\cdot)\|^j_{C^2_N}$, with $a^1_{\cdot,\cdot}$ in place of $a_{\cdot,\cdot}$. Also, by the proof of Lemma 2.4, $|\nabla^N_{e_1}a^1_{x,e_3}| \le C\|\nabla^N u^N\|_\infty \equiv C\max_e \|\nabla^N_e u^N\|_\infty$. Hence, using (2.9) again, $|\nabla^N_{e_1}\nabla^N_{e_2}\nabla^N_{e_3}I_1| \le C\sum_{j=1}^3 \|u^N(t,\cdot)\|^j_{C^3_N}$.

To same bound for $\nabla^N_{e_1}\nabla^N_{e_2}\nabla^N_{e_3}I_2$, because of the prefactor N^{-1} and $\nabla^N_{e_1}u(x) = N\big(u(x+e_1)-u(x)$, would follow from the estimate $|\nabla^N_{e_2}\nabla^N_{e_3}\big[\varphi''(u^N(t,x))\nabla^N_{e_4}u^N(t,x)\big]| \le C\sum_{j=1}^3 \|u^N(t,\cdot)\|^j_{C^3_N}$. We may write $\nabla^N_{e_3}\varphi''(u^N(t,x)) = a^2_{x,e_3}\nabla^N_{e_3}u^N(t,x)$ where a^2 is defined with φ'' instead of φ' as in the definition of a^1. Then, the same type of argument with respect to the term I_1 gives the desired bound.

The same bound for $\nabla^N_{e_1}\nabla^N_{e_2}\nabla^N_{e_3}I_3$, due to the prefactor N^{-2}, follows from the estimate $|\nabla^N_{e_3}\{\varphi'''(u^N(t,x))\big[\nabla^N_{e_4}u^N(t,x)\big]^2\}| \le C\sum_{j=1}^3 \|u^N(t,\cdot)\|^j_{C^3_N}$, shown by analogous arguments used for the term I_2.

The same bound for $\nabla^N_{e_1}\nabla^N_{e_2}\nabla^N_{e_3}I_3$, due to the prefactor is N^{-3}, follows from the estimate $|\varphi^{(4)}(z^N(x))\big[\nabla^N_{e_4}u^N(t,x)\big]^3| \le \|\varphi^{(4)}\|_\infty\|\nabla^N u^N\|^3_\infty$.

Hence, (6.30) is established. We comment that the estimate (6.30) was used to show (5.90).

Proof of Proposition 6.3. We apply (6.22) again taking $e=e_1$ and $u=\nabla^N_{e_2}u^N$ for the first term of (6.25). Then, we have

$$
\partial_t\nabla^N_{e_1}\nabla^N_{e_2}u^N = L^N_a\nabla^N_{e_1}\nabla^N_{e_2}u^N + Q^N(x),
$$

where the remainder term $Q^N(x)$ is the sum of the second and third terms in the right hand side of (6.25) and the last term in (6.22) with $e=e_1$ and $u=\nabla^N_{e_2}u^N$.

Then, by Duhamel's formula, we obtain for $v_{e_1,e_2}(t,\frac{x}{N}) := \nabla^N_{e_1}\nabla^N_{e_2}u^N(t,\frac{x}{N})$,

$$
\begin{aligned}
v_{e_1,e_2}(t,\tfrac{x}{N}) &= N^{-n}\sum_y v_{e_1,e_2}(0,\tfrac{y}{N})p(0,\tfrac{y}{N};t,\tfrac{x}{N}) \\
&+\int_0^t ds\,N^{-n}\sum_y\sum_{e'}\{\tau_{e_1-e'}\nabla^N_{e_1}\nabla^N_{e_2}u^N(s,\tfrac{y}{N})\cdot\nabla^N_{e_1}a_{y,e'}(u^N(s))\}\nabla^{N,*}_{e',y}p(s,\tfrac{y}{N};t,\tfrac{x}{N}) \\
&+\int_0^t ds\,N^{-n}\sum_y\sum_{e'}\nabla^N_{e_1}\{\nabla^N_{e_2}a_{y,e'}(u^N(s))\tau_{e_2-e'}\nabla^N_{e_2}u^N\}(s,\tfrac{y}{N})\nabla^{N,*}_{e',y}p(s,\tfrac{y}{N};t,\tfrac{x}{N}) \\
&+K\int_0^t ds\,N^{-n}\sum_y g_{y,e_2}(u^N(s))\nabla^N_{e_2}u^N(s,\tfrac{y}{N})\nabla^{N,*}_{e_1}p(s,\tfrac{y}{N};t,\tfrac{x}{N}).
\end{aligned}
$$

Let $v(t,\frac{x}{N}) = \sum_{e_1,e_2}|v_{e_1,e_2}(t,\frac{x}{N})|$ and note that $|\nabla^N u^N(s,\frac{y}{N})| \le CK^{\frac{1}{\sigma}}$ from the first Schauder estimate. Also, note that $N^{-n}\sum_y p = 1$ by the symmetry of p for the first term.

In addition, note that $g_{y,e_2}(u)$ is bounded and $K \cdot K^{1/\sigma} \le K^{3/\sigma}$ for the last term. Then, we have from (6.20), (6.23), (6.26) and (6.27) that

$$\|v(t)\|_{L^\infty} \le C\|v(0)\|_{L^\infty} + CK^{\frac{1}{\sigma}} \cdot e^{CK^{4/\sigma^2}} \Big(\int_0^t \frac{ds}{\sqrt{t-s}} \|v(s)\|_{L^\infty} + K^{\frac{2}{\sigma}}\sqrt{t} \Big), \tag{6.31}$$

for $t \in [0,T]$. This implies, by the argument in [19], p. 144 (letting $n \to \infty$), that

$$\|v(t)\|_{L^\infty} \le C(\|v(0)\|_{L^\infty} + K^{3/\sigma} e^{CK^{4/\sigma^2}}) \exp\Big\{ Ct \left(K^{\frac{1}{\sigma}} \cdot e^{CK^{4/\sigma^2}} \right)^2 \Big\} \tag{6.32}$$

concluding the proof of (6.21) by absorbing $K^{\frac{1}{\sigma}}$ and also $K^{3/\sigma}e^{CK^{4/\sigma^2}}$ in the higher order exponential by changing C. □

Remark 6.6. For the linear Laplacian (i.e., $\varphi(u) = cu$), we have $\nabla_x^N p^N = \nabla_y^N p^N$ due to $[\nabla_e^N, \Delta^N] = 0$ or $p^N = p^N(t-s, x-y)$ so that the computations made in Lemma 6.4 are unnecessary. In Lemma 6.4, especially (6.22), the second term, which is the error term obtained by computing $[\nabla_e^N, L_a^N]$, is in the form of a gradient. This is important to make the summation by parts in y and move the discrete derivative $\nabla_{e'}^N$ to p.

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