ON MICROSCOPIC DERIVATION OF A FRACTIONAL STOCHASTIC BURGERS EQUATION

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ABSTRACT. We derive, from a class of asymmetric mass-conservative interacting particle systems on $\mathbb{Z}$, with long-range jump rates of order $|\cdot|^{-(1+\alpha)}$ for $0 < \alpha < 2$, different continuum fractional SPDEs. More specifically, we show the equilibrium fluctuations of the hydrodynamics mass density field of zero-range processes, depending on the stucture of the asymmetry, and whether the field is translated with process characteristics velocity, is governed in various senses by types of fractional stochastic heat or Burgers equations.

The main result: Suppose the jump rate is such that its symmetrization is long-range but its (weak) asymmetry is nearest-neighbor. Then, when $\alpha < 3/2$, the fluctuation field in space-time scale $1/\alpha : 1$, translated with process characteristic velocity, irrespective of the strength of the asymmetry, converges to a fractional stochastic heat equation, the limit also for the symmetric process.

However, when $\alpha \geq 3/2$ and the strength of the weak asymmetry is tuned in scale $1 - 3/2\alpha$, the associated limit points satisfy a martingale formulation of a fractional stochastic Burgers equation.

1. INTRODUCTION

The purpose of this paper is to derive from a class of microscopic zero-range interacting particle systems on $\mathbb{Z}$, with asymmetric long-range jump rates, certain continuum 'fractional Burgers' and other stochastic partial differential equations (SPDE). Our motivations are three fold:

First, these results will be seen to complement recent work and conjectures in [10] which infer certain 'long-range' KPZ class variance orders from the study of occupation times in asymmetric exclusion processes on $\mathbb{Z}$ with long-range jump rates of order $|\cdot|^{-(1+\alpha)}$ for $\alpha > 0$.

Second, given the interest in anomalous scales and previous work on deterministic fractional Burgers equations [12], [34], [50], [52], it is a natural problem to try to understand the corresponding SPDEs.

Third, although with respect to nearest-neighbor mass-conservative interacting systems on $\mathbb{Z}$, there has been much interest in KPZ Burgers equation which has been interpreted and understood in several ways (cf. [1], [3], [11], [22], [23], [25], [28], [45] and references therein), there does not appear to be much work on deriving such equations in the long-range setting.

We now expand on these motivations before discussing results.

1.1. **Occupation times and KPZ class exponents.**

Consider the exclusion process on $\mathbb{Z}$ with single particle jump probability $p(x, y) = p(y - x)$. In such

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a process, each particle jumps at rate 1 and displaces according to p, except in that jumps to already occupied vertices are suppressed. The configuration η_t at time $t \geq 0$ is a collection of occupation numbers $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$ where $\eta_t(x)$ is particle number at x at time t. The system is a Markov process with a family of invariant measures ν_ρ, each indexed on configurations with asymptotic density $\rho \in [0,1]$; in fact, ν_ρ is a product of Bernoulli measures, with success probability ρ, over the lattice points in $\mathbb{Z}$, [38].

Suppose now the system is begun with distribution ν_ρ. It is known that the variance of the occupation time at the origin satisfies

$$\mathrm{Var}\int_0^t \big(\eta_s(x) - \rho\big)ds \ \sim\ 2t\int_0^t P\big(R_s = 0\big)ds,$$

where R_s is the position of a 'second-class' particle initially at the origin (cf. Subsection 1.1 [10]). Such a particle moves as a regular particle but also must exchange places when other regular particles decide to displace to its location.

When p is finite-range and with a drift $\sum_x xp(x) \neq 0$, it is known (cf. [5], [43]) that $\mathrm{Var}(R_t) = O(t^{4/3})$. Such second-class particle variances are known to connect to the variance of the height function for certain interfaces formed from the particle configuration, [4]. Now, suppose $\rho = 1/2$ so that $E[R_t] = 0$. With a Gaussian ansatz, one posits decay $P\big(R_s = 0\big) \sim \big(\mathrm{Var}(R_s)\big)^{-1/2}$, which in particular would give $\mathrm{Var}\int_0^t \big(\eta_s(0) - 1/2\big)ds \sim t^{4/3}$. Although this type of local variance estimate has not been proved, superdiffusive lower bounds have been shown [9].

For the purposes of this article, we say KPZ class variance orders are those of the second-class particle (or the occupation time), as the correspondence with a height function is not obvious in the long-range setting.

Now, when p is long-range, that is $s(x) = \big(p(x)+p(-x)\big)/2 = O(|x|^{-(1+\alpha)})$ and $a(x) = \big(p(x) - p(-x)\big)/2$ is say supported on nearest-neighbor steps $a(\pm 1) \neq 0$, one can ask about the occupation time variance orders. Surprisingly, in [10], it was shown, for several types of asymmetric jump probabilities including p, when $\alpha = 3/2$, the variance is of order $O(t^{4/3})$. Also for $\alpha > 2$, when the jump law has more than 2 moments, it was proved that the variance is of the *same* order as that for the finite-range system with a jump probability with the same drift. Then, it was conjectured (cf. Conjecture 2.17 in [10]), given that the system is more volatile as α grows, that the variance should be of order $O(t^{4/3})$ for all $\alpha \geq 3/2$, a 'long-range' extension of the standard KPZ class variance orders.

When $0 < \alpha < 3/2$, as shown in [10], the variance has the same order as for the corresponding symmetric process with symmetrized jump probability s, computed to be $O(t^{2-1/\alpha})$ for $\alpha \geq 1$ and $O(t)$ for $0 < \alpha < 1$. Therefore, in a sense, the long-range KPZ class variance orders should match those of the finite-range class when $\alpha \geq 3/2$, and those of the symmetrized system when $\alpha < 3/2$.

These are in a sense 'local' fluctuation results. One can ask whether the long-range 'bulk' fluctuations, that is those of the empirical density field, also follow such α-dependent characterizations. Given that the computations in [10] were done for the exclusion process, one can ask in addition whether the phenomenon extends to other mass-conservative interacting particle systems.

1.2. **Hydrodynamics and deterministic fractional Burgers equations.** For an array of weakly asymmetric nearest-neighbor exclusion processes on $\mathbb{Z}$, with jump probability $p(1) = 1/2 + c/n^{1/2}$ and $p(-1) = 1/2 - c/n^{1/2}$, it is well known that the diffusively scaled empirical density field,

$$\pi_t^{(n)} \ := \ \frac{1}{n^{1/2}} \sum_x \eta_{nt}(x) \delta_{\frac{x}{n^{1/2}}},$$

when started from an initial measure $\pi_0^{(n)}$ with density profile $\rho_0(\cdot)$–that is $\pi_0^{(n)}$ converges weakly to $\delta_{\rho_0(\cdot)}$–under appropriate entropy conditions, converges weakly to the unique solution of the hydrodynamic equation

$$\partial_t \rho \ = \ \frac{1}{2}\Delta\rho - 2c\nabla\rho(1-\rho); \ \ \rho(0,x) = \rho_0(x). \tag{1.1}$$

See [35] for a complete account.

However, when $1 \leq \alpha < 2$, for long-range weakly asymmetric processes, that is when $s(x) = O(|x|^{-(1+\alpha)})$ and a is nearest-neighbor, $a(1) = c/n^{1-1/\alpha}$ and $a(-1) = -c/n^{1-1/\alpha}$, the long-range density field $(1/n^{1/\alpha}) \sum_x \eta_{nt}(x)\delta_{x/n^{1/\alpha}}$, under an initial condition with density profile $\rho_0(\cdot)$, formally converges to the solution of

$$\partial_t \rho \ = \ \Delta^{\alpha/2}\rho - 2c\nabla\rho(1-\rho); \ \ \rho(0,x) = \rho_0(x).$$

Here $\Delta^{\alpha/2}$ is the standard fractional Laplacian, defined in (2.4).

When $0 < \alpha < 1$, no matter the order of the asymmetry $a(\pm 1)$, the long-range density field formally converges to the solution of

$$\partial_t \rho \ = \ \Delta^{\alpha/2}\rho; \ \ \rho(0,x) = \rho_0(x).$$

However, when $\alpha > 2$, under diffusive scaling and $a(\pm 1) = \pm c n^{-1/2}$, the density field $\pi_t^{(n)}$ formally tends to the solution of (1.1). Also, when $\alpha = 2$, under 'log' adjusted $a(\pm 1) = \pm c \log(n)/(n \log n)^{1/2}$, the field

$$\frac{1}{(n\log n)^{1/2}} \sum_x \eta_{nt}(x)\delta_{x/(n\log n)^{1/2}}$$

converges formally to the solution of (1.1).

For different particle systems, such as zero-range processes (cf. Section 2), which also have a family of invariant measures ν_ρ indexed by density, the formal long-range hydrodynamic equation, when $1 \leq \alpha < 2$, takes form

$$\partial_t \rho \ = \ \Delta^{\alpha/2}\tilde{g}(\rho) - 2c\nabla\tilde{g}(\rho); \ \ \rho(0,x) = \rho_0(x). \tag{1.2}$$

Here $\tilde{g}$ is a (nonlinear) 'flux' function defined in (2.1). When $0 < \alpha < 1$, the formal hydrodynamic equation is $\partial_t\rho = \Delta^{\alpha/2}\tilde{g}(\rho)$. When $\alpha \geq 2$, the formal equation is $\partial_t \rho = (1/2)\Delta\tilde{g}(\rho) - 2c\nabla\tilde{g}(\rho)$, corresponding to the limit of the density field seen in diffusive or 'log'-adjusted scales as for the exclusion system.

See [32] in the long-range context which addresses hydrodynamics, and also [12], [50], [52] which consider uniqueness and regularity of related equations.

It is natural to ask about the equilibrium fluctuations corresponding to these hydrodynamic limits. In particular, starting from an invariant measure ν_ρ, what are the limits of the fluctuation field $(1/n^{1/2\alpha}) \sum_x \big(\eta_{nt}(x) - \rho\big)\delta_{x/n^{1/\alpha}}$ when $0 < \alpha < 2$? When the process is symmetric, that is $p = s$, such limits were considered in [33] (cf. Proposition 2.1). The general answer, well understood from a perturbative view and in many finite range examples, is that the fluctuation limit should be a linearization of the hydrodynamic equation, forced with a certain white noise, [14], [18], [44], [48].

1.3. **KPZ and stochastic Burgers equations.** The KPZ equation,

$$\partial_t h \ = \ a\Delta h + b\big(\nabla h\big)^2 + c\dot{\mathcal{W}}_t,$$

has stimulated much recent activity in the probability/math physics literature, [17]. Here, $h(t,x)$ represents the continuum height of certain interfaces with certain growth rules. Part of the equation's mystique is that it is ill posed: The noise is not regular enough to allow a strong solution, and the square nonlinearity prevents a weak formulation.

Nevertheless, formally, the Cole-Hopf transform $Z(t,x) = e^{\lambda h(t,x)}$ with $\lambda = a/b$ satisfies the linear stochastic heat equation $\partial_t Z = a\Delta Z + (ac/b)Z\dot{\mathcal{W}}_t$ which is well-defined (cf. [36], [51]). One then declares $\log Z(t,x)$ as the 'solution' to the KPZ equation. In a recent tour-de-force, [28], what actual equation $\log Z(t,x)$ satisfies and its relation to the KPZ equation was made precise. See also [26] and [37] for recent alternative approaches.

From the microscopic point of view, the microscopic height function satisfies $h(t,x) - h(t,x+1) = \eta_t(x)$ where as before $\eta_t(x)$ is the particle number at x at time t. In nearest-neighbor exclusion processes, starting from ν_ρ, with jump probability which is weakly asymmetric in that $a(\pm 1) = O(n^{-1/4})$, instead of $O(n^{-1/2})$ as in the last subsection, using a microscopic Cole-Hopf transform, it was shown that the diffusively scaled height fluctuations converge to $\log Z(t,x)$, [11]. In [1], different initial conditions are considered, as well as importantly 'exact' statistics of the Cole-Hopf solution process.

Consider now the KPZ Burgers equation,

$$\partial_t u \ = \ a\Delta u + b\nabla u^2 + c\nabla\dot{\mathcal{W}}_t, \tag{1.3}$$

which formally governs the gradient $u = \nabla h$ of the KPZ equation solution. Again, the equation is ill posed. However, since $\eta_t(x)$ is the discrete gradient of the microscopic height function, to try to derive (1.3), it is natural to look at the fluctuation field which represents a microscopic form of u.

In [23] and [25], in a class of mass-conservative systems starting from ν_ρ, with nearest-neighbor weakly asymmetric jump probability so that $a(\pm 1) = O(n^{-1/4})$ as above, it was shown that all limit points $\mathcal{Z}_t$ of the field,

$$\mathcal{Z}^{(n)}_t \ = \ \frac{1}{n^{1/4}}\sum_x \tau_{\lfloor nvt\rfloor}\big(\eta_{nt}(x) - \rho\big)\delta_{x/n^{1/2}},$$

in a shifted frame with a characteristic speed vnt, satisfy a martingale formulation of (1.3). Namely, $\mathcal{Z}_t(H) - \mathcal{Z}_0(H) - c_1\int_0^t \mathcal{Z}_s(\Delta H)ds - c_2\mathcal{A}_t(H)$ is a martingale corresponding to $c\nabla\dot{\mathcal{W}}_t$. Here, the term $\mathcal{A}_t(H)$ is defined,

$$\mathcal{A}_t(H) \ = \ \lim_{\epsilon\downarrow 0}\int_0^t\int \nabla H(x)\mathcal{Z}_s(\tau_{-x}G_\epsilon)^2 ds dx,$$

where G_ϵ is a smoothing of the delta mass at 0, and $\tau_y\eta(x) = \eta(x+y)$ and $\tau_y G(x) = G(x+y)$. The constants c_1 and c_2 are homogenized factors reflecting the density ρ and the rates of particle interactions. Although uniqueness of a limit process has not been shown for this type of martingale formulation, it does indicate a structure corresponding to (1.3).

In this context, what is the behavior in long-range systems when $s(x) = O(|x|^{-(1+\alpha)})$ and a is nearest-neighbor of certain strength?

When $\alpha > 2$ and $a(\pm 1) = O(n^{-1/4})$, following the calculations in [25], one may see that the field $\mathcal{Z}_t^{(n)}$ has the same limit behavior, with different constants v, c_1 and c_2, as in the nearest-neighbor setting detailed above. Similarly, one can see that the same sort of behavior holds, when $\alpha = 2$ and $a(\pm 1) = O((\log n)/(n\log(n))^{1/4})$, with respect to the 'log-adjusted' field

$$\frac{1}{(n\log n)^{1/4}}\sum_x \tau_{\lfloor nvt\rfloor}(\eta_{nt}(x)-\rho)\delta_{x/(n\log n)^{1/2}}.$$

These computations are left to the interested reader.

Part of our motivation then is to ask, when $0 < \alpha < 2$ and $a(\pm 1)$ is of certain strength, whether the limits of the fluctuation field 'solve' a type of fractional KPZ Burgers equation,

$$\partial_t u \;=\; a\Delta^{\alpha/2}u + b\nabla u^2 + c\nabla^{\alpha/2}\dot{\mathcal{W}}_t. \tag{1.4}$$

Such an equation, in the finite volume, has been considered in [27] where it is shown that a 'controlled solution' exists when $\alpha > 1$ and that such a solution is unique when $\alpha > 5/2$. We comment that there does not seem to be a 'Cole-Hopf' formula to analyze (1.4). It would be of interest to understand the equation also from the point of view of Hairer's rough paths approach, [29]. In this respect, it appears that (1.4) formally can be made to make sense when $\alpha > 3/2$, [42].

1.4. **Discussion of main results.** To introduce the main ideas and to be concrete, we will concentrate in the article on zero-range processes (cf. definitions in Subsection 2) with jump probability p such that $s(x) = O(|x|^{-(1+\alpha)})$, for $0 < \alpha < 2$, and $a(\cdot)$ is nearest-neighbor with varying strengths, often depending on the scaling parameter n. It seems such systems are rich enough to capture a diverse range of fluctuation behaviors, depending on parameters.

We remark that the zero-range process is a representative system: In principle, the main results in the article should hold in a more general setting as in [25].

Although not our focus, we comment, as discussed in Subsection 1.3, that the fluctuation behavior, when $\alpha \geq 2$, is similar to that in the 'nearest-neighbor' interactions framework of [25]. Such a finding is in accordance with the 'local' fluctuation discussion in Subsection 1.1 for $\alpha \geq 2$.

Our first result (Theorem 2.3) sets the stage for later limits and identifies, in a fixed frame of reference, the equilibrium fluctuations of the density field, for long-range zero-range systems with the same nearest-neighbor weak asymmetries as in Subsection 1.2, namely $a(\pm 1) = O(n^{-(1-1/\alpha)})$, as corresponding to linearizations of the hydrodynamic limits near (1.2). The limits are two types of fractional stochastic heat equations (2.3) and (2.11), one without and one with a linear drift term, depending on whether $0 < \alpha < 1$ or $1 \leq \alpha < 2$ respectively. Such equations were considered in the literature with respect to limits of certain branching particle systems [20], [19].

Next, after absorbing linear drift terms, by observing these fluctuation fields in a moving frame with a characteristic velocity, we obtain a transition point at $\alpha = 3/2$ (Theorem 2.6). Namely, for $3/2 \leq \alpha < 2$, when $a(\pm 1) = O(n^{-1+3/2\alpha})$, the equilibrium fluctuation limit points satisfy a martingale formulation of a fractional stochastic Burgers equation (2.12). While for $0 < \alpha < 3/2$, no matter the order of the asymmetry $a(\pm 1)$, the equilibrium fluctuation limit is the unique solution of a fractional stochastic heat equation without drift.

As mentioned in Subsection 1.1, this result complements the work in [10] with respect to 'local' fluctuations of the exclusion occupation time, and shows a certain 'universality' of the transition point $\alpha = 3/2$ with respect to 'bulk' fluctuations, in a general class of zero-range systems, with 'long-range' transitions and nearest-neighbor asymmetries, across process characteristics. In particular, the presence of the 'gradient of the square' term in (2.12), when $\alpha \geq 3/2$, is more evidence for the 'strongly' asymmetric system, when $a(\pm 1)$ is a nonzero constant, to be in the standard KPZ class. In this respect, we note for the parameter $\alpha = 3/2$, the process is not weakly asymmetric but 'strongly' so. We mention also it remains open to show that the martingale formulation uniquely characterizes a limit solution of (2.12), although it suggests much of the structure of the equation (cf. Remark 2.7).

One may ask about the fluctuation field behavior of the process when its asymmetries are also 'long-range', for instance when $a(x)$ is proportional to $\mathrm{sgn}(x)|x|^{-(1+\theta)}$ for $\theta \geq \alpha$. The results in [10], also valid for such 'long-range' asymmetries, would indicate a dichotomy in the limits according to when $\alpha \geq 3/2$ or $\alpha < 3/2$, as in Theorem 2.6, should hold. Although informal calculations seem to verify such an $\alpha = 3/2$ transition point, we do not pursue this here. In this context, see the recent arXiv paper [24], which discusses some results for exclusion processes with 'long-range' asymmetries.

The methods of the article are to develop the stochastic differential of the fluctuation field $\mathcal{Z}_t^{(n)}$, and to close the equation by averaging nonlinear rate terms in terms of the field itself. Such averaging, in the fluctuation field context, known as a Boltzmann-Gibbs principle, has been proved in a sharp form in [25] for nearest-neighbor models. Taking advantage of a long-range adaptation, one can pass to the limit and obtain formally different SPDEs depending on parameters. However, to make this passage rigorous, unlike in the nearest-neighbor setting, as the fractional Laplacian $\Delta^{\alpha/2}$ does not take the class of Schwartz class functions to itself, one needs to 'lift' the processes considered to larger domains, as in [20], [19], which considered related limits.

In the next section, we define the zero-range model and state the results. In Section 3, in several subsections, the main statements are proved. In Section 4, the proof of a Boltzmann-Gibbs principle for long-range systems is given.

2. Models and Results

After defining the zero-range model and stating assumptions, we proceed to the main results.

2.1. **Notation and Assumptions.**

Define the symmetric jump probability $s = s_\alpha : \mathbb{Z} \to [0,1]$ by

$$s(x) \;=\; \frac{c_\alpha}{|x|^{1+\alpha}}; \quad x \neq 0$$

and $s(0) = 0$ for $\alpha > 0$ where c_α is a normalization constant. Let also $a : \mathbb{Z} \to \mathbb{R}$ be an anti-symmetric function given by

$$a(x) \;=\; \begin{cases} \;\;\,1 & \text{for } x = 1 \\ -1 & \text{for } x = -1 \\ \;\;\,0 & \text{otherwise.} \end{cases}$$

For $\gamma, \beta \geq 0$ and $n \geq 1$, define the jump probability $p = p_{n,\gamma,\beta,\alpha} : \mathbb{Z} \to [0,1]$ by

$$p(\cdot) \;=\; s(\cdot) + \frac{\beta}{n^\gamma} a(\cdot).$$

Here, $\beta = \beta(\alpha)$ is fixed small enough so that $0 < p(\pm 1) < 1$.

Let $g : \mathbb{N}_0 \to \mathbb{R}_+$ denote a 'rate' for the process, satisfying $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$. Informally, the zero-range system is described as follows: If there are k particles at a location, $g(k)$ is the rate at which one of these particles jumps. Then, the location to where it jumps to is governed by p. Such systems are well-studied in the literature, with respect to various applied models (cf. [21]), and include the case of independent particles when $g(k) \equiv k$.

More formally, let $\{\eta_t^{(n)} : t \geq 0\}_{n \geq 1}$ be a sequence of zero-range particle systems on the state space $\Omega = \mathbb{N}_0^{\mathbb{Z}}$ where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The configuration $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$ specifies the particle occupation numbers $\eta_t(x)$ at sites $x \in \mathbb{Z}$ at time $t \geq 0$.

To prepare for later scaling limits, we specify, in the nth process $\eta_\cdot^{(n)}$, that time is sped up in scale n. That is, the dynamics of $\eta_\cdot^{(n)}$ is given by the generator

$$L_n f(\eta) \;=\; n \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} p(y) g(\eta(x)) \big\{ f(\eta^{x,x+y}) - f(\eta) \big\}.$$

Here, $\eta^{v,w}$ is the configuration after a particle moves from v to w:

$$\eta^{v,w}(z) \;=\; \begin{cases} \eta(v) - 1 & \text{when } z = v \\ \eta(w) + 1 & \text{when } z = w \\ \quad\;\; \eta(z) & \text{otherwise.} \end{cases}$$

Since $\eta \in \Omega$ is a configuration where v is occupied, the particle number $\eta(v) \geq 1$ and so $\eta^{v,w}$ also belongs to Ω.

We will assume g satisfies the following condition, which bounds the departure rates.

(LIP) There is a constant M such that $\sup_k |g(k+1) - g(k)| \leq M < \infty$.

Under condition (LIP), the process $\eta_t^{(n)}$ can be constructed as a Markov process on Ω with a family of invariant measures $\{\bar{\nu}_\theta : 0 \leq \theta < \theta_*\}$ where $\theta_* = \liminf_{k \uparrow \infty} g(k) \leq \infty$. These probability measures, indexed by 'fugacities', are products over lattice points in $\mathbb{Z}$ with common marginals given by

$$\bar{\nu}_\theta(\eta(x) = k) \;=\; \frac{1}{Z_\theta} \frac{\theta^k}{g(k)!}$$

for $k \geq 0$. Here, $g(k)! = g(k) \cdots g(1)$ when $k \geq 1$, $g(0)! = 1$, and Z_θ is the normalization.

It will be convenient to index these measures by 'density', that is $\rho(\theta) = \int \eta(0) d\bar{\nu}_\theta$. One can see that ρ is a strictly increasing function of θ. Let $\theta = \theta(\rho)$ be the inverse function and define $\nu_\rho = \bar{\nu}_{\theta(\rho)}$ for $0 \leq \rho < \rho_*$ where $\rho_* = \lim_{\theta \uparrow \theta_*} \rho(\theta)$.

Moreover, with respect to a fixed ν_ρ, the process can be associated with a Markov semigroup on $L^2(\nu_\rho)$ with a Markov generator L_n and a core of local $L^2(\nu_\rho)$ functions. The adjoint L_n^* can be seen to be the generator with respect to reversed jump probability $p^*(\cdot) = p(-\cdot)$. Moreover, the measure ν_ρ is invariant with respect to the adjoint process, and is reversible when $p = p^* = s$. See [2] and [47] for more details about construction and invariant measures of the process.

Here, a local function is one which depends only on a finite number of occupation variables $\{\eta(x) : x \in \mathbb{Z}\}$. Also, in the following, we denote by $\mathbb{P}_\kappa$ and $\mathbb{E}_\kappa$ the measure and expectation of the process when started from initial measure κ. Also, E_κ, Cov_κ, and Var_κ will denote expectation, covariance, and variance with respect to κ. Occasionally, when the context is understood, the subscript will be dropped from these notations.

Define, for local f, the function $\tilde{f}(z) = E_{\nu_z}[f]$, when the expectation makes sense. We remark that the 'flux',

$$\tilde{g}(z) \ = \ E_{\nu_z}[g], \tag{2.1}$$

will feature in later results

The mixing properties of the system will play a role in the analysis. Consider the localized, ergodic process, corresponding to the symmetric jump probability $s(\cdot)$, on the interval $\Lambda_\ell = \{x \in \mathbb{Z} : |x| \leq \ell\}$ with $k \geq 0$ particles and generator

$$S_{k,\ell} f(\eta) \ = \ \sum_{x,y \in \Lambda_\ell} g(\eta(x))\big\{f(\eta^{x,y}) - f(\eta)\big\} s(y-x).$$

For this Markov chain, the canonical measure $\nu_{k,\ell} \ = \ \nu_\rho(\cdot| \sum_{x\in\Lambda_\ell} \eta(x) \ = \ k\}$ is reversible and invariant. Let $\lambda_{k,\ell}$ be the spectral gap, that is, the second smallest eigenvalue of $-S_{k,\ell}$ (with 0 being smallest). Denote $W(k,\ell) = \lambda_{k,\ell}^{-1}$ and recall Poincaré's inequality

$$\mathrm{Var}_{\nu_{k,\ell}}(f) \ \leq W(k,\ell) D_{k,\ell}(f)$$

where $D_{k,\ell}$ is the canonical Dirichlet form

$$D_{k,\ell}(f) \ = \ \frac{1}{2} \sum_{x,y\in\Lambda_\ell} E_{\nu_{k,\ell}}\big[g(\eta(x))\big\{f(\eta^{x,y}) - f(\eta)\big\}^2\big] s(y-x).$$

We will suppose the following condition which guarantees sufficient mixing for our purposes when $0 < \alpha < 2$:

(SG) There is a constant $C = C(\rho)$ such that

$$E_{\nu_\rho}\left[W\left(\sum_{x\in\Lambda_\ell} \eta(x), \ell\right)^2\right] \ \leq \ C\ell^{2\alpha}.$$

The condition (SG) is one on the rate g, useful in the proof of a certain 'ergodic replacement', namely the Boltzmann-Gibbs principle in Theorem 3.1.

There is a large class of rates for which this condition holds. Consider the process, on the complete graph with vertices in Λ_ℓ governing k particles, with generator

$$S^{\mathrm{unif}}_{k,\ell}(f) \ = \ \frac{1}{(2\ell)} \sum_{x,y\in\Lambda_\ell} g(\eta(x))\big\{f(\eta^{x,y}) - f(\eta)\big\}.$$

The measure $\nu_{k,\ell}$ is also reversible for this process, and the associated Dirichlet form is as follows:

$$D^{\mathrm{unif}}_{k,\ell}(f) \ = \ \frac{1}{(4\ell)} \sum_{x,y\in\Lambda_\ell} E_{\nu_{k,\ell}}\big[g(\eta(x))\big\{f(\eta^{x,y}) - f(\eta)\big\}^2\big].$$

Often, the spectral gap λ_m with respect to this mean-field process is easier to estimate. Suppose the bound $\lambda_m^{-1} \le r(k,\ell)$, in terms of a quantity $r(k,\ell)$, holds. Then, one can derive a bound on $W(k,\ell)$ for the long-range dynamics because

$$[(2\ell)^\alpha/c_\alpha]D_{k,\ell} \ \ge \ D^{\mathrm{unif}}_{k,\ell}(f).$$

Hence,

$$\mathrm{Var}_{\nu_{k,\ell}}(f) \ \le \ r(k,\ell)D^{\mathrm{unif}}_{k,\ell}(f) \ \le \ \frac{(2\ell)^\alpha r(k,\ell)}{c_\alpha}D_{k,\ell}(f)$$

which gives the estimate $W(k,\ell) \le r(k,\ell)[(2\ell)^\alpha/c_\alpha]$.

In passing, we observe that a calculation with the canonical measure $\nu_{k,\ell}$ (cf. property (MP) in [15]) shows, with respect to a universal constant C, that

$$D^{\mathrm{unif}}_{k,\ell}(f) \ \le \ C\ell^2 \sum_{\substack{|x-y|=1\\ x,y\in\Lambda_\ell}} E_{\nu_{k,\ell}}\big[g(\eta(x))\big\{f(\eta^{x,y}) - f(\eta)\big\}^2\big] \ \le \ \frac{C\ell^2}{c_\alpha}D_{k,\ell}(f).$$

Then, although not our focus, for $\alpha \ge 2$, one concludes $W(k,\ell) \le C\ell^2 c_\alpha^{-1} r(k,\ell)$.

Suitable mean-field spectral gaps, which lead to verification of (SG), have been proved for a large class of processes. In the following, C is a constant not depending on k or ℓ.

- When there exists $k_0 \in \mathbb{N}$ and $m_0 > 0$ fixed, such that $g(k+k_0) - g(k) \ge m_0$ for all $k \ge 0$, we have $r(k,\ell) \le C$, [15].
- When $g(k) = k^\beta$, for $0 < \beta < 1$, we have $r(k,\ell) \le C(1 + k/\ell)^\beta$, [41].
- When $g(k) = 1(k \ge 1)$, we have $r(k,\ell) \le C(1 + k/\ell)^2$, [40].

2.2. **Results.** We first define spaces needed to state the main theorems.

Let $\mathcal{S}(\mathbb{R})$ be the standard Schwartz space of smooth, rapidly decreasing functions equipped with the metric $d(f,g) = \sum_{k=0}^\infty 2^{-k}\min\{1, \|f-g\|_k\}$, given in terms of norms $\|f\|_k^2 = \max_{0\le k\le n}\sup_{x\in\mathbb{R}}(1+x^2)^k|\partial_x^j f(x)|$ where ∂_x^j is the jth order derivative in x. Let H_k be the completion of the compactly supported C^∞ functions with respect to $\|\cdot\|_k$. Then, $H_{k+1} \subset H_k$. Moreover, the inclusion operator $f \in H_{k+1} \mapsto f \in H_k$ is a Hilbert-Schmidt operator. We observe that $\mathcal{S}(\mathbb{R})$ is a nuclear Fréchet space.

Let H'_k be the dual of H_k equipped with the strong topology, and note $H'_{k+1} \supset H'_k$. The dual space of $\mathcal{S}(\mathbb{R})$, consisting of tempered distributions on $\mathbb{R}$, endowed with the strong topology, may be written $\mathcal{S}'(\mathbb{R}) = \cup_{k=0}^\infty H'_k$. The recent papers [6], [7], and [8] and references therein discuss in more detail these spaces, their duals, and associated topologies.

For a fixed $0 < T < \infty$, denote by $D([0,T], \mathcal{S}'(\mathbb{R}))$ and $C([0,T], \mathcal{S}'(\mathbb{R}))$ the function spaces of *càdlàg* and continuous maps respectively from $[0,T]$ to $\mathcal{S}'(\mathbb{R})$. Throughout, we equip both function spaces with the uniform topology. The bracket denotes $\langle\cdot,\cdot\rangle$ the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, but also between other pairs of spaces when the context is clear.

Further, let $\widehat{C}$ be the space of infinitely differentiable functions whose support is contained in $(-\delta_0, T)$ for some $\delta_0 > 0$ fixed. The completed tensor product $\mathcal{S}(\mathbb{R}) \otimes \widehat{C}$, topologized by norms $\|\Phi\|_k = \max_{0\leq \ell_1+\ell_2\leq k} \sup_{x\in\mathbb{R}, t\in[-\delta_0,T]}(1+x^2)^k|\partial_x^{\ell_1}\partial_t^{\ell_2}\Phi_t(x)|$ is also a nuclear Fréchet space (cf. [49] where the notation $\widehat{\otimes}$ is used in place of $\otimes$).

Throughout the article, the initial configuration $\eta_0^{(n)}$ will be distributed according to a fixed ν_ρ.

For $0 < \alpha < 2$, let now $\mathcal{Y}_\cdot^{(n)}$ be the density fluctuation field, acting on functions $H \in \mathcal{S}(\mathbb{R})$, given by

$$\mathcal{Y}_t^{(n)}(H) \ = \ \frac{1}{n^{1/2\alpha}} \sum_x H\left(\frac{x}{n^{1/\alpha}}\right)\left(\eta_t^{(n)}(x) - \rho\right).$$

We may view $\mathcal{Y}_\cdot^{(n)}$ as a member of $D([0,T], \mathcal{S}'(\mathbb{R}))$. By the central limit theorem, for each fixed $t \geq 0$, $\mathcal{Y}_t^{(n)}$ converges in distribution to $\dot{\mathcal{W}}_0$, the spatial white noise taking values in $\mathcal{S}'(\mathbb{R})$ with standard covariance

$$\mathrm{Cov}\big(\dot{\mathcal{W}}_0(G), \dot{\mathcal{W}}_0(H)\big) \ = \ \sigma^2(\rho)\int_{\mathbb{R}} G(x)H(x)dx \tag{2.2}$$

where $\sigma^2(\rho) = E_{\nu_\rho}\big[\eta(0) - \rho)^2\big]$.

Define also the space-time white noise $\dot{\mathcal{W}}_t$, which indexes spatial white noises in time, with covariance

$$\mathrm{Cov}\left(\dot{\mathcal{W}}_t(G), \dot{\mathcal{W}}_s(H)\right) \ = \ \delta(t-s)\sigma^2(\rho)\int_{\mathbb{R}} G(x)H(x)dx.$$

When $\beta = 0$ and $0 < \alpha < 2$, that is when $p = s$ and the process is symmetric, a martingale form of the following 'equilibrium fluctuations' result was shown in [33]. See also Theorem 2.3 which, when $\beta = 0$, recovers this statement. Throughout, the phrase 'with respect to' will be abbreviated 'w.r.t.'

Proposition 2.1. *When $\beta = 0$ and $0 < \alpha < 2$, starting from initial measure ν_ρ, the sequence $\mathcal{Y}_\cdot^{(n)}$, as $n \uparrow \infty$, converges weakly w.r.t. the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$ to the unique process $\mathcal{Y}_\cdot$ which solves the equation*

$$\partial_t \mathcal{Y}_t \ = \ \tilde{g}'(\rho)\Delta^{\alpha/2}\mathcal{Y}_t + \sqrt{\tilde{g}(\rho)}\nabla^{\alpha/2}\dot{\mathcal{W}}_t \tag{2.3}$$

in the generalized sense as in Definition 2.2 below.

Here, the fractional Laplacian operator $\Delta^{\alpha/2}$, acting on $H \in \mathcal{S}(\mathbb{R})$, is given by

$$(\Delta^{\alpha/2}H)(x) \ = \ \frac{1}{2}\int_{\mathbb{R}} s(y)\left[H(x+y) - 2H(x) + H(x-y)\right]dy, \tag{2.4}$$

where $s(x) = c_\alpha/|x|^{1+\alpha}$ is extended to $x \in \mathbb{R}$.

For $G, H \in \mathcal{S}(\mathbb{R})$, the noise

$$\frac{dN_t}{dt} \ := \ \nabla^{\alpha/2}\dot{\mathcal{W}}_t,$$

when integrated in time, is a spatial white noise with covariance

$$
\begin{aligned}
&E_{\nu_\rho}\left[\int_0^t \nabla^{\alpha/2}\dot{\mathcal{W}}_u(H)ds \cdot \int_0^t \nabla^{\alpha/2}\dot{\mathcal{W}}_u(G)du\right] \\
&= \sigma^2(\rho)t\int_{\mathbb{R}}\int_{\mathbb{R}} s(y)\big(H(x+y)-H(x)\big)\big(G(x+y)-G(x)\big)dydx \\
&= \sigma^2(\rho)t\int_{\mathbb{R}} G(x)\Delta^{\alpha/2}H(x)dx. \qquad (2.5)
\end{aligned}
$$

When $G=H$, we say $\|\nabla^{\alpha/2}G\|^2_{L^2(\mathbb{R}\times\mathbb{R})} := \sigma^2(\rho)\int_{\mathbb{R}}\int_{\mathbb{R}} s(y)\big(G(x+y)-G(x)\big)^2 dydx$.

Let $\{T_t : t\geq 0\}$ be the semigroup of symmetric bounded linear operators on $L^2(\mathbb{R})$ generated by $\Delta^{\alpha/2}$ (cf. [30]). Symbolically, (2.3) translates to

$$
\mathcal{Y}_t \;=\; \tilde{g}'(\rho)T_t^* Y_0 + \sqrt{\tilde{g}(\rho)}\int_0^t T^*_{t-s}\nabla^{\alpha/2}\dot{\mathcal{W}}_s ds, \qquad (2.6)
$$

where $*$ refers to the adjoint action in the definition $\langle T_t^*\mathcal{Y}_0, H\rangle = \langle \mathcal{Y}_0, T_t H\rangle$ and $\langle T^*_{t-s}\nabla^{\alpha/2}\dot{\mathcal{W}}_s, H\rangle = \langle \nabla^{\alpha/2}\dot{\mathcal{W}}_s, T_{t-s}H\rangle$.

Unfortunately, $\Delta^{\alpha/2}H$ and T_tH do not in general belong to $\mathcal{S}(\mathbb{R})$, and so the terms in (2.3) and (2.6), in weak formulation, do not make apriori sense. Nevertheless, suppose that all the terms of (2.3) and (2.6) have extension and make sense when integrated against test functions in the nuclear space, $\Phi_t(x) \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$.

Specifically, suppose the terms corresponding to maps, which take $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$ variously to

$$
\langle \mathcal{Y}_0, \Phi_0\rangle, \int_0^T \langle \mathcal{Y}_t, \Phi_t\rangle dt, \;\; \text{and} \;\; \int_0^T \langle N_t, \Phi_t\rangle dt, \qquad (2.7)
$$

and also

$$
\int_0^T \langle \mathcal{Y}_t, \Delta^{\alpha/2}\Phi_t\rangle dt, \; \Big\langle \mathcal{Y}_0, \int_0^T T_t\Phi_t dt\Big\rangle, \;\; \text{and} \;\; \int_0^T \Big\langle N_s, \int_s^T T_{t-s}\partial_t\Phi_t dt\Big\rangle ds, \quad (2.8)
$$

can be seen to define $\big(\mathcal{S}(\mathbb{R})\otimes\widehat{C}\big)'$-valued random variables.

Suppose also that the equation, which 'lifts' (2.3),

$$
\int_0^T \langle \mathcal{Y}_t, \partial_t\Phi_t + \tilde{g}'(\rho)\Delta^{\alpha/2}\Phi_t\rangle dt \;=\; -\langle \mathcal{Y}_0, \Phi_0\rangle + \sqrt{\tilde{g}(\rho)}\int_0^T \langle N_t, \partial_t\Phi_t\rangle dt \qquad (2.9)
$$

holds for $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$. Then, as discussed in Proposition 3.3 and Remark 3.4(a) in [20] (see also [19]), one can conclude that the evolution equation,

$$
\int_0^T \langle \mathcal{Y}_t, \Phi_t\rangle dt \;=\; \tilde{g}'(\rho)\Big\langle \mathcal{Y}_0, \int_0^T T_t\Phi_t dt\Big\rangle - \sqrt{\tilde{g}(\rho)}\int_0^T \Big\langle N_s, \int_s^T T_{t-s}\partial_t\Phi_t dt\Big\rangle ds, \qquad (2.10)
$$

also holds for $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$.

We remark that, by the Hahn-Banach theorem, the maps given by (2.7) and (2.8), and the associated equations above can be extended to larger domains (cf. Remark 3.4(a) in [20] and Remark 3.3 in [19]). As at most one functional $\Phi_\cdot \mapsto \int_0^T \langle \mathcal{Y}_t, \Phi_t\rangle dt$ can satisfy the evolution equation (2.10), and this map determines $\mathcal{Y}_\cdot$ (cf. Lemma 2.3 in [19], and also [13]), we obtain that (2.9) has a unique solution.

Definition 2.2. We will say that equation (2.3) holds in the 'generalized' sense when the terms in (2.9) and (2.10) define $(\mathcal{S}(\mathbb{R}) \otimes \widehat{C})'$-valued random variables on a common probability space, and the equation (2.9) is satisfied.

Part of the proof of our later results, those which aim to identify the limit behavior of fluctuation fields in terms of linear limit SPDEs, is to adopt the following strategy: One argues, with respect to any limit point, that all terms in (2.7) and (2.8) define $(\mathcal{S}(\mathbb{R}) \otimes \widehat{C})'$-valued random variables, and that the evolution equation (2.9) holds.

When $\beta > 0$, the strength of the weak-asymmetry γ should be specified. It turns out that γ should depend on α to obtain nontrivial limits.

Theorem 2.3. *Starting from initial measure ν_ρ, the sequence $\mathcal{Y}^{(n)}_\cdot$, as $n \uparrow \infty$, converges weakly w.r.t. the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$ to the unique process $\mathcal{Y}_\cdot$ which solves the following equations:*

When $\beta \geq 0$, $\gamma = 1 - 1/\alpha$, and $1 \leq \alpha < 2$,

$$\partial_t \mathcal{Y}_t \;=\; \tilde{g}'(\rho)\Delta^{\alpha/2}\mathcal{Y}_t + \beta\tilde{g}'(\rho)\nabla\mathcal{Y}_t + \sqrt{\tilde{g}(\rho)}\nabla^{\alpha/2}\dot{\mathcal{W}}_t. \tag{2.11}$$

in the generalized sense discussed in Remark 2.4.

When $\beta \geq 0$ and $0 < \alpha < 1$, no matter the value of $\gamma \geq 0$, $\mathcal{Y}_\cdot$ is symmetric and satisfies the limit equation (2.3) *in the generalized sense of Definition 2.2*

Remark 2.4. We interpret the equation (2.11) in terms of (2.3) by introducing a reference frame shift: That is, let $\mathcal{Z}_t(G) = \mathcal{Y}_t(G(\cdot - \beta\tilde{g}'(\rho)t)$ for $G \in \mathcal{S}(\mathbb{R})$ and $t \geq 0$. Then, we say $\mathcal{Y}_\cdot$ satisfies (2.11) in the 'generalized' sense if $\mathcal{Z}_\cdot$ satisfies the driftless (2.3) in the sense of Definition 2.2. Hence, well-posedness and uniqueness of the solution of (2.11) follows from that of (2.3) in the sense of Definition 2.2.

The limit equations for $\mathcal{Y}^{(n)}_\cdot$ change type according to when $\alpha < 1$ or $\alpha \geq 1$, and may be thought of as linearizations of the hydrodynamic equations mentioned already (cf. near (1.2)).

We note a 'crossover' effect is implied straightforwardly by the proof of Theorem 2.3: When $1 \leq \alpha < 2$ and $\gamma > 1 - 1/\alpha$, the extra drift term in (2.11) disappears. More precisely, the limit of $\mathcal{Y}^{(n)}_\cdot$ converges weakly w.r.t. the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$ to the unique solution of (2.3) in the sense of Definition 2.2.

To probe second-order effects, we now absorb the drift in (2.11) and observe the fluctuation field moving with a 'characteristic' velocity. Define $\mathcal{Y}^{(n),\rightarrow}_\cdot$, in terms of its action on $H \in \mathcal{S}(\mathbb{R})$, as

$$\mathcal{Y}^{(n),\rightarrow}_t(H) \;=\; \frac{1}{n^{1/2\alpha}}\sum_x H\left(\frac{x}{n^{1/\alpha}} - \frac{1}{n^{1/\alpha}}\left\{\frac{\beta\tilde{g}'(\rho)tn}{n^\gamma}\right\}\right)\left(\eta^{(n)}_t(x) - \rho\right).$$

Again, $\mathcal{Y}^{(n),\rightarrow}_\cdot$ belongs to $D([0,T], \mathcal{S}'(\mathbb{R}))$. The possible limits of $\mathcal{Y}^{(n),\rightarrow}_\cdot$, when $\beta > 0$, depend on the strength of the weak-asymmetry γ.

It will turn out that, as discussed in the introduction, when $\gamma = 1 - 3/2\alpha$ and $3/2 \leq \alpha \leq 2$, the asymmety is significant enough to introduce a 'quadratic' term in the limit. Formally, the limits of $\mathcal{Y}^{(n),\rightarrow}_t$ satisfy a type of (ill-posed) fractional KPZ-Burgers equation,

$$\partial_t \mathcal{Y}_t \;=\; \tilde{g}'(\rho)\Delta^{\alpha/2}\mathcal{Y}_t + \beta\tilde{g}''(\rho)\nabla\mathcal{Y}^2_t + \sqrt{\tilde{g}(\rho)}\nabla^{\alpha/2}\dot{\mathcal{W}}_t. \tag{2.12}$$

Note that if one replaces $\Delta^{\alpha/2}$ and $\nabla^{\alpha/2}$ by $\Delta/2$ and ∇, respectively, then the equation reduces to a KPZ-Burgers equation which governs ∇h_t, where h_t satisfies a KPZ equation. We also note that (2.12) reduces to (2.3) if $\beta\tilde{g}''(\rho) = 0$, for instance, in the case of independent particles when $g(k) \equiv k$ and $\tilde{g}(\rho) \equiv \rho$.

To give sense to this equation, as in [25], we define the notion of an 'L^2-energy' martingale formulation of (2.12). Let $\iota : \mathbb{R} \to [0,\infty)$ be given by $\iota(z) = (1/2)1_{[-1,1]}(z)$ and, for $\varepsilon > 0$, let $\iota_\varepsilon(z) = \varepsilon^{-1}\iota(\varepsilon^{-1}z)$. Let $G_\varepsilon : \mathbb{R} \to [0,\infty)$ be a smooth compactly supported approximating function in $\mathcal{S}(\mathbb{R})$ such that

$$\|G_\varepsilon\|^2_{L^2(\mathbb{R})} \leq 2\|\iota_\varepsilon\|^2_{L^2(\mathbb{R})} = \varepsilon^{-1} \text{ and } \lim_{\varepsilon\downarrow 0} \varepsilon^{-1/2}\|G_\varepsilon - \iota_\varepsilon\|_{L^2(\mathbb{R})} \;=\; 0. \tag{2.13}$$

Such approximating functions can be found by convoluting ι_ε with smooth kernels. For $x \in \mathbb{R}$, let τ_x denote the shift so that $\tau_x\eta(z) = \eta(z+x)$ and $\tau_x G_\varepsilon(z) = G_\varepsilon(x+z)$.

With respect to a process $\mathcal{Y}_\cdot \in C([0,T];\mathcal{S}'(\mathbb{R}))$, define the process $\mathcal{A}^\varepsilon_\cdot \in C([0,T],\mathcal{S}'(\mathbb{R}))$, for $\varepsilon > 0$, by its action on $H \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{A}^\varepsilon_t(H) \;=\; \int_0^t \int_{\mathbb{R}} \nabla H(x)\Big[\mathcal{Y}_u(\tau_{-x}G_\varepsilon)\Big]^2 dx du. \tag{2.14}$$

We say the process $\mathcal{Y}_\cdot$ satisfies an L^2 *energy condition* if, for $H \in \mathcal{S}(\mathbb{R})$, $A^\varepsilon_\cdot(H)$ is a 'uniformly L^2 Cauchy' sequence, as $\varepsilon \downarrow 0$, with respect to the space of random trajectories equipped with the complete metric $d(x_\cdot, y_\cdot) = \mathbb{E}[\sup_{t\in[0,T]}(x_t - y_t)^2]^{1/2}$, that is,

$$\lim_{\varepsilon_1,\varepsilon_2\downarrow 0} \mathbb{E}\Big[\sup_{t\in[0,T]} \big(\mathcal{A}^{\varepsilon_1}_t(H) - \mathcal{A}^{\varepsilon_2}_t(H)\big)^2\Big] \;=\; 0, \tag{2.15}$$

and the limit does not depend on the specific smoothing family $\{G_\varepsilon\}$. The limit process $\mathcal{A}_\cdot(H)$ belongs to $C([0,T])$, and is defined by the uniformly L^2 Cauchy limit

$$\mathcal{A}_t(H) \;:=\; \lim_{\varepsilon\downarrow 0} \mathcal{A}^\varepsilon_t(H).$$

Definition 2.5. We will say that $\mathcal{Y}_\cdot$ is a *fractional L^2-energy solution* of (2.12) if the following holds.

(i) For each fixed $t \in [0,T]$, $\mathcal{Y}_t$ is a spatial white noise with covariance (2.2).
(ii) The process $\mathcal{Y}_\cdot$ satisfies the L^2-energy condition (2.15).
(iii) There exists a process $\mathcal{A}_\cdot \in C([0,T],\mathcal{S}'(\mathbb{R}))$ whose action on $H \in \mathcal{S}'(\mathbb{R})$ is the uniformly L^2 Cauchy limit $\mathcal{A}_\cdot(H)$.
(iv) There is a process $\mathcal{M}_\cdot \in C([0,T],\mathcal{S}'(\mathbb{R}))$, such that, for $H \in \mathcal{S}(\mathbb{R})$, $\mathcal{M}_\cdot(H)$ is a continuous martingale with quadratic variation $\langle\mathcal{M}(H)\rangle_t = \tilde{g}(\rho)t\|\nabla^{\alpha/2}H\|^2_{L^2(\mathbb{R}\times\mathbb{R})}$.
(v) The maps, which take variously $H_\cdot \in \mathcal{S}(\mathbb{R}) \otimes \widehat{C}$ to $\int_0^T \mathcal{Y}_t(\Delta^{\alpha/2}H_t)dt$, $\int_0^T \mathcal{Y}_t(H_t)dt$, $\int_0^T \mathcal{M}_t(H_t)dt$, $\int_0^T \mathcal{A}_t(H_t)dt$, and $\mathcal{Y}_0(H_0)$, can be seen to define $(\mathcal{S}(\mathbb{R}) \otimes \widehat{C})'$-valued random variables.

(vi) Moreover, for $\Phi_\cdot \in \mathcal{S}(\mathbb{R}) \otimes \widehat{C}$, we have a.s. on a common probability space that

$$\begin{aligned}\int_0^T \mathcal{M}_t(\partial_t \Phi_t)dt \;&=\; \mathcal{Y}_0(\Phi_0) + \int_0^T \mathcal{Y}_t(\partial_t\Phi_t)dt \\ &+\tilde{g}'(\rho)\int_0^T \mathcal{Y}_t(\Delta^{\alpha/2}\Phi_t)dt - \beta\tilde{g}''(\rho)\int_0^T \mathcal{A}_t(\partial_t\Phi_t)dt.\end{aligned} \tag{2.16}$$

We remark that one can interpret (2.16) as a way to lift the formal equation $\mathcal{M}_t(H) = \mathcal{Y}_t(H) - \mathcal{Y}_0(H) + \tilde{g}'(\rho)\int_0^t \mathcal{Y}_s(\Delta^{\alpha/2}H)ds + \beta\tilde{g}''(\rho)\int_0^t \mathcal{A}_s(H)ds$ or (2.12), much as (2.9) lifts (2.3), to give the equation a precise meaning, in view of the ambiguity of the term $\int_0^t \mathcal{Y}_s(\Delta^{\alpha/2}H)ds$ as a process in time. An intermediate form of item (vi), concentrating on functions $\Phi_t(x) = f(t)H(x)$, where $H \in \mathcal{S}(\mathbb{R})$ and $f \in \widehat{C}$, would also suffice in this regard, in which case (2.16) reduces to

$$\begin{aligned}\int_0^T f'(t)\mathcal{M}_t(H)dt \;&=\; f(0)\mathcal{Y}_0(H) + \int_0^T f'(t)\mathcal{Y}_t(H)dt \\ &+\tilde{g}'(\rho)\int_0^T f(t)\mathcal{Y}_t(\Delta^{\alpha/2}H)dt - \beta\tilde{g}''(\rho)\int_0^T f'(t)\mathcal{A}_t(H)dt.\end{aligned} \tag{2.17}$$

Although, we have pursued the more general form, as it might be useful in later development, to verify part (vi), our strategy will be to show (2.17) and then to derive (2.16) by approximations. We also note that $\mathcal{M}_\cdot(H)$ is a Brownian motion by Levy's theorem, and so $\mathcal{M}_\cdot$ is a version of the noise $\sqrt{\tilde{g}(\rho)}N_\cdot$.

Theorem 2.6. *Starting from initial measure ν_ρ, when $3/2 \leq \alpha < 2$, $\gamma = 1 - 3/2\alpha$, and $\beta\tilde{g}''(\rho) \neq 0$, the sequence $\{\mathcal{Y}_\cdot^{(n),\rightarrow} : n \geq 1\}$ is tight w.r.t. the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$, and any limit point $\mathcal{Y}_\cdot$ is a fractional L^2-energy solution of* (2.12).

However, when $0 < \alpha < 3/2$ or $\beta\tilde{g}''(\rho) = 0$, no matter the value of $\gamma \geq 0$, the sequence $\mathcal{Y}_\cdot^{(n),\rightarrow}$, as $n \uparrow \infty$, converges weakly w.r.t. the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$ to the unique symmetric process $\mathcal{Y}_\cdot$ which solves the limit equation (2.3) *in the generalized sense of Definition 2.2*

Remark 2.7. The result indicates a transition point with respect to the long-range strength parameter value $\alpha = 3/2$, consistent with the 'local' fluctuation results in [10] (cf. Subsection 1.1). Namely, when $\alpha < 3/2$, the characteristic velocity translated fluctuation field limit, no matter the strength of the asymmetry, is the limit for the symmetric process.

However, when $\alpha \geq 3/2$, under an appropriate asymmetry scale $\gamma = \gamma(\alpha)$, the limit points satisfy a martingale formulation of a 'fractional' KPZ-Burgers equation, involving a 'quadratic gradient' term. Note, although quite suggestive, the uniqueness of this martingale formulation is still open. Part of the difficulty is that the process $\mathcal{A}_\cdot(H)$ is understood only as a Cauchy limit, and how it relates further to $\mathcal{Y}_\cdot$ is not clear.

Again, by the proof of Theorem 2.6, there is a 'crossover' effect in that, for $3/2 \leq \alpha \leq 2$ and $\gamma > 1 - 3/2\alpha$, the 'quadratic' term drops out and the sequence $\mathcal{Y}_\cdot^{(n),\rightarrow}$ converges weakly w.r.t. the uniform topology on $D([0,T], \mathcal{S}'(\mathbb{R}))$ to the unique solution of (2.3).

3. Proofs

The arguments for Theorems 2.3 and 2.6 adapt the 'hydrodynamics' scheme of [25], with some new features, to the long-range context, developing the stochastic differential of $\mathcal{Y}_t^{(n)}$ and $\mathcal{Y}_t^{(n),\rightarrow}$ into drift and martingale terms, before analyzing their limits. Since the arguments of the two theorems are similar, to simplify the discussion, we only prove in detail Theorem 2.6, the most involved.

In Subsection 3.1, various generator actions are computed in general. Then, in Subsection 3.2, a general 'Boltzmann-Gibbs' principle is stated which will help close equations. In Subsection 3.3, tightness of the processes in Theorem 2.6 is shown. In Subsection 3.4, we identify several features of the limit points. In Subsection 3.5, we discuss essential notions which put the fractional stochastic heat equation (2.3) and the energy solution equation (2.16) on a firm footing. In Subsection 3.6, we finish the proof of Theorem 2.6.

3.1. **Stochastic differentials.** For $H \in \mathcal{S}(\mathbb{R})$, $x \in \mathbb{Z}$, $0 < \alpha < 2$, and $n \geq 1$, define scaled and unscaled operators:

$$\begin{aligned}
\Delta_{x,y}^{(n)} H &= H\left(\frac{x+y}{n^{1/\alpha}}\right) + H\left(\frac{x-y}{n^{1/\alpha}}\right) - 2H\left(\frac{x}{n^{1/\alpha}}\right), \\
\nabla_x^{(n)} H &= \frac{n^{1/\alpha}}{2}\left\{H\left(\frac{x+1}{n^{1/\alpha}}\right) - H\left(\frac{x-1}{n^{1/\alpha}}\right)\right\}, \\
d_{x,y}^{(n)} H &= H\left(\frac{x+y}{n^{1/\alpha}}\right) - H\left(\frac{x}{n^{1/\alpha}}\right) \\
\mathfrak{d}_x^{(n)} H &= n^{1/\alpha}\left\{H\left(\frac{x+1}{n^{1/\alpha}}\right) - H\left(\frac{x}{n^{1/\alpha}}\right)\right\}.
\end{aligned}$$

Define, for $\gamma, s \geq 0$,

$$\begin{aligned}
H_{\gamma,s}(\cdot) &= H\Big(\cdot - \frac{1}{n^{1/\alpha}}\Big\lfloor \frac{2\beta\tilde{g}'(\rho)sn}{n^\gamma}\Big\rfloor\Big) \\
\text{and } \widetilde{H}_{\gamma,s}(\cdot) &= H\Big(\cdot - \frac{1}{n^{1/\alpha}}\Big\{ \frac{2\beta\tilde{g}'(\rho)sn}{n^\gamma}\Big\}\Big),
\end{aligned} \tag{3.1}$$

functions seen in frames along $n^{-1/\alpha}\mathbb{Z}$ and $\mathbb{R}$ respectively which will be useful.

3.1.1. *Fields in a fixed frame.* We develop

$$\begin{aligned}
L_n \mathcal{Y}_s^{(n)}(H) &= \frac{n}{2n^{1/2\alpha}} \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} s(y) g(\eta_s^{(n)}(x)) \Delta_{x,y}^{(n)} H \\
&\quad + \frac{2n\beta}{n^{\gamma+3/2\alpha}} \sum_{x\in\mathbb{Z}} g(\eta_s^{(n)}(x)) \nabla_x^{(n)} H.
\end{aligned}$$

Then, we have

$$\mathcal{M}_t^{(n)}(H) \quad := \quad \mathcal{Y}_t^{(n)}(H) - \mathcal{Y}_0^{(n)}(H) - \int_0^t L_n \mathcal{Y}_s^{(n)}(H) ds$$

is a martingale. In these and following calculations, we note $(\mathcal{Y}_s^{(n)}(H))^k$ and later below $F(s, \eta_s^{(n)}; H, n)^k$ for $k \geq 1$, although not local, are $L^2(\nu_\rho)$ functions which can be approximated by local ones and are in the domain of L_n.

Noting

$$s(y) \;=\; \frac{1}{n^{1+1/\alpha}} s\left(\frac{y}{n^{1/\alpha}}\right),$$

we may decompose

$$\mathcal{M}_t^{(n)}(H) \;=\; \mathcal{Y}_t^{(n)}(H) - \mathcal{Y}_0^{(n)}(H) - \mathcal{I}_t^{(n)}(H) - \mathcal{B}_t^{(n)}(H) \tag{3.2}$$

where

$$\begin{aligned}
\mathcal{I}_t^{(n)}(H) &= \frac{1}{2}\int_0^t \frac{1}{n^{1/2\alpha}} \sum_{x\in\mathbb{Z}} \Big[\frac{1}{n^{1/\alpha}} \sum_{y\in\mathbb{Z}} s(y/n^{1/\alpha})\big(g(\eta_s^{(n)}(x)) - \tilde{g}(\rho)\big)\Delta_{x,y}^{(n)} H\Big] ds \\
\mathcal{B}_t^{(n)}(H) &= \frac{2n\beta}{n^{\gamma+3/2\alpha}} \int_0^t \sum_{x\in\mathbb{Z}} \big(g(\eta_s^{(n)}(x)) - \tilde{g}(\rho)\big)\nabla_x^{(n)} H ds.
\end{aligned}$$

In the last two lines, centering constants were inserted noting $\sum_x \Delta_{x,y}^{(n)} = \sum_x \nabla_x^{(n)} = 0$.

The integrand of the quadratic variation $\langle \mathcal{M}^{(n)}(H)\rangle_t$ equals

$$\begin{aligned}
&L_n\big(\mathcal{Y}_s^{(n)}(H)\big)^2 - 2\mathcal{Y}_s^{(n)}(H) L_n \mathcal{Y}_s^{(n)}(H) \\
&\quad= \frac{1}{n^{2/\alpha}} \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} s(y/n^{1/\alpha}) g(\eta_s^{(n)}(x))(d_{x,y}^{(n)} H)^2 \\
&\qquad + \frac{n\beta}{n^{\gamma+3/\alpha}} \sum_{x\in\mathbb{Z}} \big(g(\eta_s^{(n)}(x)) - g(\eta_s^{(n)}(x+1))\big)(\mathfrak{d}_x^{(n)} H)^2.
\end{aligned}$$

Then, $(\mathcal{M}_t^{(n)}(H))^2 - \langle \mathcal{M}^{(n)}(H)\rangle_t$ is a martingale with

$$\begin{aligned}
\langle \mathcal{M}^{(n)}(H)\rangle_t &= \int_0^t \frac{1}{n^{2/\alpha}} \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} s(y/n^{1/\alpha}) g(\eta_s^{(n)}(x))(d_{x,y}^{(n)} H)^2 ds \\
&\qquad + \int_0^t \frac{n\beta}{n^{\gamma+3/\alpha}} \sum_{x\in\mathbb{Z}} \big[g(\eta_s^{(n)}(x)) - g(\eta_s^{(n)}(x+1))\big](\mathfrak{d}_x^{(n)} H)^2 ds.
\end{aligned}$$

We remark, in passing, as $s(1) = c_\alpha > \beta$, the negative terms in the last line are compensated by the positive ones in the first line of the above display. Given $0 < \alpha < 2$, the $L^2(\nu_\rho)$ norm of the second term in the quadratic variation is of small order $O(n^{1-2/\alpha})$.

By stationarity of ν_ρ and the Burkholder-Davis-Gundy bound, $E[\mathcal{M}^4(t)] \le CE[\langle\mathcal{M}\rangle_t^2]$, we have

$$\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{M}_t^{(n)}(H) - \mathcal{M}_s^{(n)}(H)\big)^4\big] \;\le\; C(\beta,\alpha,\rho,g,H)|t-s|^2.$$

3.1.2. *Fields in a moving frame.* Let $F(s,\eta_s^{(n)};H,n) = \mathcal{Y}_s^{(n),\rightarrow}(H)$ and write, as before in the fixed frame,

$$\begin{aligned}
L_n F(s,\eta_s^{(n)};H,n) &= \frac{n}{2n^{1/2\alpha}} \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} s(y) g(\eta_s^{(n)}(x))\Delta_{x,y}^{(n)} \widetilde{H}_{\gamma,s} \\
&\qquad + \frac{2n\beta}{n^{\gamma+3/2\alpha}} \sum_{x\in\mathbb{Z}} g(\eta_s^{(n)}(x))\nabla_x^{(n)} \widetilde{H}_{\gamma,s}.
\end{aligned}$$

Also,

$$\frac{\partial}{\partial_s}F(s,\eta_s^{(n)};H,n) \quad = \quad \Big\{\frac{-2\beta\tilde{g}'(\rho)n}{n^\gamma}\Big\}\frac{1}{n^{3/2\alpha}}\sum_{x\in\mathbb{Z}}\nabla\widetilde{H}_{\gamma,s}\Big(\frac{x}{n^{1/\alpha}}\Big)\Big(\eta_s^{(n)}(x)-\rho\Big).$$

Then,

$$\begin{aligned}\mathcal{M}_t^{(n),\to}(H) \quad := \quad & F(t,\eta_t^{(n)};H,n)-F(0,\eta_0^{(n)};H,n)\\ & -\int_0^t \frac{\partial}{\partial_s}F(s,\eta_s^{(n)};H,n)+L_nF(s,\eta_s^{(n)};H,n)ds\end{aligned}$$

is a martingale. We write

$$\begin{aligned}&\mathcal{M}_t^{(n),\to}(H)\\ &\quad = \mathcal{Y}_t^{(n),\to}(H)-\mathcal{Y}_0^{(n),\to}(H)-\mathcal{I}_t^{(n),\to}(H)-\mathcal{B}_t^{(n),\to}(H)-\mathcal{K}_t^{(n),\to}(H)\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mathcal{I}_t^{(n),\to}(H) &= \int_0^t \frac{1}{2n^{1/2\alpha}}\sum_{x\in\mathbb{Z}}\Big[\frac{1}{n^{1/\alpha}}\sum_{y\in\mathbb{Z}}s\big(\frac{y}{n^{1/\alpha}}\big)\big(g(\eta_s^{(n)}(x))-\tilde{g}(\rho)\big)\Delta_{x,y}^{(n)}H_{\gamma,s}\Big]ds\\ \mathcal{B}_t^{(n),\to}&(H)\\ &= \frac{2n\beta}{n^{\gamma+3/2\alpha}}\int_0^t\sum_{x\in\mathbb{Z}}\big(g(\eta_s^{(n)}(x))-\tilde{g}(\rho)-\tilde{g}'(\rho)(\eta_s^{(n)}(x)-\rho)\big)\nabla_x^{(n)}H_{\gamma,s}ds\\ \mathcal{K}_t^{(n),\to}(H) &= \int_0^t \Big[\frac{1}{2n^{1/2\alpha}}\sum_{x\in\mathbb{Z}}\Big[\frac{1}{n^{1/\alpha}}\sum_{y\in\mathbb{Z}}s\big(\frac{y}{n^{1/\alpha}}\big)\kappa_{x,y}^{n,1}(H,s)\big(g(\eta_s^{(n)}(x))-\tilde{g}(\rho)\big)\Big]\\ &\quad +\frac{2n\beta}{n^{\gamma+3/2\alpha}}\sum_{x\in\mathbb{Z}}\kappa_x^{n,2}(H,s)\big(g(\eta_s^{(n)}(x))-\tilde{g}(\rho)-\tilde{g}'(\rho)(\eta_s^{(n)}(x)-\rho)\big)\Big]ds.\end{aligned}$$

Here, as $\sum_x \Delta_{x,y}^{(n)}H_{\gamma,s}=\sum_x\nabla_x^{(n)}H_{\gamma,s}=0$, centering constants were introduced in $\mathcal{I}_t^{(n),\to}$ and $\mathcal{B}_t^{(n),\to}$. By Taylor expansion,

$$\begin{aligned}\kappa_{x,y}^{n,1}(H,s) \;&=\; \Delta_{x,y}^{(n)}\big(\widetilde{H}_{\gamma,s}-H_{\gamma,s}\big)\\ &= O(n^{-1/\alpha})\cdot\Delta_{x,y}^{(n)}H'_{\gamma,s}\\ &\quad +O(n^{-2/\alpha})\cdot\Big[H_{\gamma,s}^{(4)}((x+y+z_1)/n^{1/\alpha})\\ &\qquad +H_{\gamma,s}^{(4)}((x-y+z_2)/n^{1/\alpha})+2H_{\gamma,s}^{(4)}((x+z_3)/n^{1/\alpha})\Big]\end{aligned}$$

and

$$\kappa_x^{n,2}(H,s) \quad = \quad O(n^{-1/\alpha})\cdot\Delta H_{\gamma,s}(x/n^{1/\alpha})+O(n^{-2/\alpha})\cdot H'''_{\gamma,s}((x+z_4)/n^{1/\alpha})$$

where $|z_k|\le 1$ for $1\le k\le 4$.

As in the fixed frame calculation, $(\mathcal{M}_t^{(n),\to}(H))^2-\langle\mathcal{M}^{(n),\to}(H)\rangle_t$ is a martingale with

$$\begin{aligned}\langle\mathcal{M}^{(n),\to}(H)\rangle_t \quad = \quad & \int_0^t \frac{1}{n^{2/\alpha}}\sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}}s\big(\frac{y}{n^{1/\alpha}}\big)g(\eta_s^{(n)}(x))(d_{x,y}^{(n)}\widetilde{H}_{\gamma,s})^2ds\\ & +\int_0^t \frac{n\beta}{n^{\gamma+3/\alpha}}\sum_{x\in\mathbb{Z}}\big[g(\eta_s^{(n)}(x))-g(\eta_s^{(n)}(x+1))\big](\mathfrak{d}_x^{(n)}\widetilde{H}_{\gamma,s})^2ds.\end{aligned}$$

Also, we have the bound, as in the fixed frame,

$$\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{M}_t^{(n),\rightarrow}(H) - \mathcal{M}_s^{(n),\rightarrow}(H)\big)^4\big] \ \le\ C(\beta,\alpha,\rho,g,H)|t-s|^2. \tag{3.4}$$

3.2. **Boltzmann-Gibbs principle.** We will need to approximate terms in the stochastic differential of $\mathcal{Y}_t^{(n),\rightarrow}$ in order to close and recover limiting equations. The main tool for this approximation is the 'Boltzmann-Gibbs principle'. Define, for $\eta\in\Omega$ and $\ell\ge 1$, that

$$\eta^{(\ell)}(x) \ :=\ \frac{1}{2\ell+1}\sum_{y\in\Lambda_\ell}\eta(x+y).$$

For $h:\mathbb{Z}\to\mathbb{R}$, $K>0$ and $c\in\mathbb{R}$, define $\bar h_{c,s}:[0,K]\times\mathbb{Z}\to\mathbb{R}$ by $\bar h_{c,s}(x) = \bar h(x-\lfloor cs\rfloor)$. Also, for $\ell\ge 1$ and $0<\alpha<2$, denote

$$w_\alpha(\ell) \ :=\ \begin{cases} \ell^{\alpha-1} & \text{for } 1<\alpha<2\\ \log(\ell)+1 & \text{for } \alpha=1\\ 1 & \text{for } 0<\alpha<1.\end{cases} \tag{3.5}$$

Theorem 3.1. *Suppose $0<\alpha<2$. Let f be a local $L^5(\nu_\rho)$ function supported on sites Λ_{ℓ_0} such that $\tilde f(\rho)=\tilde f'(\rho)=0$. There exists a constant $C=C(\rho,\alpha,\ell_0)$ such that, for $K>0$, $\ell\ge\ell_0$ and $h\in\ell^1(\mathbb{Z})\cap\ell^2(\mathbb{Z})$,*

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}\Big(&f(\tau_x\eta_s^{(n)})\\ &-\frac{\tilde f''(\rho)}{2}\Big\{\Big(\big(\eta_s^{(n)}\big)^{(\ell)}(x)-\rho\Big)^2-\frac{\sigma^2(\rho)}{2\ell+1}\Big\}\Big)\bar h_{c,s}(x)ds\Big)^2\Big]\\ \le\ C\|f\|^2_{L^5(\nu_\rho)}\Big(\frac{Kw_\alpha(\ell)}{n^{1-1/\alpha}}&\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}h^2(x)\Big)+\frac{K^2n^{2/\alpha}}{\ell^3}\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}|h(x)|\Big)^2\Big).\end{aligned}$$

On the other hand, when only $\tilde f(\rho)=0$ is known,

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}&\Big(f(\tau_x\eta_s^{(n)})-\tilde f'(\rho)\Big\{\big(\eta_s^{(n)}\big)^{(\ell)}(x)-\rho\Big\}\bar h_{c,s}(x)ds\Big)^2\Big]\\ \le\ C\|f\|^2_{L^5(\nu_\rho)}&\Big(\frac{K\ell^\alpha}{n^{1-1/\alpha}}\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}h^2(x)\Big)+\frac{K^2n^{2/\alpha}}{\ell^2}\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}|h(x)|\Big)^2\Big).\end{aligned}$$

We remark Theorem 3.1 is a long-range version of Theorem 3.2 in [25]. A proof of the theorem is given in Section 4.

3.3. **Tightness.** We now prove tightness of the fluctuation fields in Theorem 2.6, using this Boltzmann-Gibbs principle.

Proposition 3.2. *Starting from ν_ρ, with respect to the range of parameters in Theorem 2.6, the sequences $\{\mathcal{Y}_\cdot^{(n),\rightarrow}:n\ge1\}$, $\{\mathcal{M}_\cdot^{(n),\rightarrow}:n\ge1\}$, $\{\mathcal{I}_\cdot^{(n),\rightarrow}:n\ge1\}$, $\{\mathcal{B}_\cdot^{(n),\rightarrow}:n\ge1\}$, $\{\mathcal{K}_\cdot^{(n),\rightarrow}:n\ge1\}$, and $\{\langle\mathcal{M}^{(n),\rightarrow}\rangle_\cdot:n\ge1\}$ are tight w.r.t. the uniform topology on $D([0,T],\mathcal{S}'(\mathbb{R}))$.*

Proof. By Mitoma's criterion [39], for each $H\in\mathcal{S}(\mathbb{R})$, it is enough to show tightness w.r.t. the uniform topology of $\{\mathcal{Y}_\cdot^{(n),\rightarrow}(H):n\ge1\}$, $\{\mathcal{M}_\cdot^{(n),\rightarrow}(H):n\ge1\}$, $\{\mathcal{I}_\cdot^{(n),\rightarrow}(H):n\ge1\}$, $\{\mathcal{B}_\cdot^{(n),\rightarrow}(H):n\ge1\}$, $\{\mathcal{K}_\cdot^{(n),\rightarrow}(H):n\ge1\}$ and $\{\langle\mathcal{M}^{(n),\rightarrow}(H)\rangle_\cdot:n\ge1\}$. Note that all initial values vanish, except $\mathcal{Y}_0^{(n),\rightarrow}(H)$.

Tightness of

$$\mathcal{Y}_t^{(n),\rightarrow}(H) \;=\; \mathcal{Y}_0^{(n),\rightarrow}(H)+\mathcal{I}_t^{(n),\rightarrow}(H)+\mathcal{B}_t^{(n),\rightarrow}(H)+\mathcal{K}_t^{(n),\rightarrow}(H)+\mathcal{M}_t^{(n),\rightarrow}(H),$$

is accomplished by showing each term is tight. All initial values of the constituents vanish except $\mathcal{Y}_0^{(n),\rightarrow}(H)$, which is tight as it converges weakly to a Gaussian random variable given that we start from ν_ρ.

Tightness of the martingale term follows from the standard technique of dividing into subintervals of size δ^{-1}, stationarity of ν_ρ, $\mathcal{M}_0^{(n),\rightarrow}(H)=0$, Doob's inequality, and (3.4):

$$\begin{aligned}
&\mathbb{P}_{\nu_\rho}\Big(\sup_{\substack{|t-s|\le\delta\\ 0\le s,t\le T}}|\mathcal{M}_t^{(n),\rightarrow}(H)-\mathcal{M}_s^{(n),\rightarrow}(H)|>\varepsilon\Big)\\
&\qquad\qquad\le\;\frac{3T}{\delta}\mathbb{P}_{\nu_\rho}\Big(\sup_{0\le t\le T}|\mathcal{M}_t^{(n),\rightarrow}(H)|>\varepsilon\Big)\\
&\qquad\qquad\le\;\frac{3TC}{\varepsilon^4\delta}\mathbb{E}_{\nu_\rho}\big[\big(\mathcal{M}_\delta^{(n),\rightarrow}(H)\big)^4\big]\;\le\;C(\beta,\alpha,\rho,\varepsilon,g,H)\delta.
\end{aligned}$$

The proof of tightness for $\mathcal{B}_t^{(n),\rightarrow}(H)$ makes use of Theorem 3.1. One obtains for $\ell\ge 1$ and $0\le t\le T$ that

$$\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le t}(\mathcal{B}_s^{(n),\rightarrow}(H))^2\Big]\;\le\;\frac{C(\rho,\alpha,\beta,g,H)}{n^{2\gamma+3/\alpha-2}}\Big\{\frac{tw_\alpha(\ell)}{n^{1-1/\alpha}}+\frac{t^2n^{2/\alpha}}{\ell^3}+\frac{t^2n^{2/\alpha}}{\ell^2}\Big\}. \tag{3.6}$$

Indeed, since $V(\eta(0))=g(\eta(0))-\tilde g(\rho)-\tilde g'(\rho)(\eta(0)-\rho)$ is a function of single site, and $\tilde V(\rho)=\tilde V'(\rho)=0$ and $\tilde V''(\rho)=\tilde g''(\rho)$, we apply Theorem 3.1, with respect to $\mathcal{B}_t^{(n),\rightarrow}(H)=2\beta n^{1-\gamma-3/2\alpha}\int_0^t\sum_x\big(\nabla_x^{(n)}H_{\gamma,s}\big)V(\tau_x\eta_s)ds$ and $\ell_0=1$, to estimate

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Big[\Big(\mathcal{B}_t^{(n),\rightarrow}(H)\\
&\quad-\frac{\beta\tilde g''(\rho)n}{n^{\gamma+3/2\alpha}}\int_0^t\sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H_{\gamma,s})\Big\{\Big(\big(\eta_s^{(n)}\big)^{(\ell)}(x)-\rho\Big)^2-\frac{\sigma^2(\rho)}{2\ell+1}\Big\}ds\Big)^2\Big]\\
&\le\;\frac{C(\alpha,\rho,\beta,g)}{n^{2\gamma+3/\alpha-2}}\Big\{\frac{tw_\alpha(\ell)}{n^{1-1/\alpha}}+\frac{t^2n^{2/\alpha}}{\ell^3}\Big\}\\
&\quad\times\Big[\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H)^2\Big)+\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}|\nabla_x^{(n)}H|\Big)^2\Big].
\end{aligned}$$

However, squaring the integral, noting the fourth moment, $E_{\nu_\rho}[(\eta^{(\ell)}(0)-\rho)^4]\le C\ell^{-2}$, gives the third term on the right-side of (3.6).

For the sequence in Theorem 2.6, when $3/2\le\alpha<2$ and $\gamma=1-3/2\alpha$, or $1<\alpha<3/2$, choose $\ell=\lfloor t^{1/(\alpha+1)}n^{1/\alpha}\rfloor\ge 1$. Since $w_\alpha(\ell)=\ell^{\alpha-1}$ in these cases, we have $\mathbb{E}_{\nu_\rho}\big[(\mathcal{B}_t^{(n),\rightarrow}(H))^2\big]\le Ct^{2\alpha/(\alpha+1)}$ where the exponent $2\alpha/(\alpha+1)>1$. However, when $\ell=\lfloor t^{1/(\alpha+1)}n^{1/\alpha}\rfloor\le 1$, by squaring, taking expectation and using independence under ν_ρ, one gets a similar bound:

$$\mathbb{E}_{\nu_\rho}\big[(\mathcal{B}_t^{(n),\rightarrow}(H))^2\big]\le C(\rho,\beta,g)t^2n^{1/\alpha}\big[\frac{1}{n^{1/\alpha}}\sum_x|\nabla_x^{(n)}H|^2\big]\le C(\rho,\alpha,\beta,g,H)t^{\frac{2\alpha+1}{\alpha+1}}.$$

One may now apply the Kolmogorov-Centsov criterion and stationarity of ν_ρ to obtain tightness of $\mathcal{B}_t^{(n),\rightarrow}$ in these cases.

However, when $0 < \alpha \leq 1$, we use the following argument. By stationarity of ν_ρ, the standard technique of dividing into subintervals of size δ^{-1}, $\mathcal{B}_0^{(n),\rightarrow}(H) = 0$, and (3.6), for $\ell > 1$, we obtain

$$\mathbb{P}_{\nu_\rho}\Big(\sup_{\substack{|s-t|\leq\delta\\ 0\leq s,t\leq T}} \big|\mathcal{B}_t^{(n),\rightarrow}(H) - \mathcal{B}_s^{(n),\rightarrow}(H)\big| > \varepsilon\Big) \leq \frac{3T}{\delta}\mathbb{P}_{\nu_\rho}\big(\sup_{0\leq t\leq\delta}\big|\mathcal{B}_t^{(n),\rightarrow}(H)\big| > \varepsilon\big)$$

$$\leq \frac{3T}{\delta\varepsilon^2}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\leq t\leq\delta}\big|\mathcal{B}_t^{(n),\rightarrow}(H)\big|^2\Big] \leq \frac{C}{n^{2\gamma+\frac{3}{\alpha}-2}}\Big\{\frac{w_\alpha(\ell)}{n^{1-\frac{1}{\alpha}}} + \frac{\delta n^{\frac{2}{\alpha}}}{\ell^2}\Big\}. \tag{3.7}$$

Choose now $\ell = \lfloor n^{1/\alpha}\rfloor$. From the definition of $w_\alpha(\ell)$ in (3.5), we see (3.7) is variously of order $O(n^{-2\gamma-1}\log(n))$ when $\alpha = 1$, and of order $O(n^{-2\gamma-2\alpha^{-1}+1})$ when $0 < \alpha < 1$. Hence, for $0 < \alpha \leq 1$, the bound (3.7) vanishes as $n \uparrow \infty$.

The tightness arguments for $\mathcal{I}_t^{(n),\rightarrow}(H)$, $\langle\mathcal{M}_t^{(n),\rightarrow}(H)\rangle$ and $\mathcal{K}_t^{(n),\rightarrow}(H)$ are simpler and follow by squaring all terms, using independence and stationarity under ν_ρ, and the Kolmogorov-Centsov criterion. For instance,

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\big[(\mathcal{I}_t^{(n),\rightarrow}(H))^2\big] \quad &\leq \quad C(\rho,g)t^2\Big\{\frac{1}{n^{1/\alpha}}\sum_x\Big[\frac{1}{n^{1/\alpha}}\sum_y s(y/n^{1/\alpha})\Delta_{x,y}^{(n)}H\Big]^2\Big\}\\ &\leq \quad C(\rho,\alpha,g,H)t^2. \qquad \square\end{aligned}$$

3.4. **Identification.** We now identify some features of the structure of the limit points with respect to Theorem 2.6. Let $\mu^{(n)}$ be the distribution of

$$\Big(\mathcal{Y}_t^{(n),\rightarrow}, \mathcal{M}_t^{(n),\rightarrow}, \mathcal{I}_t^{(n),\rightarrow}, \mathcal{B}_t^{(n),\rightarrow}, \mathcal{K}_t^{(n),\rightarrow}, \langle\mathcal{M}^{(n),\rightarrow}\rangle_t : t\in[0,T]\Big).$$

By Proposition 3.2, we have that $\{\mu^{(n)}\}$ is tight w.r.t. the uniform topology on $D([0,T],(\mathcal{S}'(\mathbb{R}))^6)$. Suppose n' is a subsequence where $\mu^{n'}$ converges weakly to a limit point μ. Let also $\mathcal{Y}_\cdot$, $\mathcal{M}_\cdot$, $\mathcal{I}_\cdot$, $\mathcal{B}_\cdot$, $\mathcal{K}_\cdot$ and $\mathcal{D}_\cdot$ be the respective limits in distribution of the components, all realized on a probability space, which we will call the 'underlying common probability space' forthwith. Since tightness of the components in Proposition 3.2 is shown w.r.t. the uniform topology on $D([0,T],\mathcal{S}'(\mathbb{R}))$, we have that $\mathcal{Y}_\cdot$, $\mathcal{M}_\cdot$, $\mathcal{I}_\cdot$, $\mathcal{B}_\cdot$, $\mathcal{K}_\cdot$ and $\mathcal{D}_\cdot$ have a.s. continuous paths, and therefore belong to $C([0,T],\mathcal{S}'(\mathbb{R}))$.

For $\varepsilon > 0$, let $G_\varepsilon : \mathbb{R}\to[0,\infty)$ be a smooth compactly supported function which approximates $\iota_\varepsilon(z) = \varepsilon^{-1}1_{[-1,1]}(z\varepsilon^{-1})$ in the sense (2.13). For $H\in\mathcal{S}(\mathbb{R})$ and $0 < \alpha < 2$, define the process $\mathcal{A}_\cdot^{(n),\varepsilon} \in C([0,T],\mathcal{S}'(\mathbb{R}))$ by its action on $H\in\mathcal{S}(\mathbb{R})$:

$$\mathcal{A}_t^{(n),\varepsilon}(H) := \int_0^t \frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H)\big[\tau_x\mathcal{Y}_u^{(n),\rightarrow}(G_\varepsilon)\big]^2du.$$

Lemma 3.3. *For fixed $0 < \varepsilon \leq 1$ and $H\in\mathcal{S}(\mathbb{R})$, the transformation $F_\cdot$: $D([0,T];\mathcal{S}'(\mathbb{R}))\to C([0,T])$, given by*

$$F_t(\pi_\cdot) = \int_0^t du\int_{\mathbb{R}}dx\big(\nabla H(x)\big)\big\{\pi_u(\tau_{-x}G_\varepsilon)\big\}^2,$$

is continuous w.r.t. the uniform topology.

We prove this lemma at the end of this subsection.

Lemma 3.4. *The sequence $\mathcal{A}_\cdot^{(n'),\varepsilon}$, as $n' \uparrow \infty$ converges weakly w.r.t. the uniform topology on $C([0,T], \mathcal{S}'(\mathbb{R}))$ to $\mathcal{A}_\cdot^\varepsilon$, defined in* (2.14)*, with respect to the limit point $\mathcal{Y}_\cdot$. Moreover, for $H \in \mathcal{S}(\mathbb{R})$, we have*

$$\sup_{t\in[0,T]} \mathrm{Cov}\big(\mathcal{A}_t^\varepsilon(H), \mathcal{A}_t^\varepsilon(H)\big) \ \le\ C(\rho)\varepsilon^{-2}T^2\|\nabla H\|_{L^1(\mathbb{R})}^2. \tag{3.8}$$

Proof. By Lemma 3.3, and the weak convergence $\mathcal{Y}_\cdot^{(n'),\rightarrow} \Rightarrow \mathcal{Y}_\cdot$, for $H \in \mathcal{S}(\mathbb{R})$, we have in distribution w.r.t. the uniform topology that

$$\lim_{n'\uparrow\infty} \mathcal{A}_\cdot^{(n'),\varepsilon}(H) \quad = \quad \int_0^\cdot du \int_{\mathbb{R}} dx\big(\nabla H(x)\big)\big\{\mathcal{Y}_u(\tau_{-x}G_\varepsilon)\big\}^2 \ = \ \mathcal{A}_\cdot^\varepsilon(H).$$

Also, by the weak convergence w.r.t. the uniform topology, $\mathcal{Y}_\cdot^{(n'),\rightarrow} \Rightarrow \mathcal{Y}_\cdot$, we have for times $t_1, \ldots, t_k$ and $H_1, \ldots, H_k \in \mathcal{S}(\mathbb{R})$ the weak convergence,

$$\big(\mathcal{A}_{t_t}^{(n'),\varepsilon}(H_1), \ldots, \mathcal{A}_{t_k}^{(n'),\varepsilon}(H_k)\big) \ \Rightarrow\ \big(\mathcal{A}_{t_1}^\varepsilon(H_1), \ldots, \mathcal{A}_{t_k}^\varepsilon(H_k)\big).$$

Then, invoking the proof of Theorem 5.3 in [39] (or Theorem 6.15 in [51]), we conclude $\mathcal{A}_\cdot^{(n'),\varepsilon}$ converges weakly w.r.t. the uniform topology to $\mathcal{A}_\cdot^\varepsilon$.

We now show (3.8). Note that the fourth moment

$$\mathbb{E}\big(\mathcal{Y}_t(G_\varepsilon)\big)^4 \ \le\ \liminf_{n'\uparrow\infty} \mathbb{E}_{\nu_\rho}\big(\mathcal{Y}_t^{(n'),\rightarrow}(G_\varepsilon)\big)^4 \ \le\ C(\rho)\varepsilon^{-2}$$

holds by Fatou's lemma for weakly converging variables, using Skorohod's theorem. Then, the covariance of $\mathcal{A}_t^\varepsilon(H)$, noting $\mathbb{E}\big(\mathcal{Y}_t(\tau_{-x}G_\varepsilon)\big)^4 = \mathbb{E}\big(\mathcal{Y}_t(G_\varepsilon)\big)^4$, is bounded

$$\begin{aligned}\sup_{t\in[0,T]} \mathrm{Cov}\big(\mathcal{A}_t^\varepsilon(H), \mathcal{A}_t^\varepsilon(H)\big) \quad &\le \quad T^2\|\nabla H\|_{L^1(\mathbb{R})}^2\mathbb{E}\big(\mathcal{Y}_t(G_\varepsilon)\big)^4 \\ &\le \quad C(\rho)\varepsilon^{-2}T^2\|\nabla H\|_{L^1(\mathbb{R})}^2. \end{aligned}$$

□

Proposition 3.5. *Consider the systems in Theorem 2.6. Let $H \in \mathcal{S}(\mathbb{R})$.*

(1) When $3/2 \le \alpha < 2$, $\gamma = 1 - 3/2\alpha$, and $\beta\tilde{g}''(\rho) \neq 0$, there is a constant $C = C(\beta, \alpha, \rho, g, T)$ such that

$$\begin{aligned}&\lim_{n\uparrow\infty} \mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le T}\Big|\mathcal{B}_s^{(n),\rightarrow}(H) - \beta\tilde{g}''(\rho)\mathcal{A}_s^{(n),\varepsilon}(H)\Big|^2\Big] \\ &\qquad \le\ C\Big(\varepsilon^{\alpha-1} + \varepsilon^{-1}\|G_\varepsilon - \iota_\varepsilon\|_{L^2(\mathbb{R})}^2\Big)\Big[\|\nabla H\|_{L^2(\mathbb{R})}^2 + \|\nabla H\|_{L^1(\mathbb{R})}^2\Big].\end{aligned}$$

Consequently, on the underlying common probability space, $\{A_\cdot^\varepsilon(H)\}$ is a uniformly L^2 Cauchy sequence, as $\varepsilon \downarrow 0$, as specified in (2.15)*. Therefore, the limit $\lim_{\varepsilon\downarrow 0}\mathcal{A}_\cdot^\varepsilon(H) =: \mathcal{A}_\cdot(H) \in C([0,T])$, does not depend on the specific family $\{G_\varepsilon\}$, and is stationary, $\mathcal{A}_t(H) - \mathcal{A}_s(H) \stackrel{d}{=} \mathcal{A}_{t-s}(H)$ for $0 \le s \le t \le T$. Also, a.s. on the underlying common probability space, for $0 \le t \le T$, we have $\beta\tilde{g}''(\rho)\mathcal{A}_t(H) = \mathcal{B}_t(H)$.*

(2) When $0 < \alpha < 3/2$ or $\beta\tilde{g}''(\rho) = 0$, we have $\lim_{n\uparrow\infty}\mathcal{B}_t^{(n),\rightarrow}(H) = \mathcal{B}_t(H) = 0$, uniformly over $0 \le t \le T$, in L^2 with respect to the underlying common probability space.

(3) When $0 < \alpha < 2$,

$$\lim_{n\uparrow\infty} \sup_{t\in[0,T]} \mathbb{E}_{\nu_\rho}\Big[\Big|\mathcal{I}_t^{(n),\rightarrow}(H) - \tilde{g}'(\rho)\int_0^t \mathcal{Y}_s^{(n),\rightarrow}(\Delta^{\alpha/2}H)ds\Big|^2\Big] \;=\; 0$$

$$\lim_{n\uparrow\infty} \sup_{t\in[0,T]} \mathbb{E}_{\nu_\rho}\Big[\Big|\langle\mathcal{M}^{(n),\rightarrow}(H)\rangle_t - \tilde{g}(\rho)t\|\nabla^{\alpha/2}H\|^2_{L^2(\mathbb{R}\times\mathbb{R})}\Big|^2\Big] \;=\; 0$$

$$\lim_{n\uparrow\infty} \sup_{t\in[0,T]} \mathbb{E}_{\nu_\rho}\Big[\Big|\mathcal{K}_t^{(n),\rightarrow}(H)\Big|^2\Big] \;=\; 0.$$

Then, in L^2***, with respect to the underlying common probability space, we have*** $\mathcal{K}_t(H) \equiv 0$ ***and*** $D_t(H) \equiv \tilde{g}(\rho)t\|\nabla^{\alpha/2}H\|^2_{L^2(\mathbb{R}\times\mathbb{R})}$***. Moreover,*** $\mathcal{M}_\cdot(H)$ ***is a continuous martingale with quadratic variation*** $D_\cdot(H)$***, and hence, by Levy's theorem,*** $\mathcal{M}_\cdot$ ***is a version of the noise in*** (2.3).

Proof. We first verify (1) and (3). Suppose the limit display for $\mathcal{B}_\cdot^{(n),\rightarrow}(H)$ holds. Recall Lemma 3.3 and the definition of F_t. Since $\mu^{n'}$ converges weakly w.r.t. the uniform topology, we have, by the continuous mapping theorem, with respect to continuous function $(b_\cdot, y_\cdot) \in C([0,T], (\mathcal{S}'(\mathbb{R}))^2) \mapsto \sup_{0\leq s\leq T}|b_s(H) - \beta\tilde{g}''(\rho)F_t(y_\cdot)|$, that $\sup_{0\leq s\leq T}\big|\mathcal{B}_s^{(n'),\rightarrow}(H) - \beta\tilde{g}''(\rho)\mathcal{A}_s^{(n'),\varepsilon}(H)\big|$ converges weakly to $\sup_{0\leq s\leq T}\big|\mathcal{B}_s(H) - \beta\tilde{g}''(\rho)\mathcal{A}_s^{\varepsilon}(H)\big|$, as $n' \uparrow \infty$.

Hence, by Fatou's lemma, valid with respect to weakly converging random variables using Skorohod's theorem, we conclude

$$\begin{aligned}&\mathbb{E}\big[\sup_{0\leq s\leq T}\big|\mathcal{B}_s(H) - \beta\tilde{g}''(\rho))\mathcal{A}_s^{\varepsilon}(H)\big|^2\big] \\ &\quad\leq\; C\big[\varepsilon^{\alpha-1} + \varepsilon^{-1}\|G_\varepsilon - \iota_\varepsilon\|^2_{L^2(\mathbb{R})}\big]\big\{\|\nabla H\|^2_{L^2(\mathbb{R})} + \|\nabla H\|^2_{L^1(\mathbb{R})}\big\}. \qquad (3.9)\end{aligned}$$

Therefore, by the triangle inequality, $\mathcal{A}_\cdot^{\varepsilon}(H)$, as $\varepsilon \downarrow 0$, is a uniform L^2 Cauchy sequence satisfying (2.15), and the following limit holds: $\beta\tilde{g}''(\rho)\mathcal{A}_t(H) = \mathcal{B}_t(H)$ for all $0 \leq t \leq T$ a.s. with respect to the underlying common probability space.

We now argue the limit for $\mathcal{B}_\cdot^{(n),\rightarrow}$ assumed before. Let $\ell = \lfloor\varepsilon n^{1/\alpha}\rfloor$. To move the shift by $n^{-1/\alpha}\lfloor a\tilde{g}'(\rho)sn/n^\gamma\rfloor$ in $\tau_x\mathcal{Y}_s^{(n),\rightarrow}(\iota_\varepsilon)$ to $\nabla_x^{(n)}H_{\gamma,s}$ (cf. (3.1)), write

$$\begin{aligned}&\sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H_{\gamma,s})\Big(\big(\eta_s^{(n)}\big)^{(\ell)}(x) - \rho\Big)^2 \\ &\quad=\; \sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H_{\gamma,s})\Big(\frac{1}{2n^{1/\alpha}\varepsilon+1}\sum_{|z|\leq\varepsilon n^{1/\alpha}}(\eta_s^{(n)}(z+x) - \rho)\Big)^2 \\ &\quad=\; \frac{1+O(n^{-1/\alpha})}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H)\big[\tau_x\mathcal{Y}_s^{(n),\rightarrow}(\iota_\varepsilon)\big]^2.\end{aligned}$$

Since $\gamma = 1 - 3/2\alpha$ and $w_\alpha(\ell) = \ell^{\alpha-1}$ for $3/2 \leq \alpha < 2$, by Theorem 3.1, as in the proof of estimate (3.6), we have

$$\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\leq s\leq T}\Big(\mathcal{B}_s^{(n),\rightarrow}(H) - \beta\tilde{g}''(\rho)\int_0^s\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}(\nabla_x^{(n)}H)\tau_x\mathcal{Y}_u^{(n),\rightarrow}(\iota_\varepsilon)^2du\Big)^2\Big]$$

equals

$$\lim_{n\uparrow\infty} \mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le T}\Big(\mathcal{B}^{(n),\to}_s(H)$$
$$-\beta\tilde g''(\rho)\int_0^s \sum_{x\in\mathbb{Z}} (\nabla^{(n)}_x H_{\gamma,s})\Big\{\big((\eta^{(n)}_s)^{(\ell)}(x)-\rho\big)^2 - \frac{\sigma^2(\rho)}{2\ell+1}\Big\}du\Big)^2\Big]$$
$$\le \lim_{n\uparrow\infty} C(\beta,\rho,g,T)\Big(\varepsilon^{\alpha-1}+\frac{1}{\varepsilon^3 n^{1/\alpha}}\Big)$$
$$\times\Big[\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}\big(\nabla^{(n)}_x H\big)^2\Big)+\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}\big|\nabla^{(n)}_x H\big|\Big)^2\Big].$$

Here, as the sum of $\nabla^{(n)}_x H_{\gamma,s}$ on x vanishes, the centering constant $(2\ell+1)^{-1}\sigma^2(\rho)$ was inserted above.

Now,

$$\mathcal{Y}^{(n),\to}_u(\iota_\varepsilon)^2 - \mathcal{Y}^{(n),\to}_u(G_\varepsilon)^2 = \big[\mathcal{Y}^{(n),\to}_u(\iota_\varepsilon)-\mathcal{Y}^{(n),\to}_u(G_\varepsilon)\big]\cdot\big[\mathcal{Y}^{(n),\to}_u(\iota_\varepsilon)+\mathcal{Y}^{(n),\to}_u(G_\varepsilon)\big].$$

By Schwarz inequality and stationarity of ν_ρ,

$$\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le T}\Big(\int_0^s \frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}(\nabla^{(n)}_x H)\tau_x\mathcal{Y}^{(n),\to}_u(\iota_\varepsilon)^2 du - \mathcal{A}^{(n),\varepsilon}_s(H)\Big)^2\Big]$$

is less than

$$\lim_{n\uparrow\infty}\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^T \frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}|\nabla^{(n)}_x H|\big|\tau_x\mathcal{Y}^{(n),\to}_u(\iota_\varepsilon)^2-\tau_x\mathcal{Y}^{(n),\to}_u(G_\varepsilon)^2\big|du\Big)^2\Big]$$
$$\le C(\rho)\varepsilon^{-1}\|G_\varepsilon-\iota_\varepsilon\|^2_{L^2(\mathbb{R})}T^2\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}\big|\nabla^{(n)}_x H\big|\Big)^2.$$

These estimates with the inequality $(a+b)^2\le 2a^2+2b^2$ finish the proof of the $\mathcal{B}^{(n),\to}_t$ limit.

The proof for the limit of $\mathcal{I}^{(n),\to}_\cdot$ is analogous, since $\Delta^{\alpha/2}G\in C_{p,0}(\mathbb{R})$ is uniformly continuous, and is omitted.

The arguments for $\mathcal{K}^{(n),\to}_\cdot(H)$ and $\langle\mathcal{M}^{(n),\to}\rangle_\cdot$ and identification of limits $\mathcal{K}_\cdot$ and $\mathcal{D}_\cdot(H)$, noting their forms, and that the process starts from invariant product measure ν_ρ, follow by straightforward L^2 calculations.

We now address the martingale convergence. By the identification given before, any limit point of the quadratic variation sequence equals $\mathcal{D}_\cdot(H)$. Also, the limit of martingale sequence, w.r.t. to the uniform topology, $\mathcal{M}_\cdot(H)$, is a continuous martingale. Now, by the triangle inequality, Doob's inequality and the quadratic variation bound (3.4),

$$\sup_n \mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le t}|\mathcal{M}^{(n),\to}_s(H)-\mathcal{M}^{(n),\to}_{s^-}(H)|\Big]$$
$$\le 2\sup_n\mathbb{E}_{\nu_\rho}\Big[\sup_{u\in[0,t]}|\mathcal{M}^{(n),\to}_u(H)|^2\Big]^{1/2}$$
$$\le 2\sup_n\mathbb{E}_{\nu_\rho}\Big[\langle M^{(n),\to}(H)\rangle_t\Big]^{1/2} \le C(\rho,\alpha,\beta,g,H,T).$$

Then, by Corollary VI.6.30 of [31], $(\mathcal{M}^{(n'),\rightarrow}_\cdot(H), \langle \mathcal{M}^{(n'),\rightarrow}(H)\rangle_\cdot)$ converges in distribution to $(\mathcal{M}_\cdot(H), \langle \mathcal{M}(H)\rangle_\cdot)$. Since $\langle \mathcal{M}^{(n'),\rightarrow}(H)\rangle_\cdot$ converges in distribution to $\mathcal{D}_\cdot(H)$, we have $\langle \mathcal{M}(H)\rangle_t = \tilde{g}(\rho)t\|\nabla^{\alpha/2}H\|^2_{L^2(\mathbb{R}\times\mathbb{R})}$ for $0 \le t \le T$. Hence, by Levy's theorem, $\mathcal{M}_\cdot$ is a version of the noise $\sqrt{\tilde{g}(\rho)}N_\cdot$ desired. This finishes the proof of (1) and (3).

We now consider part (2). When $\beta\tilde{g}''(\rho) = 0$, trivially $\mathcal{B}^{(n),\rightarrow}_\cdot(H) \equiv 0$ and the statement follows. When $0 < \alpha < 3/2$, by the use of Fatou's lemma, as above in the argument for (1), we need only show $\lim_{n\uparrow\infty}\sup_{t\in[0,T]}|\mathcal{B}^{(n),\rightarrow}_t(H)| = 0$ in $L^2(\mathbb{P}_{\nu_\rho})$ to finish. To this end, we invoke the estimate (3.6) with $\ell = \lfloor n^{1/\alpha}\rfloor$. Noting (3.5), since $w_\alpha(\ell) = \ell^{\alpha-1}$ for $1 < \alpha < 3/2$, and $w_\alpha(\ell) \le \log(\ell) + 1$ for $0 < \alpha \le 1$ and $\ell \ge 1$, we have, uniformly over $0 \le t \le T$, that

$$\mathcal{B}^{(n),\rightarrow}_t(H) \ = \ \begin{cases} O(n^{-\gamma+1-3/2\alpha}) & \text{for } 1 < \alpha < 3/2 \\ O(n^{-\gamma+1/2-1/\alpha}\log(n)) & \text{for } 0 < \alpha \le 1. \end{cases}$$

Hence, for $0 < \alpha < 3/2$, the desired limit of $\mathcal{B}^{(n),\rightarrow}_t(H)$ holds as $n \uparrow \infty$. □

We now identify a version of the limit of $\mathcal{A}^\varepsilon_\cdot$ as $\varepsilon \downarrow 0$ in terms of $\mathcal{B}_t$, which will be used later in the proof of Theorem 2.6 to verify Definition 2.5.

Proposition 3.6. *When $3/2 \le \alpha < 2$, $\gamma = 1 - 3/2\alpha$, and $\beta\tilde{g}''(\rho) \neq 0$, there is a unique process $\mathcal{A}_\cdot \in C([0,T], \mathcal{S}'(\mathbb{R}))$ such that $\mathcal{A}^\varepsilon_\cdot$ converges weakly to $\mathcal{A}_\cdot$ in $C([0,T], \mathcal{S}'(\mathbb{R}))$, and whose action on $H \in \mathcal{S}(\mathbb{R})$ is the uniform L^2 Cauchy limit $\mathcal{A}_\cdot(H)$ given in Proposition 3.5. Also, $(\beta\tilde{g}''(\rho))^{-1}\mathcal{B}_\cdot$ is a version of $\mathcal{A}_\cdot$ on the underlying common probability space. Moreover, the covariance of $\mathcal{A}_t(H)$ is bounded*

$$\sup_{t\in[0,T]} \mathrm{Cov}\big(\mathcal{A}_t(H), \mathcal{A}_t(H)\big) \ \le \ C(T,\beta,\alpha,\rho,g)\big\{\|\nabla H\|^2_{L^2(\mathbb{R})} + \|\nabla H\|^2_{L^1(\mathbb{R})}\big\}. \quad (3.10)$$

Proof. By part (1) of Proposition 3.5, for each $H \in \mathcal{S}(\mathbb{R})$, we have that $\mathcal{A}^\varepsilon_s(H)$ converges to $\mathcal{A}_s(H)$ uniformly over $s \in [0,T]$, as $\varepsilon \downarrow 0$, in L^2, and also that $\mathcal{A}_s(H) = (\beta\tilde{g}''(\rho))^{-1}\mathcal{B}_s(H)$ for $0 \le s \le T$ a.s. with respect to the underlying probability space. Therefore, trivially, $\{\mathcal{A}^\varepsilon_\cdot(H) : 0 < \varepsilon \le 1\}$ is tight in $C([0,T])$. Also, we may also conclude that the vector $(\mathcal{A}^\varepsilon_{t_i}(H_i) : 1 \le i \le k)$ converges weakly to $(\mathcal{A}_{t_i}(H_i) : 1 \le i \le k)$, for $H_1, \ldots, H_k \in \mathcal{S}(\mathbb{R})$, $t_1, \ldots, t_k \ge 0$, which has the same distribution as $((\beta\tilde{g}''(\rho))^{-1}\mathcal{B}_{t_i}(H_i) : 1 \le i \le k)$.

Then, the existence of $\mathcal{A}_\cdot \in C([0,T], \mathcal{S}'(\mathbb{R}))$, and convergence of $\mathcal{A}^\varepsilon_\cdot$ to it w.r.t. the uniform topology, now follows by Theorem 5.3 in [39] (or Theorem 6.15 in [51]). Also, since $\mathcal{A}_\cdot$ and $(\beta\tilde{g}''(\rho))^{-1}\mathcal{B}_\cdot \in C([0,T], \mathcal{S}'(\mathbb{R}))$ have the same finite dimensional distributions, $(\beta\tilde{g}''(\rho))^{-1}\mathcal{B}_\cdot$ is a version of $\mathcal{A}_\cdot$ on the underlying common probability space.

To show the relation (3.10), write $\mathcal{A}_t(H) = \big[\mathcal{A}_t(H) - \mathcal{A}^1_t(H)\big] + \mathcal{A}^1_t(H)$ with respect to $\varepsilon = 1$. The term in square brackets is bounded through (3.9), using $\beta\tilde{g}''(\rho)\mathcal{A}_t(H) = \mathcal{B}_t(H)$ a.s. But, the other term is bounded through (3.8). The covariance bound follows by combining the bounds on the two terms. □

Proof of Lemma 3.3. We first make a preliminary bound. Suppose $\pi^{(m)}_\cdot$ converges to $\pi^{(\infty)}_\cdot$ w.r.t. the uniform topology on $D([0,T]; \mathcal{S}'(\mathbb{R}))$. Then, for each $\phi \in \mathcal{S}(\mathbb{R})$, we have that $\sup_{m\le\infty}\sup_{0\le t\le T}|\langle\pi^{(m)}_t, \phi\rangle| < \infty$. Hence, $\{\pi^{(m)}_t : m \le$

$\infty, t \in [0,T]\}$ is weakly bounded. By the Banach-Steinhaus theorem, the family is strongly bounded on a neighborhood U of 0:

$$\sup_{\phi\in U}\sup_{m\leq\infty}\sup_{t\in[0,T]} |\langle \pi_t^{(m)}, \phi\rangle| \;\leq\; L < \infty.$$

As the neighborhood U contains a base set $A = \{\phi : \|\phi\|_k < \delta\}$ for some $k \geq 0$ and $\delta > 0$, we have $\sup_{\phi\in H_k}\sup_{m\leq\infty}\sup_{t\in[0,T]} |\langle \pi_t^{(m)}, \phi\rangle| \leq (2L/\delta)\|\phi\|_k$. In other words, $\|\pi_t^{(m)}\|_{H_k'} \leq (2L/\delta)$ uniformly over m and t. In particular, as a consequence, for each $\pi_\cdot \in D([0,T];\mathcal{S}'(\mathbb{R}))$ there is a $k \geq 0$ such that $\sup_{t\in[0,T]}\|\pi_t\|_{H_k'} < \infty$.

We now show, for $\pi_\cdot \in D([0,T],\mathcal{S}'(\mathbb{R}))$, that $F_\cdot(\pi_\cdot)$ belongs to $C([0,T])$. Note that

$$\|\tau_{-x}G_\varepsilon\|_k = O(|x|^k) \;\text{ and }\; H \in \mathcal{S}(\mathbb{R}) \text{ rapidly decreases.} \tag{3.11}$$

Then, as $\sup_{t\in[0,T]}\|\pi_t\|_{H_k'} < \infty$ for some $k \geq 0$, we have, uniformly over $s, t \in [0,T]$, that $|F_t(\pi_\cdot) - F_s(\pi_\cdot)| = O(|t-s|)$.

We now prove that $F_\cdot$ is continuous w.r.t. the uniform topology. For a sequence $\pi_\cdot^{(m)} \to \pi^{(\infty)}$ in $D([0,T],\mathcal{S}'(\mathbb{R}))$, converging w.r.t. the uniform topology, we have $\sup_{m\leq\infty}\sup_{t\in[0,T]}\|\pi_t\|_{H_k'} < \infty$ for some $k \geq 0$. Then, noting (3.11), we may approximate, uniformly in $m \leq \infty$ and $t \in [0,T]$, for a J large that

$$F_t(\pi_\cdot^{(m)}) \;\sim\; \int_0^t du \int_{-J}^J dx\big(\nabla H(x)\big)\big\{\pi_u^{(m)}(\tau_{-x}G_\varepsilon)\big\}^2.$$

Further, as the family $\{\tau_{-x}G_\varepsilon : |x| \leq J\}$ is bounded in H_k, and $\mathcal{S}'(\mathbb{R})$ is equipped with the strong topology, we have

$$\lim_{m\uparrow\infty}\sup_{|x|\leq J}\sup_{u\in[0,T]} |\langle \pi_u^{(m)} - \pi_u^{(\infty)}, \tau_{-x}G_\varepsilon\rangle| \;=\; 0.$$

Hence, by the relation $a^2 - b^2 = (a-b)(a+b)$, uniformly over $t \in [0,T]$, we have

$$\begin{aligned}\lim_{m\to\infty}\int_0^t du \int_{-J}^J dx\big(\nabla H(i/r)\big)\big\{\pi_u^{(m)}(\tau_{-x}G_\varepsilon)\big\}^2 \\ = \int_0^t du \int_{-J}^J dx\big(\nabla H(i/r)\big)\big\{\pi_u^{(\infty)}(\tau_{-x}G_\varepsilon)\big\}^2,\end{aligned}$$

from which the desired continuity of $F_\cdot$ follows. □

3.5. **Generalized domains.** The goal here is to develop results which will allow to determine that the terms in (2.7) and (2.8), also in part (v) of Definition 2.5 are well-defined. Recall the discussion after Proposition 2.1. We now state some notions from [19] and [20]. For $p > 0$, define the Banach space

$$C_{p,0}(\mathbb{R}) \;=\; \Big\{\phi \in C(\mathbb{R}) : \phi/\phi_p \in C_0(\mathbb{R})\Big\},$$

where $\phi_p(x) = (1+|x|^2)^{-p}$ and $C_0(\mathbb{R})$ is the set of functions vanishing at infinity. The norm on $C_{p,0}$ is $\|\phi\|_p = \sup_x |\phi(x)/\phi_p(x)|$.

We now state part of Proposition 2.1 in [20] and Lemma 2.5 in [19]: For $1/2 < p < (1+\alpha)/2$ and $t \geq 0$,

- The space $C_{p,0}(\mathbb{R})$ and its dual $C_{p,0}'(\mathbb{R})$ are intermediate in that $\mathcal{S}(\mathbb{R}) \subset C_{p,0}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset C_{p,0}'(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$.
- $\mathcal{S}(\mathbb{R})$ is densely and continuously embedded in $C_{p,0}(\mathbb{R})$.

- $\Delta^{\alpha/2}, T_t : \mathcal{S}(\mathbb{R}) \to C_{p,0}(\mathbb{R})$ are continuous linear mapings.
- $t \mapsto T_t\phi$ is a continuous map in $C_{p,0}(\mathbb{R})$ for each $\phi \in \mathcal{S}(\mathbb{R})$.

Note also, by the density of $\mathcal{S}(\mathbb{R})$ in $C_{p,0}(\mathbb{R})$, that $\mathcal{S}(\mathbb{R})\otimes\widehat{C}$ is dense in $C_{p,0}(\mathbb{R})\otimes\widehat{C}$ (cf. [49]).

Let now $\mathcal{V}_t$ be a process belonging to $C([0,T],\mathcal{S}'(\mathbb{R}))$. For $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$ and fixed $t \in [0,T]$, we may extend straightforwardly the domain of $\mathcal{V}_t$, and define $\mathcal{V}_t(\Phi_t) = \langle \mathcal{V}_t, \Phi_t\rangle$.

Lemma 3.7. *The map $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C} \mapsto \int_0^T \langle \mathcal{V}_t, \Phi_t\rangle dt$ defines a $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variable on the probability space where $\mathcal{V}_\cdot$ lives.*

Then, for processes $\mathcal{Y}_\cdot$ and $N_\cdot$ in $C([0,T],\mathcal{S}'(\mathbb{R}))$ on a common probability space, the maps involving integrals in (2.7) *define $(\mathcal{S}(R)\otimes\widehat{C})'$-valued random variables. The map $\Phi_\cdot \mapsto \langle \mathcal{Y}_0, \Phi_0\rangle$ also defines a $(\mathcal{S}(R)\otimes\widehat{C})'$-valued random variable. All of these random variables are defined on the common probability space where $\mathcal{Y}_\cdot$ and $N_\cdot$ reside.*

Proof. The main statement is a part of the proof of Corollary 2.2 in [19] (which refers to the statements and proofs of Lemma 3.3 and Theorem 3.4 in [13]). For the convenience of the reader, we give the details.

We first argue, as in Lemma 3.3, that $\sup_{t\in[0,T]}\|\mathcal{V}_t\|_{H'_k} < \infty$ for some k. Since $\mathcal{V}_\cdot$ belongs to $C([0,T],\mathcal{S}'(\mathbb{R}))$, for each $\Phi_\cdot \in \mathcal{S}(R)$, we have $\sup_{t\in[0,T]}|\langle\mathcal{V}_t,\Phi\rangle| < \infty$. Then, the family $\{\mathcal{V}_t : t \in [0,T]\}$ is weakly bounded. Therefore, by the Banach-Steinhaus theorem, the family is strongly bounded on a neighborhood $U \subset \mathcal{S}(\mathbb{R})$ of 0:

$$\sup_{\Phi\in U}\sup_{t\in[0,T]}|\langle\mathcal{V}_t,\Phi\rangle| \ \le\ L < \infty.$$

The neighborhood U must contain a base set $A = \{\Phi : \|\Phi\|_k < \delta\}$ for some $k \ge 0$ and $\delta > 0$. Hence,

$$\sup_{\Phi\in H_k}\sup_{t\in[0,T]}|\langle\mathcal{V}_t,\Phi\rangle| \ \le\ (2L/\delta)\|\Phi\|_k.$$

In other words, $\|\mathcal{V}_t\|_{H'_k} \le (2L/\delta)$ uniformly over $t \in [0,T]$.

Now, we show that the map $t \mapsto \langle\mathcal{V}_t,\Phi_t\rangle$ is continuous so that the integral $\int_0^T\langle\mathcal{V}_t,\Phi_t\rangle dt$ is well-defined. Write

$$\begin{aligned}|\langle\mathcal{V}_t,\Phi_t\rangle - \langle\mathcal{V}_s,\Phi_s\rangle| \ &\le\ |\langle\mathcal{V}_t,\Phi_t-\Phi_s\rangle| + |\langle\mathcal{V}_t-\mathcal{V}_s,\Phi_s\rangle|\\ &\le\ \|\mathcal{V}_t\|_{H'_k}\|\Phi_t-\Phi_s\|_k + |\langle\mathcal{V}_t-\mathcal{V}_s,\Phi_s\rangle|.\end{aligned}$$

Since $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\hat{C}$, we have that $\{\Phi_s : s\in[0,T]\}$ is a bounded family in $\mathcal{S}(\mathbb{R})$. Then, by continuity of $\mathcal{V}_\cdot$ (in the strong topology on $\mathcal{S}'(\mathbb{R})$), $\lim_{s\to t}\sup_r|\langle\mathcal{V}_t - \mathcal{V}_s,\Phi_r\rangle| = 0$. Also, we have $\|\Phi_t-\Phi_s\|_k \le |t-s|\sup_r\|\partial_r\Phi_r\|_k$. Therefore, $\lim_{s\to t}\langle\mathcal{V}_s,\Phi_s\rangle = \langle\mathcal{V}_t,\Phi_t\rangle$.

Finally, define the linear functional $\tilde{\mathcal{V}}(\Phi_\cdot) := \int_0^T\langle\mathcal{V}_t,\Phi_t\rangle dt$ for $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$. Let $\Phi_\cdot^{(n)}$ be a sequence converging to $\Phi_\cdot$ in $\mathcal{S}(\mathbb{R})\otimes\widehat{C}$. Then, as $\sup_{t\in[0,T]}\|\mathcal{V}_t\|_{H'_k} \le 2L/\delta$, we have $|\tilde{\mathcal{V}}(\Phi_\cdot^{(n)} - \Phi_\cdot)| \le T(2L/\delta)\|\Phi_\cdot^{(n)} - \Phi_\cdot\|_k$. Hence, $\tilde{\mathcal{V}}(\cdot)$ is a linear continuous random functional on the probability space where $\mathcal{V}_\cdot$ is defined. By Itô's regularlization theorem (cf. Lemma 2.4 in [19]), as $\mathcal{S}(\mathbb{R})\otimes\widehat{C}$ is a nuclear Fréchet space, there is a unique $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variable $\tilde{\mathcal{V}}$ on the same probability space such that $\langle\tilde{\mathcal{V}},\Phi_\cdot\rangle = \tilde{\mathcal{V}}(\Phi_\cdot)$ a.s. for $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C}$. ☐

Denote, by $Q_t(G,H) = \mathrm{Cov}\big(\mathcal{V}_t(G), \mathcal{V}_t(H)\big)$, the covariance of the process $\mathcal{V}_\cdot$ with respect to time $t \in [0,T]$ and $G, H \in \mathcal{S}(\mathbb{R})$.

Proposition 3.8. *Suppose, with respect to a constant C_V and $1/2 < p < (1+\alpha)/2$, that the covariance satisfies, for $H_\cdot \in \mathcal{S}(\mathbb{R}) \otimes \widehat{C}$, the estimate*

$$\sup_{t\in[0,T]} |Q_t(H_t,H_t)| \ \le\ C_V \sup_{t\in[0,T]} \|H_t\|_p^2. \tag{3.12}$$

Then, the L^2 limit

$$\lim_{k\uparrow\infty} \int_0^T \langle \mathcal{V}_t, \Phi_t^{(k)}\rangle dt \ =: \ \int_0^T \langle \mathcal{V}_t, \Phi_t\rangle dt,$$

where $\{\Phi_\cdot^{(k)}\} \subset \mathcal{S}(\mathbb{R}) \otimes \widehat{C}$ approximates $\Phi_\cdot \in C_{p,0}(\mathbb{R}) \otimes \widehat{C}$, is well-defined and does not depend on the approximating sequence.

As a consequence, all maps given in (2.8)*, in terms of processes $\mathcal{Y}_\cdot$ and $N_\cdot$ in $C([0,T], \mathcal{S}'(\mathbb{R}))$ on a common probability space with covariances* (2.2) *and* (2.5) *respectively, are linear continuous random functionals on $\mathcal{S}(\mathbb{R})\otimes\widehat{C}$, which define unique $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variables on the common probability space.*

Proof. The argument follows part of the proof of Theorem 4.1 on pages 59-61 in [20] for similar processes. For the convenience of the reader, we give the argument.

By Proposition 3.7, for each $k \ge 1$, the map $\Phi_\cdot \in (\mathcal{S}(\mathbb{R})\otimes\widehat{C}) \mapsto \int_0^T \langle \mathcal{V}_t, \Phi_t^{(k)}\rangle dt$ defines a $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variable. Write

$$\begin{aligned}
\mathbb{E}\Big|\int_0^T \langle \mathcal{V}_t, \Phi_t^{(k)}\rangle dt - \int_0^T \langle \mathcal{V}_t, \Phi_t^{(\ell)}\rangle dt\Big|^2 \ &= \ \mathbb{E}\Big|\int_0^T \langle \mathcal{V}_t, \Phi_t^{(k)} - \Phi_t^{(\ell)}\rangle dt\Big|^2 \\
&\le\ T^2 \sup_{0\le t\le T} Q_t\big(\Phi_t^{(k)} - \Phi_t^{(\ell)}, \Phi_t^{(k)} - \Phi_t^{(\ell)}\big) \\
&\le\ T^2 C_V \sup_{0\le t\le T} \|\Phi_t^{(k)} - \Phi_t^{(\ell)}\|_p^2.
\end{aligned}$$

Hence, $\big\{ \int_0^T \langle \mathcal{V}_t, \Phi_t^{(k)}\rangle dt\big\}$ is an L^2-Cauchy sequence. The limit $\int_0^T \langle \mathcal{V}_t, \Phi_t\rangle dt$ does not depend on the approximation taken, and is linear and continuous in $\Phi_t \in C_{p,0}(\mathbb{R})\otimes\widehat{C}$.

To finish the proof of the proposition, note that

$$\begin{aligned}
\int_{\mathbb{R}} G^2(x)dx \quad &\le \quad \Big(\int_{\mathbb{R}} (1+|x|^2)^{-2p} dx\Big)\|G\|_p^2 \quad \text{and} \\
\int_{\mathbb{R}} G(x)\Delta^{\alpha/2}G(x)dx \quad &\le \quad \Big(\int_{\mathbb{R}} (1+|x|^2)^{-2p} dx\Big)\|G\|_p \|\Delta^{\alpha/2}G\|_p.
\end{aligned} \tag{3.13}$$

Also, recall that $\Delta^{\alpha/2}$ is a continuous operator from $\mathcal{S}(\mathbb{R})$ to $C_{p,0}(\mathbb{R})$. Then, the covariances (2.2) and (2.5) satisfy (3.12) for any $1/2 < p < (1+\alpha)/2$.

Since also T_t is a continuous operator from $\mathcal{S}(\mathbb{R})$ to $C_{p,0}(\mathbb{R})$ and $t \mapsto T_t\phi$ is continuous in $C_{p,0}$ for $\phi \in \mathcal{S}(\mathbb{R})$, one observes, from the above Cauchy limit development, that the maps in (2.8) define linear continuous random functionals. Hence, by Itô's regularization theorem (cf. Lemma 2.4 in [19]), the maps correspond to $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variables on the probability space where $\mathcal{Y}_\cdot$ and $N_\cdot$ are defined. $\square$

3.6. **Proof of Theorem 2.6.** We now combine the previous results in two helping propositions, and prove Theorem 2.6 at the end of the Subsection.

Proposition 3.9. *Consider the processes in Theorem 2.6 when* $0 < \alpha < 3/2$ *or* $\beta\tilde{g}''(\rho) = 0$*. All limit points are such that the terms in* (2.7) *and* (2.8) *define* $(\mathcal{S}(\mathbb{R}) \otimes \widehat{C})'$*-valued random variables on the underlying common probability space. Moreover, equation* (2.9) *holds for* $\Phi_\cdot \in \mathcal{S}(\mathbb{R}) \otimes \widehat{C}$*.*

Proof. As discussed in the beginning of Subsection 3.4, let n' be a subsequence so that $\mu^{(n')}$ converges weakly to μ. By Proposition 3.2, the limit points $\mathcal{Y}_\cdot$ and $\mathcal{M}_\cdot$ belong to $C([0,T];\mathcal{S}'(\mathbb{R}))$. Also, since the particle systems start from ν_ρ, the covariance of $\mathcal{Y}_t$ is given by (2.2). In addition, by Proposition 3.5, $\mathcal{M}_\cdot$ is a version of the noise $\sqrt{\tilde{g}(\rho)}N_\cdot$ in (2.3) corresponding to covariance (2.5). Then, by Proposition 3.7 and Proposition 3.8, the terms in (2.7) and (2.8) define $(\mathcal{S}(\mathbb{R}) \otimes \widehat{C})'$-valued random variables on the underlying common probability space.

We now claim that (2.9) holds for a function of the form $\Phi_t(x) = \Phi(x)f(t)$ for $\Phi \in \mathcal{S}(\mathbb{R})$ and $f \in \widehat{C}$. Linear combinations of such functions are dense in $\mathcal{S}(\mathbb{R}) \otimes \widehat{C}$. Hence, given this claim, by L^2 approximations, as in the proof of Proposition 3.8, one may verify (2.9), for $\Phi_\cdot \in \mathcal{S}(\mathbb{R}) \otimes \widehat{C}$.

To show the claim, we multiply the decomposition (3.3) by $f'(t)$ and then integrate over $t \in [0,T]$ to obtain

$$\begin{aligned}\int_0^T f'(t)\mathcal{Y}_t^{(n),\rightarrow}(\Phi)dt \;&=\; \int_0^T f'(t)dt\mathcal{Y}_0^{(n),\rightarrow}(\Phi) + \int_0^T f'(t)\mathcal{B}_t^{(n),\rightarrow}(\Phi)dt \\ &+ \int_0^T f'(t)\int_0^t \mathcal{Y}_s^{(n),\rightarrow}(\Delta^{\alpha/2}\Phi)dsdt + \int_0^T f'(t)\mathcal{M}_t^{(n),\rightarrow}(\Phi)dt + \mathcal{E}_1(n) \quad (3.14)\end{aligned}$$

where $\mathcal{E}_1(n)$ represents the errors. By parts (2) and (3) of Proposition 3.5, we conclude that

$$\lim_{n\uparrow\infty} \mathbb{E}_{\nu_\rho}\Big|\int_0^T f'(t)\mathcal{B}_t^{(n),\rightarrow}(\Phi)dt\Big|^2 = 0 \;\text{ and }\; \lim_{n\uparrow\infty} \mathbb{E}_{\nu_\rho}[\mathcal{E}_1(n)^2] = 0.$$

Since $f(T) = 0$, we have $\int_0^T f'(t)dt\mathcal{Y}_0^{(n),\rightarrow}(\Phi) = -f(0)\mathcal{Y}_0^{(n),\rightarrow}(\Phi)$ and, for $\Psi \in \mathcal{S}(\mathbb{R})$, we write

$$\int_0^T f'(t)\int_0^t \mathcal{Y}_s^{(n),\rightarrow}(\Delta^{\alpha/2}\Phi)dsdt \;=\; -\int_0^T f(t)\mathcal{Y}_t^{(n),\rightarrow}(\Psi)dt + \mathcal{E}_2(n).$$

Here, by the covariance structure (2.2), and (3.13), we have $\mathbb{E}_{\nu_\rho}\big[\mathcal{E}_2(n)^2\big] \leq C(f,T,\rho,p)\|\Psi - \Delta^{\alpha/2}\Phi\|_p^2$.

Now, we observe that $\pi_\cdot \in D([0,T],\mathcal{S}'(\mathbb{R})) \mapsto \int_0^T h(t)\pi_t(\Psi)dt$ is continuous w.r.t. the uniform topology for fixed $\Psi \in \mathcal{S}(\mathbb{R})$ and $h \in \widehat{C}$. Then, in distribution,

$$\begin{aligned}\lim_{n'\uparrow\infty}\int_0^T f'(t)\mathcal{Y}_t^{(n'),\rightarrow}(\Phi)dt \;&=\; \int_0^T f'(t)\mathcal{Y}_t(\Phi)dt \;\text{ and} \\ \lim_{n'\uparrow\infty}\int_0^T f'(t)\mathcal{M}_t^{(n'),\rightarrow}(\Phi)dt \;&=\; \int_0^T f'(t)\mathcal{M}_t(\Phi)dt.\end{aligned}$$

Let now $\{\Psi_k\}$ be a sequence in $\mathcal{S}(\mathbb{R})$ which approximates $\Delta^{\alpha/2}\Phi$ in $C_{p,0}$ for $1/2 < p < (1+\alpha)/2$. Then, by Proposition 3.8, we have in L^2, that

$$\lim_{k\uparrow\infty}\int_0^T f(t)\mathcal{Y}_t(\Psi_k)dt \;=\; \int_0^T f(t)\mathcal{Y}_t(\Delta^{\alpha/2}\Phi)dt.$$

Finally, noting the development starting from (3.14), we may take the limit as $n = n' \uparrow \infty$ and then $k \uparrow \infty$ to obtain

$$\int_0^T f'(t)\mathcal{Y}_t(\Phi)dt \;=\; -f(0)\mathcal{Y}_0(\Phi) - \int_0^T f(t)\mathcal{Y}_t(\Delta^{\alpha/2}\Phi)dt + \int_0^T f'(t)\mathcal{M}_t(\Phi_t)dt$$

a.s. with respect to the underlying probability space. □

Proposition 3.10. *Consider the processes in Theorem 2.6 when* $3/2 \le \alpha < 2$, $\gamma = 1 - 3/2\alpha$ *and* $\beta\tilde{g}''(\rho) \neq 0$. *All limit points are such that the corresponding maps in part (v) of Definition 2.5 define a* $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$*-valued random variables on the underlying common probability space, and that the equation* (2.16) *holds.*

Proof. We begin as in the proof of Proposition 3.9, and consider a subsequential limit in distribution $\mu^{(n')} \Rightarrow \mu$. With respect to this limit point, the proof of Proposition 3.9 shows that all the maps in part (v) of Definition 2.5, except for the map with respect to $\mathcal{A}_\cdot$, which was not considered there, define $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variables on the underlying common probability space.

By Proposition 3.6, $(\beta\tilde{g}''(\rho))^{-1}\mathcal{B}_\cdot$ is a version of $\mathcal{A}_\cdot \in C([0,T],\mathcal{S}'(\mathbb{R}))$ where, for each $H \in \mathcal{S}(\mathbb{R})$, $\mathcal{A}_\cdot(H)$ is the uniform L^2 Cauchy limit of $\mathcal{A}^\varepsilon_\cdot(H)$. In the following development, we will use this version of $\mathcal{A}_\cdot$, which belongs to the underlying common probability space.

We now consider the map, $\Phi_\cdot \in \mathcal{S}(\mathbb{R})\otimes\widehat{C} \mapsto \int_0^T \mathcal{A}_t(\partial_t\Phi_t)dt$, with respect to $\mathcal{A}_\cdot$. For $3/2 \le \alpha < 2$ and $1/2 < p < (1+\alpha)/2$, we have

$$\int_{\mathbb{R}} H^2(x)dx \le \|H\|_p^2\int_{\mathbb{R}}(1+|x|^2)^{-2p}dx \;\;\text{and}\;\; \int_{\mathbb{R}}|H(x)|dx \le \|H\|_p\int_{\mathbb{R}}(1+|x|^2)^{-p}dx.$$

Then, the covariance of $\mathcal{A}_t(H)$, given in (3.10), satisfies (3.12). Hence, by Proposition 3.8, the map associated to $\mathcal{A}_\cdot$, in part (v) of Definition 2.5, defines a $(\mathcal{S}(\mathbb{R})\otimes\widehat{C})'$-valued random variable on the underlying common probability space.

To show that equation (2.16) holds, we again follow the argument of Proposition 3.9: We will establish (2.16) with respect to a function of the form $\Phi_t(x) = \Phi(x)f(t)$ for $H \in \mathcal{S}(\mathbb{R})$ and $f \in \widehat{C}$. Then, as linear combinations of such functions are dense in $\mathcal{S}(\mathbb{R})\otimes\widehat{C}$, L^2 approximations, as in the proof of Proposition 3.8, will derive (2.16) on functions in $\mathcal{S}(\mathbb{R})\otimes\widehat{C}$ with respect to the underlying common probability space.

The only difference with the proof of Proposition 3.9, in establishing (2.16) on such product functions, that is equation (2.17), is that $\mathcal{B}_t^{(n),\to}(\Phi)$ is not negligible, and needs development.

By part (1) of Proposition 3.5, we have

$$\int_0^T f'(t)\mathcal{B}_t^{(n'),\to}(\Phi)dt \;=\; \beta\tilde{g}''(\rho)\int_0^T f'(t)\mathcal{A}_t^{(n'),\varepsilon}(\Phi)dt + \mathcal{E}_3(n,\varepsilon),$$

where $\lim_{\varepsilon\downarrow 0}\lim_{n'\uparrow\infty}\mathbb{E}_{\nu_\rho}\mathcal{E}_3(n',\varepsilon)^2 = 0$. Now, by Lemma 3.4, $\mathcal{A}_\cdot^{(n'),\varepsilon}(\Phi)$ converges weakly in $C([0,T])$, as $n' \uparrow \infty$, to $\mathcal{A}^\varepsilon_\cdot(\Phi)$, defined with respect to limit point $\mathcal{Y}_\cdot$ in the underlying common probability space. Since $\pi_\cdot \in C([0,T]) \mapsto \int_0^T f'(t)\pi_t dt$ is a continuous function, we have in distribution that

$$\lim_{n'\uparrow\infty}\int_0^T f'(t)\mathcal{A}_t^{(n'),\varepsilon}(\Phi)dt \;=\; \int_0^T f'(t)\mathcal{A}_t^{\varepsilon}(\Phi)dt.$$

By the uniform L^2 limit, $\lim_{\varepsilon\downarrow 0}\mathcal{A}_\cdot^\varepsilon(\Phi) = \mathcal{A}_\cdot(\Phi)$, in part (1) of Proposition 3.5, we have in L^2 of the underlying common probability space that

$$\lim_{\varepsilon\downarrow 0}\int_0^T f'(t)\mathcal{A}_t^{\varepsilon}(\Phi)dt \;=\; \int_0^T f'(t)\mathcal{A}_t(\Phi)dt.$$

Following the sequence of steps in the proof of Proposition 3.9, starting from (3.14), with the input above on the analysis of $\int_0^T f'(t)\mathcal{B}_t^{(n'),\to}(\Phi)dt$, we recover (2.17), by passing to the limit as $n'\uparrow\infty$, and then $k\uparrow\infty$ and $\varepsilon\downarrow 0$. $\square$

Proof of Theorem 2.6. When $0<\alpha<3/2$ or $\beta\tilde{g}''(\rho)=0$, by Proposition 3.9, any limit point $\mathcal{Y}_\cdot$ is such that the terms in (2.7) and (2.8) are well-defined on the underlying common probability space, and the equation (2.9) is satisfied. Hence, by uniqueness of the 'generalized' solution to (2.9) (cf. discussion after (2.10)), the limit point $\mathcal{Y}_\cdot$ is identified as the solution of (2.3) in the sense of Definition 2.2.

On the other hand, when $3/2\le\alpha<2$, $\gamma = 1-3/2\alpha$ and $\beta\tilde{g}''(\rho)\neq 0$, we claim that any limit point $\mathcal{Y}_\cdot$ satisfies Definition 2.5, and is therefore a fractional L^2 energy solution of (2.12). First, for part (i), since we start from an invariant measure ν_ρ, for each $t\in[0,T]$, $\mathcal{Y}_t$ has covariance (2.2) and is therefore a white noise. Next, for part (ii), by Proposition 3.5, $\mathcal{Y}_\cdot$ satisfies the energy condition (2.15). Parts (iii) and (iv), on specifications of $\mathcal{A}_\cdot$ and $\mathcal{M}_\cdot$, hold by Propositions 3.5 and 3.6; the latter proposition also identifies the version of $\mathcal{A}_\cdot$ on the underlying common probability space that we use. Finally, by Proposition 3.10, the maps, in part (v), are well-defined and the equation (2.16), in part (vi), holds on the underlying common probability space. $\square$

4. Proof of the generalized Boltzmann-Gibbs Principle

We adapt the proof of Theorem 3.2 in [25] to the long-range setting. We first recall some preliminary notions and estimates, before proving Theorem 3.1 at the end of the section.

We will need an 'equivalence of ensembles' estimate for the zero-range invariant measure ν_ρ. Since ν_ρ is a product measure which satisfies the assumptions of Proposition 5.1 in [25], the following case of this result holds.

In the following, we will abbreviate $\eta^{(\ell)}(0)=\eta^{(\ell)}$ for $\ell\ge 1$.

Proposition 4.1. *Let f be a local $L^5(\nu_\rho)$ function, supported on sites Λ_{ℓ_0}, such that $\tilde{f}(\rho)=\tilde{f}'(\rho)=0$. Then, there exists a constant $C=C(\rho,\ell_0)$ such that for $\ell\ge\ell_0$ we have*

$$\Big\|E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}] - \Big\{(\eta^{(\ell)}-\rho)^2 - \frac{\sigma^2(\rho)}{2\ell+1}\Big\}\frac{\tilde{f}''(\rho)}{2}\Big\|_{L^4(\nu_\rho)} \;\le\; \frac{C\|f\|_{L^5(\nu_\rho)}}{\ell^{3/2}}.$$

On the other hand, when only $\tilde{f}(\rho)=0$ is known,

$$\Big\|E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}] - \{\eta^{(\ell)}-\rho\}\tilde{f}'(\rho)\Big\|_{L^4(\nu_\rho)} \;\le\; \frac{C\|f\|_{L^5(\nu_\rho)}}{\ell}.$$

We now define notions of H_1 and H_{-1} norms which will be useful. Let $S_n = (L_n + L_n^*)/2$ be the generator of the process with respect to jump probability $s(\cdot)$. Define the $H_{1,n}$ semi-norm $\|\cdot\|_{1,n}$ on local, bounded functions by

$$\|f\|_{1,n}^2 \;:=\; E_{\nu_\rho}\big[f(-S_n)f\big] \;=\; nD_{\nu_\rho}(f),$$

where $D_{\nu_\rho}(f)$ is the grand-canonical Dirichlet form

$$D_{\nu_\rho}(f) \;=\; \frac{1}{2}\sum_{x,y\in\mathbb{Z}} \mathbb{E}_{\nu_\rho}\Big[g(\eta(x))\big(\nabla_{x,x+y}f(\eta)\big)^2\Big]s(y).$$

Let $H_{1,n}$ be the Hilbert space consisting of the completion of functions with finite $H_{1,n}$ norm modulo norm-zero functions. Note that local bounded functions are dense in $H_{1,n}$.

Let $\|\cdot\|_{-1,n}$ be the dual semi-norm with respect to the $L^2(\nu_\rho)$ inner-product given by

$$\|f\|_{-1,n} \;:=\; \sup\Big\{\frac{E_{\nu_\rho}[f\phi]}{\|\phi\|_{1,n}} : \phi\neq 0 \text{ local, bounded}\Big\}.$$

Denote $H_{-1,n}$ as the Hilbert space corresponding to the completion over those functions with finite $\|\cdot\|_{-1,n}$ norm modulo norm-zero functions.

We now state a long-range form of Proposition 4.1 in [25], specialized to the case when ν_ρ is a product measure. Denote the Λ_ℓ-restricted Dirichlet form, on local, bounded functions, by

$$D_{\nu_\rho,\ell}(\phi) \;=\; \frac{1}{2}\sum_{x,y\in\Lambda_\ell} E_{\nu_\rho}\Big[g(\eta(x))\big(\nabla_{x,x+y}\phi(\eta)\big)^2\Big]s(y).$$

Proposition 4.2. *Let $r:\Omega\to\mathbb{R}$ be a local $L^4(\nu_\rho)$ function, supported on sites in Λ_{ℓ_0}, for $\ell_0\geq 1$. Suppose that $E_{\nu_\rho}[r|\eta^{(\ell_0)}]=0$ a.s. Then, for local, bounded functions ϕ, we have*

$$\big|E_{\nu_\rho}[r(\eta)\phi(\eta)]\big| \;\leq\; E_{\nu_\rho}\Big[W\Big(\sum_{x\in\Lambda_{\ell_0}}\eta(x),\ell_0\Big)^2\Big]^{1/4}\|r\|_{L^4(\nu_\rho)}D^{1/2}_{\nu_\rho,\ell_0}(\phi).$$

Proof. Recall the notation for the canonical process in Subsection 2.1. Since $E_{\nu_\rho}[r|\sum_{|x|\leq\ell_0}\eta(x)=k] = E_{\nu_{k,\ell_0}}[r]=0$, the function r restricted to $\mathcal{G}_{k,\ell_0}=\{\eta:\sum_{x\in\Lambda_{\ell_0}}\eta(x)=k\}$ is orthogonal to constant functions. Hence, r belongs to the range of $-S_{k,\ell_0}$, and the equation $r=-S_{k,\ell_0}u$ holds for a function u on $\mathcal{G}_{k,\ell_0}$.

With $k=\sum_{x\in\Lambda_{\ell_0}}\eta(x)$, write

$$\begin{aligned}
\big|E_{\nu_\rho}[r\phi]\big| &= \big|E_{\nu_\rho}\big[E_{\nu_\rho}[r\phi|\eta^{(\ell_0)}]\big]\big|\\
&= \big|E_{\nu_\rho}\big[E_{\nu_\rho}[(-S_{k,\ell_0}u)\phi|\eta^{(\ell_0)}]\big]\big|\\
&\leq E_{\nu_\rho}\Big[E_{\nu_\rho}[u(-S_{k,\ell_0}u)|\eta^{(\ell_0)}]^{1/2}E_{\nu_\rho}[\phi(-S_{k,\ell_0}\phi)|\eta^{(\ell_0)}]^{1/2}\Big].
\end{aligned}$$

To obtain the last line, as $-S_{n,\mathcal{G}}$ is a nonnegative symmetric operator, we used that it has a square root.

Since $W(k,\ell_0)$ is the reciprocal of the spectral gap for $-S_{k,\ell_0}$, we have

$$E_{\nu_\rho}\big[ru|\eta^{(\ell_0)}\big] \;\leq\; W(k,\ell_0)E_{\nu_\rho}\big[r^2|\eta^{(\ell_0)}\big].$$

Inputting this estimate into the previous display, we obtain

$$\big|E_{\nu_\rho}[r\phi]\big| \quad \le \quad E_{\nu_\rho}\Big[W\Big(\sum_{x\in\Lambda_{\ell_0}}\eta(x)\Big)E_{\nu_\rho}[r^2|\eta^{(\ell_0)}]\Big]^{1/2}D^{1/2}_{\nu_\rho,\ell_0}(\phi).$$

We now use Schwarz inequality to finish the proof. □

To simplify notation, for the rest of the section, we will drop the superscript 'n' and write $\eta^{(n)} = \eta$.

We now give a useful estimate on the variance of additive functionals. See the proof of Theorem 2.2 in [46], Lemma 4.3 in [16] and Appendix 1.6 in [35] for similar estimates. Recall the translation $\tau_z\eta$ given by $\tau_z\eta(x) = \eta(x+z)$. For $c\in\mathbb{R}$ and $r\in L^2(\nu_\rho)$, let $r(i,\eta):\mathbb{Z}\times\Omega\to\mathbb{R}$ be a function such that $\|r(i,\cdot)\|_{-1,n}\le\|r\|_{-1,n}$ for each $i\in\mathbb{Z}$.

Proposition 4.3. *Let $r:\Omega\to\mathbb{R}$ be a mean-zero $L^2(\nu_\rho)$ function, $\tilde r(\rho)=0$. Then, there exists a universal constant C such that*

$$\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le K}\Big(\int_0^s r(\lfloor cs\rfloor,\eta_u)ds\Big)^2\Big] \;\le\; CK\|r\|^2_{-1,n}.$$

We give a proof of the proposition at the end of the section.

We now adapt several lemmas in [25] to the long-range setting. Their arguments are similar to those for nearest-neighbor processes in [25], although there are some differences. Recall the definition of $\bar h_{c,s}$, for $c\in\mathbb{R}$ and $s\in[0,K]$, above the statement of Theorem 3.1.

Lemma 4.4. *Let $f:\Omega\to\mathbb{R}$ be a local $L^4(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\tilde f(\rho)=0$. Then, there exists a constant $C=C(\rho)$ such that, for $\ell\ge\ell_0$, and $h\in\ell^1(\mathbb{Z})\cap\ell^2(\mathbb{Z})$,*

$$\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}\bar h_{c,s}(x)\Big\{f(\tau_x\eta_s)-E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(\ell)}(x)]\Big\}ds\Big)^2\Big]$$
$$\le\; CK\frac{\ell^{\alpha+1}}{n}\|f\|^2_{L^4(\nu_\rho)}\sum_{x\in\mathbb{Z}}h^2(x).$$

Proof. Since $h\in\ell^1(\mathbb{Z})$, the expectation is well-defined. We may write the integrand function as follows:

$$\sum_{x\in\mathbb{Z}}\bar h_{c,s}(x)\Big\{f(\tau_x\eta)-E_{\nu_\rho}[f(\tau_x\eta)|\eta^{(\ell)}(x)]\Big\} \;=\; r(\lfloor cs\rfloor,\eta)$$

where

$$r(i,\eta)\;:=\;\sum_{x\in\mathbb{Z}}h(x)\Big\{f(\tau_{i+x}\eta_s)-E_{\nu_\rho}[f(\tau_{i+x}\eta_s)|\eta_s^{(\ell)}(i+x)]\Big\}.$$

By translation of invariance of ν_ρ and $s(\cdot)$, we have that $\|r(i,\cdot)\|_{-1,n}=\|r\|_{-1,n}$ where $r(\eta)=r(0,\eta)$. Hence, by Proposition 4.3, we need only a sufficient estimate of the $H_{-1,n}$ norm of r to finish.

Using Proposition 4.2, bound n times the $H_{-1,n}$ norm as follows:

$$\begin{aligned}
&\sup_{\phi}\Big\{D_{\nu_\rho}^{-\frac{1}{2}}(\phi)E_{\nu_\rho}\big[\sum_{x\in\mathbb{Z}}h(x)\Big\{f(\tau_x\eta)-E_{\nu_\rho}[f(\tau_x\eta)|\eta^{(\ell)}(x)]\Big\}\phi\big]\Big\}\\
&=\ \sup_{\phi}\sum_{x\in\mathbb{Z}}D_{\nu_\rho}^{-\frac{1}{2}}(\phi)E_{\nu_\rho}\Big[h(x)\big(f(\eta)-E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}]\big)\phi(\tau_{-x}\eta)\Big] \qquad (4.1)\\
&\le\ \sup_{\phi}D_{\nu_\rho}^{-\frac{1}{2}}(\phi)\sum_{x\in\mathbb{Z}}|h(x)|E_{\nu_\rho}\big[W\big(\sum_{z\in\Lambda_\ell}\eta(z),\ell\big)^2\big]^{\frac{1}{4}}\|f\|_{L^4(\nu_\rho)}D^{\frac{1}{2}}_{\nu_\rho,\ell}(\phi(\tau_{-x}\eta)).
\end{aligned}$$

By translation-invariance of ν_ρ, and counting the number of repetitions,

$$\sum_{x\in\mathbb{Z}}D_{\nu_\rho,\ell}(\phi(\tau_{-x}\eta))\ \le\ (2\ell+1)D_{\nu_\rho}(\phi).$$

In passing, we comment that, in the nearest-neighbor setting, the last inequality could be taken as an equality.

Then, by the spectral gap assumption (SG), and $2ab=\inf_{\kappa>0}[\kappa a^2+\kappa^{-1}b^2]$, we bound (4.1) by

$$\begin{aligned}
&\sup_{\phi}D_{\nu_\rho}^{-\frac{1}{2}}(\phi)\inf_{\kappa>0}\Big\{\kappa C\ell^\alpha\|f\|^2_{L^4(\nu_\rho)}\sum_{x\in\mathbb{Z}}h^2(x)+\kappa^{-1}C\ell D_{\nu_\rho}(\phi)\Big\}\\
&\quad\le\Big(C\ell^{\alpha+1}\|f\|^2_{L^4(\nu_\rho)}\sum_{x\in\mathbb{Z}}h^2(x)\Big)^{\frac{1}{2}}. \qquad \square
\end{aligned}$$

The size of the box in the conditional expectation is now doubled.

Lemma 4.5. *Let $f:\Omega\to\mathbb{R}$ be a local $L^5(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\tilde f(\rho)=\tilde f'(\rho)=0$. There exists a constant $C=C(\rho,\ell_0)$ such that, for $\ell\ge\ell_0$, and $h\in\ell^1(\mathbb{Z})\cap\ell^2(\mathbb{Z})$,*

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}\bar h_{c,s}(x)\\
&\qquad\qquad\times\Big\{E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(\ell)}(x)]-E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(2\ell)}(x)]\Big\}ds\Big)^2\Big]\\
&\qquad\le\ CK\|f\|^2_{L^5(\nu_\rho)}\frac{\ell^{\alpha-1}}{n}\sum_{x\in\mathbb{Z}}h^2(x).
\end{aligned}$$

On the other hand, when only $\tilde f(\rho)=0$ is known,

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}\bar h_{c,s}(x)\\
&\qquad\qquad\times\Big\{E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(\ell)}(x)]-E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(2\ell)}(x)]\Big\}ds\Big)^2\Big]\\
&\qquad\le\ CK\|f\|^2_{L^5(\nu_\rho)}\frac{\ell^{\alpha}}{n}\sum_{x\in\mathbb{Z}}h^2(x).
\end{aligned}$$

Proof. We prove only the first statement since the second has a similar argument. Define the sigma-field $\mathcal{F}_\ell=\sigma\{\eta^{(\ell)},\eta^c_\ell\}$ where $\eta^c_\ell=\{\eta(x):x\notin\Lambda_\ell\}$ for $\ell\ge1$. Since $\mathcal{F}_{2\ell}\subset\mathcal{F}_\ell$, $\ell\ge\ell_0$, f is supported on sites Λ_{ℓ_0}, and ν_ρ is a product measure, we have that

$$\begin{aligned} E_{\nu_\rho}\Big[E_{\nu_\rho}\big[f(\eta)\big|\eta^{(\ell)}\big]\big|\eta^{(2\ell)}\Big] &= E_{\nu_\rho}\Big[E_{\nu_\rho}\big[f(\eta)\big|\eta^{(\ell)},\eta^c_\ell\big]\big|\eta^{(2\ell)},\eta^c_{2\ell}\Big]\\ &= E_{\nu_\rho}\big[f(\eta)\big|\eta^{(2\ell)},\eta^c_{2\ell}\big] \;=\; E_{\nu_\rho}\big[f(\eta)\big|\eta^{(2\ell)}\big]. \end{aligned}$$

Therefore, we may follow similar steps as in the proof of Lemma 4.4 to the last line, with

$$r(i,\eta) \;=\; \sum_{x\in\mathbb{Z}} h(x)\Big\{E_{\nu_\rho}[f(\tau_{i+x}\eta)|\eta^{(\ell)}(i+x)] - E_{\nu_\rho}[f(\tau_{i+x}\eta)|\eta^{(2\ell)}(i+x)]\Big\}$$

and $r(\eta) = r(0,\eta)$.

We now claim that the following variance is bounded as

$$\|E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}] - E_{\nu_\rho}[f(\eta)|\eta^{(2\ell)}]\|^2_{L^4(\nu_\rho)} \;\le\; C\|f\|^2_{L^5(\nu_\rho)}\ell^{-2}.$$

Indeed, by the inequality $(a+b+c)^2 \le 3a^2+3b^2+3c^2$, the variance is bounded by

$$\begin{aligned} &3\Big\|E_{\nu_\rho}\Big[f(\eta) - \frac{\varphi_f''(\rho)}{2}\Big\{(\eta^{(\ell)}-\rho)^2 - \frac{\sigma^2(\rho)}{2\ell+1}\Big\}\Big|\eta^{(\ell)}\Big]\Big\|^2_{L^4(\nu_\rho)}\\ &\quad+3\Big\|E_{\nu_\rho}\Big[f(\eta) - \frac{\varphi_f''(\rho)}{2}\Big\{(\eta^{(2\ell)}-\rho)^2 - \frac{\sigma^2(\rho)}{2(2\ell)+1}\Big\}\Big|\eta^{(2\ell)}\Big]\Big\|^2_{L^4(\nu_\rho)}\\ &\quad+3\Big\|\frac{\varphi_f''(\rho)}{2}\Big\{E_{\nu_\rho}\Big[(\eta^{(\ell)}-\rho)^2 - \frac{\sigma^2(\rho)}{2\ell+1}\Big|\eta^{(\ell)}\Big]\\ &\qquad\qquad+E_{\nu_\rho}\Big[(\eta^{(2\ell)}-\rho)^2 - \frac{\sigma^2(\rho)}{2(2\ell)+1}\Big|\eta^{(2\ell)}\Big]\Big\}\Big\|^2_{L^4(\nu_\rho)}. \end{aligned}$$

The first two terms are bounded by Proposition 4.1 of order $O(\|f\|^2_{L^5(\nu_\rho)}\ell^{-3})$. However, the last term, by a fourth moment bound of $(\eta^{(k)}-\rho)^2$ with variously $k=\ell$ and $k=2\ell$ is of order $O(\|f\|^2_{L^2(\nu_\rho)}\ell^{-2})$. Combining the two estimates, we obtain that the variance is of order $O(\|f\|^2_{L^5(\nu_\rho)}\ell^{-2})$ as desired.

Replacing now $\|f\|^2_{L^4(\nu_\rho)}$ in the last line of Lemma 4.4 by this variance estimate, one recovers the bound in the first statement. □

Lemma 4.6. *Let $f:\Omega\to\mathbb{R}$ be a local $L^5(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\tilde f(\rho) = \tilde f'(\rho) = 0$. Then, there exists a constant $C = C(\rho,\alpha,\ell_0)$ such that, for $\ell\ge\ell_0$, and $h\in\ell^1(\mathbb{Z})\cap\ell^2(\mathbb{Z})$,*

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}\bar h_{c,s}(x)\Big\{E_{\nu_\rho}[f(\tau_x\eta)|\eta^{(\ell_0)}(x)] - E_{\nu_\rho}[f(\tau_x\eta)|\eta^{(\ell)}(x)]\Big\}ds\Big)^2\Big]\\ &\qquad\le CK\|f\|^2_{L^5(\nu_\rho)}\frac{w_\alpha(\ell)}{n}\sum_{x\in\mathbb{Z}}h^2(x). \end{aligned}$$

On the other hand, when only $\tilde f(\rho) = 0$ is known,

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t\sum_{x\in\mathbb{Z}}\bar h_{c,s}(x)\Big\{E_{\nu_\rho}[f(\tau_x\eta)|\eta^{(\ell_0)}(x)] - E_{\nu_\rho}[f(\tau_x\eta)|\eta^{(\ell)}(x)]\Big\}ds\Big)^2\Big]\\ &\qquad\le CK\|f\|^2_{L^5(\nu_\rho)}\frac{\ell^\alpha}{n}\sum_{x\in\mathbb{Z}}h^2(x). \end{aligned}$$

Proof. As the second statement has a similar proof, we only demonstrate the first display. Write $\ell = 2^{m+1}\ell_0 + r$ where $0 \le r \le 2^{m+1}\ell_0 - 1$. Then,

$$E_{\nu_\rho}[f(\eta)|\eta^{(\ell_0)}] - E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}] \;=\; E_{\nu_\rho}[f(\eta)|\eta^{(2^{m+1}\ell_0)}] - E_{\nu_\rho}[f(\eta)|\eta^{(\ell)}] + \sum_{i=0}^{m}\big\{E_{\nu_\rho}[f(\eta)|\eta^{(2^i\ell_0)}] - E_{\nu_\rho}[f(\eta)|\eta^{(2^{i+1}\ell_0)}]\big\}.$$

Now, by Minkowski's inequality and Lemma 4.5, we obtain that the left-side of the display in the lemma statement is bounded by

$$\Big\{\big(\frac{CK(2^{m+1}\ell_0)^{\alpha-1}}{n}\big)^{1/2} + \sum_{i=0}^{m}\big(\frac{CK(2^i\ell_0)^{\alpha-1}}{n}\big)^{1/2}\Big\}^2 \|f\|^2_{L^5(\nu_\rho)} \sum_{x\in\mathbb{Z}} h^2(x).$$

When $\alpha > 1$, one obtains an estimate for the last display of order $O(\ell^{\alpha-1})$. However, when $\alpha = 1$, the order is $O(\log(\ell)+1)$. When $0 < \alpha < 1$, the series in the display is summable, and so the order is $O(1)$. These orders correspond to the different cases in the definition of $w_\alpha(\cdot)$ in (3.5). Hence, the last display is further bounded by

$$\frac{CK\|f\|^2_{L^5(\nu_\rho)} w_\alpha(\ell)}{n}\sum_{x\in\mathbb{Z}} h^2(x),$$

to finish the proof. □

The last step is an 'equivalence of ensembles' estimate.

Lemma 4.7. *Let $f : \Omega \to \mathbb{R}$ be a local $L^5(\nu_\rho)$ function supported on sites in Λ_{ℓ_0} such that $\tilde f(\rho) = \tilde f'(\rho) = 0$. Then, there exists a constant $C = C(\rho, \ell_0)$ such that, for $\ell \ge \ell_0$, and $h \in \ell^1(\mathbb{Z})$,*

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t \sum_{x\in\mathbb{Z}} \bar h_{c,s}(x)\Big\{E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(\ell)}(x)] \\ - \frac{\tilde f''(\rho)}{2}\Big((\eta_s^{(\ell)}(x)-\rho)^2 - \frac{\sigma^2(\rho)}{2\ell+1}\Big)\Big\}ds\Big)^2\Big] \\ \le\; CK^2\|f\|^2_{L^5(\nu_\rho)}\frac{n^{2/\alpha}}{\ell^3}\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}|h(x)|\Big)^2\end{aligned}$$

On the other hand, when only $\tilde f(\rho) = 0$ is known,

$$\begin{aligned}\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le t\le K}\Big(\int_0^t \sum_{x\in\mathbb{Z}} \bar h_{c,s}(x)\Big\{E_{\nu_\rho}[f(\tau_x\eta_s)|\eta_s^{(\ell)}(x)] - \tilde f'(\rho)\big(\eta_s^{(\ell)}(x)-\rho\big)\Big\}ds\Big)^2\Big] \\ \le\; CK^2\|f\|^2_{L^5(\nu_\rho)}\frac{n^{2/\alpha}}{\ell^2}\Big(\frac{1}{n^{1/\alpha}}\sum_{x\in\mathbb{Z}}|h(x)|\Big)^2.\end{aligned}$$

Proof. By squaring and using stationarity and translation-invariance of ν_ρ, the left-hand side of each display is bounded by

$$\mathbb{E}_{\nu_\rho}\Big[\Big(\int_0^K \sum_{x\in\mathbb{Z}} |\bar h_{c,s}(x)||\psi(x,\eta_s)|ds\Big)^2\Big] \;\le\; K^2 E_{\nu_\rho}\Big[\Big(\sum_{x\in\mathbb{Z}}|h(x)||\psi(x,\eta)|\Big)^2\Big]$$

where $\psi(x,\eta)$ is the expression in curly braces. Now, for each η, by Schwarz inequality,

$$\Big(\sum_{x\in\mathbb{Z}}|h(x)|\psi(x,\eta)\Big)^2 \ \le\ \Big(\sum_{x\in\mathbb{Z}}|h(x)|\Big)\sum_{x\in\mathbb{Z}}|h(x)|\psi^2(x,\eta).$$

Since ν_ρ is translation-invariant, $E_{\nu_\rho}[\psi^2(x,\eta)] = E_{\nu_\rho}[\psi^2(0,\eta)]$. The desired bound now follows, noting the form of $\psi(0,\eta)$, from Proposition 4.1. □

Proof of Theorem 3.1. With the above ingredients in place, the estimate follows now by first applying Lemma 4.4 with $\ell=\ell_0$, and then Lemmas 4.6 and 4.7. □

Proof of Proposition 4.3. Recall the canonical measure $\nu_{k,\ell}$ in Subsection 2.1. Let $\Lambda_{k,\ell}=\{\eta:\sum_{x\in\Lambda_\ell}\eta(x)=k\}$.

Divide $[0,K)$ into $J+1=\lceil cK\rceil$ intervals, $[0,K)=\cup_{i=0}^J[t_i,t_{i+1})$, where $t_0=0$, $t_{J+1}=K$, and $t_{i+1}-t_i=1/c$ for $0\le i\le J-1$. Let $r^{(i)}(\eta)=r(i,\eta)$ and define the resolvent $u^{(i)}_\lambda$ by $\lambda u^{(i)}_\lambda - S_n u^{(i)}_\lambda = r^{(i)}$ for $0\le i\le J$ and $\lambda>0$. By multiplying by $u^{(i)}_\lambda$ and taking expectation, we have

$$\lambda\|u^{(i)}_\lambda\|^2_{L^2(\nu_\rho)}+\|u^{(i)}_\lambda\|^2_{1,n} \ =\ E_{\nu_\rho}\big[r^{(i)}u^{(i)}_\lambda\big] \ \le\ \|r^{(i)}\|_{-1,n}\|u^{(i)}_\lambda\|_{1,n}.$$

Hence, both $\lambda\|u^{(i)}_\lambda\|^2_{L^2(\nu_\rho)}, \|u^{(i)}_\lambda\|^2_{1,n}\le\|r^{(i)}\|^2_{-1,n}\le\|r\|^2_{-1,n}$.

Write the integral

$$\begin{aligned}\int_0^s r(\lfloor cu\rfloor,\eta_u)du \ &=\ \sum_{i:t_{i+1}\le s}\int_{t_i}^{t_{i+1}} r^{(i)}(\eta_u)du+\int_{t_{\lfloor sc\rfloor}}^s r^{(\lfloor sc\rfloor)}(\eta_u)du\\ &=\ \sum_{i:t_{i+1}\le s}\int_{t_i}^{t_{i+1}}\big(\lambda u^{(i)}_\lambda - S_n u^{(i)}_\lambda\big)(\eta_u)du+\int_{t_{\lfloor sc\rfloor}}^s\big(\lambda u^{(\lfloor sc\rfloor)}_\lambda - S_n u^{(\lfloor sc\rfloor)}_\lambda\big)(\eta_u)du.\end{aligned}$$

We comment that the index set $\{i:t_{i+1}\le s\}$ equals $\emptyset$, $\{i:i+1\le\lfloor sc\rfloor\}$, and $\{i:i+1\le J+1\}$ when $s<c^{-1}$, $s<K$ and $s=K$, correspondingly.

Now,

$$M^{(i)}(v) \ :=\ u^{(i)}_\lambda(\eta_v)-u^{(i)}_\lambda(\eta_0)-\int_0^v L_n u^{(i)}_\lambda(\eta_u)du$$

is a martingale with respect to the process $\eta_\cdot$. On the other hand, with respect to the time-reversed process $\eta^*(\cdot):=\eta(K-\cdot)$, the process

$$M^{*,(i)}(w) \ :=\ u^{(i)}_\lambda(\eta^*_w)-u^{(i)}_\lambda(\eta^*_0)-\int_0^w L^*_n u^{(i)}_\lambda(\eta^*_u)du$$

is a martingale. By the invariance of ν_ρ, we have

$$\mathbb{E}_{\nu_\rho}\big(M^{(i)}(s)-M^{(i)}(u)\big)^2 \ =\ 2|s-u|\|u^{(i)}_\lambda\|^2_{1,n} \ \le\ 2|s-u|\|r^{(i)}\|^2_{-1,n} \ \le\ 2|s-u|\|r\|^2_{-1,n}$$

and similarly $\mathbb{E}_{\nu_\rho}\big(M^{*,(i)}(s)-M^{*,(i)}(u)\big)^2\le 2|s-u|\|r\|^2_{-1,n}$.

Given $L_n+L^*_n=2S_n$, we may write

$$M^{(i)}(t_{i+1})-M^{(i)}(t_i)+M^{*,i}(K-t_i)-M^{*,i}(K-t_{i+1}) \ =\ -2\int_{t_i}^{t_{i+1}} S_n u^{(i)}_\lambda(\eta_u)du.$$

Then,

$$
\begin{aligned}
2\int_0^s r(\lfloor cu\rfloor,\eta_u)du &= M(s)+M^*(K)-M^*(K-s)\\
&\quad +2\sum_{i:t_{i+1}\le s}\int_{t_i}^{t_{i+1}}\lambda u_\lambda^{(i)}(\eta_u)du+2\int_{t_{\lfloor sc\rfloor}}^{s}\lambda u_\lambda^{(\lfloor sc\rfloor)}(\eta_u)du
\end{aligned}
$$

where

$$
\begin{aligned}
M(u) &:= \sum_{i:t_{i+1}\le u}\left\{M^{(i)}(t_{i+1})-M^{(i)}(t_i)\right\}+M^{(\lfloor uc\rfloor)}(u)-M^{(\lfloor uc\rfloor)}(t_{\lfloor uc\rfloor})\\
M^*(u) &:= \sum_{j:t_{j+1}>K-u}\left\{M^{*,(j)}(K-t_j)-M^{*,(j)}(K-t_{j+1})\right\}\\
&\quad +M^{*,(\lfloor (K-u)c\rfloor)}(u)-M^{*,(\lfloor (K-u)c\rfloor)}(K-t_{\lfloor (K-u)c\rfloor})
\end{aligned}
$$

are martingales respectively with respect to the forward and time-reversed processes.

By stationarity of ν_ρ and orthogonality of increments, the variances of both $M(K)$ and $M^*(K)$ are less than $2[J/c+(K-c^{-1}\lfloor cK\rfloor)]\|r\|^2_{-1,n}=2K\|r\|^2_{-1,n}$.

Hence, by Doob's inequality and $(a+b+c)^2\le 3(a^2+b^2+c^2)$, we have

$$
\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le K}\big(M(s)+M^*(K)-M^*(K-s)\big)^2\Big] \ \le\ 72K\|r\|^2_{-1,n}.
$$

Also, by Schwartz inequality and stationarity of ν_ρ, when λ is chosen as $\lambda=K^{-1}$, we have

$$
\begin{aligned}
&\mathbb{E}_{\nu_\rho}\Big[\sup_{0\le s\le K}\Big(\sum_{i:t_{i+1}\le s}\int_{t_i}^{t_{i+1}}\lambda u_\lambda^{(i)}(\eta_u)du+\int_{t_{\lfloor sc\rfloor}}^{s}\lambda u_\lambda^{(\lfloor sc\rfloor)}(\eta_u)du\Big)^2\Big]\\
&\quad\le\ \lambda^2K^2\sup_{0\le i\le J}\|u_\lambda^{(i)}\|^2_{L^2(\nu_\rho)}\ \le\ K\|r\|^2_{-1,n}.
\end{aligned}
$$

The desired statement, with $C=152/4=38$, now follows by the inequality $(a+b)^2\le 2a^2+2b^2$. $\square$

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