The Annals of Probability
2013, Vol. 41, No. 3A, 1461–1512
DOI: 10.1214/11-AOP703

LARGE DEVIATIONS FOR THE CURRENT AND TAGGED PARTICLE IN 1D NEAREST-NEIGHBOR SYMMETRIC SIMPLE EXCLUSION

BY SUNDER SETHURAMAN[1] AND S. R. S. VARADHAN[2]

University of Arizona and New York University

Laws of large numbers, starting from certain nonequilibrium measures, have been shown for the integrated current across a bond, and a tagged particle in one-dimensional symmetric nearest-neighbor simple exclusion [*Ann. Inst. Henri Poincaré Probab. Stat.* **42** (2006) 567–577]. In this article, we prove corresponding large deviation principles and evaluate the rate functions, showing different growth behaviors near and far from their zeroes which connect with results in [*J. Stat. Phys.* **136** (2009) 1–15].

1. Introduction and results. The one-dimensional nearest-neighbor symmetric simple exclusion process follows a collection of nearest-neighbor random walks on the lattice $\mathbb{Z}$, each of which is equally likely to move left or right, except in that jumps to already occupied sites are suppressed. More precisely, the model is a Markov process $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$, evolving on the configuration space $\Sigma = \{0,1\}^{\mathbb{Z}}$, with generator

$$(L\phi)(\eta) = (1/2)\sum_x \bigl[\eta(x)\bigl(1-\eta(x+1)\bigr) + \eta(x+1)\bigl(1-\eta(x)\bigr)\bigr]\bigl(\phi(\eta^{x,x+1}) - \phi(\eta)\bigr),$$

where $\eta^{x,y}$, for $x \neq y$, is the configuration obtained from η by exchanging the values at x and y,

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & \text{when } z \neq x, y, \\ \eta(x), & \text{when } z = y, \\ \eta(y), & \text{when } z = x. \end{cases}$$

A detailed treatment can be found in Liggett [23].

As the process is "mass conservative," that is, no birth or death, one expects a family of invariant measures corresponding to particle density. In fact, for each $\rho \in [0,1]$, the product over $\mathbb{Z}$ of Bernoulli measures ν_ρ which independently puts a particle at locations $x \in \mathbb{Z}$ with probability ρ, that is, $\nu_\rho(\eta_x = 1) = 1 - \nu_\rho(\eta_x = 0) = \rho$, are invariant. We will denote E_ρ as expectation under ν_ρ.

Received January 2011; revised July 2011.
[1]Supported in part by NSF-0906713 and NSA Q H982301010180.
[2]Supported in part by NSF Grants DMS-09-04701 and OISE-0730136.
MSC2010 subject classifications. 60K35.
Key words and phrases. Symmetric, exclusion, current, tagged particle, large deviations, one dimensional.

Consider now the integrated current across the bond $(-1, 0)$, and a distinguished, or tagged particle, say initially at the origin. Let $J_{-1,0}(t)$ and X_t be the current and position of the tagged particle at time t, respectively. The problem of characterizing the asymptotic behavior of the current and tagged particle in interacting systems has a long history (cf. Spohn [37], Chapters 8.I, 6.II), and was mentioned in Spitzer's seminal paper [36].

The goal of this paper is to investigate the large deviations of $J_{-1,0}(t)$ and X_t when the initial distribution of particles is part of a large class of nonequilibrium measures. Our initial motivation was to understand better laws of large numbers (LLN) and central limit theorems (CLT) in Jara and Landim [16] for the current and tagged particle when the process starts from a class of "local equilibrium" initial measures. It turns out that recent formal expansions of the large deviation "pressure" for the current in Derrida and Gerschenfeld [10, 11] might also be recovered in such a study.

The article [16] is a nonequilibrium generalization of CLTs in Arratia [1], Rost and Vares [32], and De Masi and Ferrari [8], which established "subdiffusive" behaviors in the 1D nearest-neighbor symmetric simple exclusion model. Namely, starting under an equilibrium ν_ρ, $t^{-1/4}J_{-1,0}(t) \Rightarrow \mathrm{N}(0, \sigma_J^2)$ and $t^{-1/4}X_t \Rightarrow \mathrm{N}(0, \sigma_X^2)$, where $\sigma_J^2 = \sqrt{2/\pi}(1-\rho)\rho$ and $\sigma_X^2 = \sqrt{2/\pi}(1-\rho)/\rho$. Physically, the "subdiffusive" scale in the CLT is explained as being due to "trapping" induced from the nearest-neighbor dynamics which enforces a rigid ordering of particles. Recently, the CLTs were extended to an invariance principle with respect to a fractional Brownian motion, $\lambda^{-1/4}J_{-1,0}(\lambda t) \Rightarrow \sigma_J fBM_{1/4}(t)$ and $\lambda^{-1/4}X_{\lambda t} \Rightarrow \sigma_X fBM_{1/4}(t)$, in Peligrad and Sethuraman [26].

We now specify the class of initial measures considered, that is, "deterministic initial configurations" and "local equilibrium product measures." Let M_1 be the space of functions $\gamma : \mathbb{R} \to [0, 1]$, and let $M_1(\rho_*, \rho^*)$ be those functions in M_1 which equal ρ_* for all $x \le x_*$, and which equal ρ^* for all $x \ge x^*$, for some $x_* \le x^*$.

We will consider on M_1 the topology induced by $C_K(\mathbb{R})$, the set of continuous, compactly supported functions on $\mathbb{R}$, with the duality $\langle \cdot ; \cdot \rangle$ where $\langle \gamma ; G \rangle = \int G(x)\gamma(x)\,dx$ for $\gamma \in M_1$ and $G \in C_K(\mathbb{R})$. This topology, if M_1 is thought of as a measure space, is the vague topology which is metrizable.

Local equilibrium measure (LEM). For $0 < \rho_*, \rho^* < 1$, let $\gamma \in M_1(\rho_*, \rho^*)$ be a piecewise continuous function, such that $0 < \gamma(x) < 1$ for all $x \in \mathbb{R}$. With respect to γ and a scaling parameter $N \ge 1$, we define a sequence of local equilibrium product measures $\nu^{(N)}_{\gamma(\cdot)}$ as those formed from the marginals $\nu^{(N)}_{\gamma(\cdot)}(\eta(x) = 1) = \gamma(x/N)$ for $x \neq 0$, and $\nu^{(N)}_{\gamma(\cdot)}(\eta(0) = 1) = 1$.

Deterministic initial configuration (DIC). For $0 < \rho_*, \rho^* < 1$, let γ be a piecewise continuous function in $M_1(\rho_*, \rho^*)$. Then, a sequence of deterministic initial configurations $\xi^{\gamma,N}$ is one such that $\xi^{\gamma,N}(0) = 1$ and for all continuous, compactly supported G, $\lim_{N\to\infty} \frac{1}{N}\sum_x \xi^{\gamma,N}(x)G(x/N) = \int G(x)\gamma(x)\,dx$.

We remark particular examples of local equilibrium measures $\nu^{(N)}_{\gamma(\cdot)}$ are the equilibrium measures $\nu_\rho(\cdot|\eta(0) = 1)$ conditioned to have a particle at the origin for

$0 < \rho < 1$. Suitable deterministic configurations $\xi^{\gamma,N}$, for instance, include the "alternating" configuration where every other vertex is occupied corresponding to $\gamma(x) \equiv 1/2$. Nonequilibrium initial measures, corresponding to step profiles $\gamma(x) = \rho_* 1_{(-\infty,0]}(x) + \rho^* 1_{(0,\infty)}(x)$, can also be constructed. The condition that the origin is occupied in these configurations allows us to distinguish the corresponding particle as the "tagged" particle.

In a sense, the profiles γ, associated to the local equilibria and deterministic profiles above, are "nondegenerate," in that γ is asymptotically bounded strictly between 0 and 1. Also, the property that $\gamma(x)$ is constant for large $|x|$, and with respect to (LEM) specifications that $0 < \gamma < 1$, is useful to establish later Proposition 1.3, although some modifications, for instance, in terms of profiles sufficiently close to being constant for large $|x|$, should be possible with more work. However, under "degenerate" profiles, different current and tagged particle large deviation behaviors might occur. See comments after Theorems 1.7 and 1.8 for an "example."

We now describe the LLNs, proved in Jara and Landim [16] (stated under a class of local equilibrium measures, but the same proof also works starting from the initial measures above):

$$\lim_{N\to\infty} \frac{1}{N} J_{-1,0}(N^2 t) = v_t \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} X_{N^2 t} = u_t, \tag{1.1}$$

in probability, where v_t and u_t satisfy

$$\frac{dv_t}{dt} = -\frac{1}{2}\partial_x \rho(t,0) \quad \text{and} \quad \frac{du_t}{dt} = -\frac{1}{2}\frac{\partial_x \rho(t,u_t)}{\rho(t,u_t)}$$

and $\partial_t \rho = (1/2)\partial_{xx}\rho$ and $\rho(0,x) = \gamma(x)$, that is, $\rho(t,x) = \sigma_t * \gamma(x)$ where $\sigma_t(y) = (2\pi t)^{-1/2} \exp\{-y^2/2t\}$. Note that

$$v_t = -\frac{1}{2}\int_0^t \partial_x \rho(s,0)\,ds = \int_0^\infty [\rho(t,x) - \rho(0,x)]\,dx,$$

and u_t is also the unique number α, where

$$\int_0^\alpha \rho(t,x)\,dx = -\frac{1}{2}\int_0^t \partial_x \rho(s,0)\,ds = \int_0^\infty [\rho(t,x) - \rho(0,x)]\,dx.$$

To explain the last equation, the right-hand side, as already indicated, is the integrated macroscopic current across the origin up to time t. As the microscopic dynamics is nearest-neighbor with enforced ordering of particles, the tagged particle, initially at the origin, will be at the head of the flow through the origin. So, to compute its macroscopic position u_t at time t, we find α so that the mass at time t between positions $x = 0$ and $x = \alpha$, the left-hand side of the equation, equals the integrated current, and conclude $u_t = \alpha$.

We remark, starting from a class of local equilibrium measures, corresponding invariance principles in subdiffusive $t^{1/4}$ scale, in the sense of finite-dimensional distributions, with respect to fractional Brownian motion-type Gaussian processes, was also proved in [16]. Also, for the current, starting from a large class of product measures, self-normalized CLTs have been shown in Liggett [24] and Vandenberg-Rodes [38].

In this context, we derive large deviation principles (LDPs) (Theorem 1.5), in diffusive scale, corresponding to the laws of large numbers (1.1) when starting from (LEM) or (DIC) measures. We give also lower and upper bounds on the associated rate functions, starting from various nondegenerate initial conditions (Theorem 1.6). A consequence of these rate function bounds, say when starting from deterministic initial configurations, is that the following growth structure can be deduced: Namely, the rate functions are quadratic near their zeroes, but are third order far away from the zeroes.

In particular, the third order asymptotics we derive confirm the formal third-order expansions in Derrida–Gerschenfeld [10] for the probability distribution of the current across the origin at large times; cf. discussion after Theorem 1.6. On the other hand, starting from a "degenerate" deterministic initial configuration with $\gamma(x) = 1_{[-1,1]}(x)$, we show that the large deviations behavior is, at most, quadratic (Theorem 1.8).

Moreover, in Theorem 1.7, starting under deterministic configurations when $\gamma(x) \equiv \rho$, we find the exact asymptotic behavior of the rate functions near their zeroes.

The main idea for the LDPs is to relate, through several "entropy" and "energy" estimates, the current and tagged particle deviations to those established in Kipnis, Olla and Varadhan [18], Landim [20] and Landim and Yau [22], with respect to the hydrodynamic limit of the process empirical density; cf. Propositions 1.1, 1.4. The growth order asymptotics are proved in part by estimations of currents and calculus of variations arguments.

At this point, we remark that the behavior of the tagged particle, in contrast to the subdiffusive $d = 1$ nearest-neighbor result, scales differently in symmetric exclusion models in $d \geq 2$, and also in $d = 1$ when the underlying jump probability is not nearest-neighbor, that is, when particles are free to pass by other particles. Namely, in Kipnis and Varadhan [19], starting under an equilibrium $\nu_\rho\{\cdot|\eta(0) = 1)$, in diffusive scale, invariance principles for the tagged particle to Brownian motion were proved. Later, in Rezakhanlou [31], starting from local equilibrium measures, in diffusive scale, an invariance principle with respect to a diffusion with a drift given in terms of the profile γ is proved for the "averaged" tagged particle position, averaging over all the positions of $O(N)$ particles in a sequence of tori with N vertices. In Quastel, Rezakhanlou and Varadhan [30], in $d \geq 3$, a corresponding large deviations principle is proved for the "averaged" tagged particle position with rate function, which is finite on processes with finite relative entropy, with respect to diffusions which, in some sense, add an additional drift to the limit diffusion in [31]. This LDP for the "averaged" tagged particle would seem also to hold in $d \le 2$ (nonnearest-neighbor in $d = 1$), given regularity results on the self-diffusion coefficient in Landim, Olla and Varadhan [21] not available when [30] was written.

We also mention, other large deviation works with respect to empirical densities and currents in related interacting systems are Benois, Landim and Kipnis [2], Bertini et al. [3, 4], Bertini, Landim and Mourragui [5], Farfan, Landim and Mourragui [12], Quastel [29], and Grigorescu [14]; see also Kipnis and Landim [17], Chapter 10, and references therein. Also, we note, with respect to totally asymmetric nearest-neighbor exclusion in $d = 1$, large deviation "lower tail" bounds for tagged particles are found in Seppäläinen [35].

We now give the hydrodynamic limit and rate function for the process empirical density $\mu^N(s, x; \eta) \in D([0, T]; M_1)$,

$$\mu^N(s, x; \eta) = \sum_{k\in\mathbb{Z}} \eta_{N^2 s}(k) 1_{[k/N,(k+1)/N)}(x)$$

where $x \in \mathbb{R}$, $s \in [0, T]$, and $0 < T < \infty$ is a fixed time.

PROPOSITION 1.1. *Starting from local equilibrium measures or deterministic configurations, we have for $t \in [0, T]$, $\epsilon > 0$ and smooth, compactly supported ϕ, that*

$$\lim_{N\uparrow\infty} P\left\{\left|\int \phi(x)\mu^N(t, x)\,dx - \int \phi(x)m(t, x)\,dx\right| > \epsilon\right\} = 0,$$

where m satisfies $\partial_t m = (1/2)\partial_{xx} m$ with initial data $m(0, x) = \gamma(x)$.

A reference for the proof of Proposition 1.1, among other places, is Theorem 8.1 in Seppäläinen [34].

The rate functions for the process empirical density differ depending on the type of initial distribution. First, following [18, 20], suppose the process starts from a local equilibrium measure $\nu^{(N)}_{\gamma(\cdot)}$. For $\mu \in D([0, T]; M_1)$, define the linear functional on $C_K^{1,2}([0, T] \times \mathbb{R})$:

$$\begin{aligned} l(\mu; G) &= \int G(T, x)\mu_T(x)\,dx - \int G(0, x)\mu_0(x)\,dx \\ &\quad - \int_0^T \int \mu_t(x)\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)G(t, x)\,dx\,dt. \end{aligned}$$

Let

$$I_0(\mu) = \sup_{G\in C_K^{1,2}([0,T]\times\mathbb{R})} \left\{l(\mu; G) - \frac{1}{2}\int_0^T \int \mu_t(1 - \mu_t)(x)G_x^2(t, x)\,dx\,dt\right\},$$

$$\begin{aligned} h(\mu_0; \gamma) = \sup_{\phi_0,\phi_1\in C_K(\mathbb{R})} \bigg\{&\int \mu_0(x)\phi_0(x)\,dx + \int \big(1 - \mu_0(x)\big)\phi_1(x)\,dx \\ &- \int \log\big[\gamma(x)e^{\phi_0(x)} + \big(1 - \gamma(x)\big)e^{\phi_1(x)}\big]\,dx\bigg\}, \end{aligned}$$

and form the rate function

$$I_\gamma^{\mathrm{LE}}(\mu) = I_0(\mu) + h(\mu_0; \gamma).$$

Here, $C_K^{\alpha,\beta}$ is the space of compactly supported functions, α and β-times continuously differentiable in t and x, respectively. In addition, we will use the notation $\mu_t(x) = \mu(t,x)$.

Next, starting from deterministic configurations $\xi^{\gamma,N}$, the rate function in [22] (written for zero-range systems, but the methods straightforwardly apply to our exclusion context) is given by

$$I_\gamma^{\mathrm{DC}}(\mu) = \begin{cases} I_0(\mu), & \text{when } \mu_0 = \gamma, \\ \infty, & \text{otherwise.} \end{cases}$$

To simplify notation, we call both I_γ^{LE} and I_γ^{DC} as I_γ, omitting the super scripts "LE" and "DC," when statements apply to both and the context clear. For $0 \le \alpha, \beta \le 1$, let $h_d(\alpha;\beta) = \alpha \log[\alpha/\beta] + (1-\alpha)\log[(1-\alpha)/(1-\beta)]$ with usual conventions $0 \log 0 = 0/0 = 0$ and $\log 0 = -\infty$.

From the definition, I_γ is a convex function. Also, a main point in [18] was to note that when $I_\gamma(\mu) < \infty$ is finite, that first

$$h(\mu_0;\gamma) = \int h_d(\mu_0(x); \gamma(x))\, dx < \infty.$$

[Of course, starting from deterministic configurations, $\mu_0 = \gamma$.] Also second, μ corresponds to a function $H_x \in L^2([0,T] \times \mathbb{R}, \mu(1-\mu)\, dx\, dt)$ and satisfies a "weakly asymmetric hydrodynamic equation,"

$$\partial_t \mu = \tfrac{1}{2}\partial_{xx}\mu - \partial_x[H_x \mu(1-\mu)] \tag{1.2}$$

in the weak sense. That is, for $G \in C_K^{1,2}([0,T]\times\mathbb{R})$, we have

$$l(\mu; G) = \int_0^T \int G_x H_x \mu(1-\mu)(t,x)\, dx\, dt \tag{1.3}$$

and

$$I_0(\mu) = \frac{1}{2}\int_0^T \int H_x^2 \mu(1-\mu)\, dx\, dt. \tag{1.4}$$

Reciprocally, if for a density $\mu \in D([0,T]; M_1)$, there exists $H_x \in L^2([0,T] \times \mathbb{R}, \mu(1-\mu)\, dx\, dt)$, such that μ satisfies (1.2) weakly, then $I_0(\mu)$ is given by (1.4).

Recall a function $\mathcal{I} : \mathcal{X} \to [0,\infty]$ on a complete, separable metric space $\mathcal{X}$ is a rate function if it has closed level sets $\{x : \mathcal{I}(x) \le a\}$. It is a *good* rate function if the level sets are also compact. Also, a sequence $\{X_n\}$ of random variables with values in $\mathcal{X}$ satisfies a large deviation principle (LDP) with speed n and rate function $\mathcal{I}$ if for every Borel set $U \in \mathcal{B}_{\mathcal{X}}$,

$$-\inf_{x \in \bar{U}} \mathcal{I}(x) \geq \limsup_{n \to \infty} \frac{1}{n} \log \Pr(X_n \in U)$$
$$\geq \liminf_{n \to \infty} \frac{1}{n} \log \Pr(X_n \in U) \geq -\inf_{x \in U^\circ} \mathcal{I}(x),$$

where U° is the interior of U, and $\bar{U}$ is the closure of U.

Let $\mathcal{A} = \mathcal{A}(\gamma)$ be the space of all densities μ, such that $I_\gamma(\mu) < \infty$, which can be approximated in $D([0, T]; M_1)$ by a sequence of densities $\{\mu^n\}$ satisfying (1.2) corresponding to $\{H_x^n\} \subset C_K^{1,2}([0, T] \times \mathbb{R})$, such that $I_\gamma(\mu^n) \to I_\gamma(\mu)$.

For general local equilibrium measures (LEM) and deterministic initial configurations (DIC), only a weak large deviation principle is available. The next proposition follows straightforwardly from the methods of [18] (see also [17], Chapter 10), and replacement estimates in [22], namely Theorem 6.1 and Claims 1, 2 [22], Section 6.

PROPOSITION 1.2. *With respect to initial local equilibrium measures* (*LEM*) *or deterministic configurations* (*DIC*), *corresponding to profile* γ, I_γ *is a good convex rate function*, *and for* $U \subset D([0, T]; M_1)$,

$$-\inf_{\mu \in \bar{U}} I_\gamma(\mu) \geq \limsup_{N \uparrow \infty} \frac{1}{N} \log P[\mu^N \in U]$$
$$\geq \liminf_{N \uparrow \infty} \frac{1}{N} \log P[\mu^N \in U] \geq -\inf_{\mu \in U^\circ \cap \mathcal{A}} I_\gamma(\mu).$$

The last proposition raises the question when $\mathcal{A}(\gamma)$ is large enough so that the lower bound matches the upper bound. However, with respect to the profiles considered, the following containment is true, so that, as a corollary, the full LDP holds.

PROPOSITION 1.3. *With respect to profiles* γ *associated to local equilibrium measures* (*LEM*) *and deterministic configurations* (*DIC*),

$$\mathcal{A}(\gamma) \supset \{\mu : I_\gamma(\mu) < \infty\}.$$

COROLLARY 1.4. *With respect to initial local equilibrium measures* (*LEM*) *and deterministic configurations* (*DIC*), *the LDP with speed* N *holds for* $\{\mu^N\}$ *with good convex rate function* I_γ.

We note Proposition 1.3, for continuous profiles $\gamma \in M_1(\rho, \rho)$ with $0 < \rho < 1$ and $0 < \gamma(\cdot) < 1$ corresponding to local equilibrium measures, was stated in [20], and the associated LDP in Corollary 1.4 with respect to these initial measures is

Theorems 3.2, 3.3 [20]. In Section 5, we prove Proposition 1.3, generalizing the initial states allowed.

It will be convenient to rewrite (1.2) in terms of a macroscopic "current" or "flux" J: That is, when $I_\gamma(\mu) < \infty$, define J so that weakly,

$$\partial_x J + \partial_t \mu = 0; \qquad J = -\tfrac{1}{2}\partial_x \mu + H_x \mu(1-\mu).$$

It turns out such currents have nice properties and relations; cf. Propositions 2.4 and 2.6. Namely, the time integrated current $x \mapsto \int_0^T J(x,t)\,dt$ is a well-defined function on $\mathbb{R}$. Also, the limit

$$\int_0^\infty [\mu_T(x) - \mu_0(x)]\,dx := \lim_{L\to\infty} \int_0^L [\mu_T(x) - \mu_0(x)]\,dx \text{ converges} \tag{1.5}$$

and

$$\int_0^T J(0,t)\,dt = \int_0^\infty \mu_T(x) - \mu_0(x)\,dx. \tag{1.6}$$

In addition, for $\alpha, \beta \in \mathbb{R}$, $\int_0^T [J(\beta,t) - J(\alpha,t)]\,dt = \int_\beta^\alpha [\mu_T(x) - \mu_0(x)]\,dx$.

We now write the current and tagged particle rate function in terms of I_γ. Define the functions $\mathbb{J} = \mathbb{J}_\gamma$ and $\mathbb{I} = \mathbb{I}_\gamma$, for $a \in \mathbb{R}$, by

$$\begin{aligned}\mathbb{J}(a) &= \inf\left\{ I_\gamma(\mu) : \int_0^T J(0,t)\,dt = a \right\}\\ &= \inf\left\{ I_\gamma(\mu) : \int_0^\infty \mu_T(x) - \mu_0(x)\,dx = a \right\}\end{aligned}$$

and

$$\begin{aligned}\mathbb{I}(a) &= \inf\left\{ I_\gamma(\mu) : \int_0^T J(0,t)\,dt = \int_0^a \mu_T(x)\,dx \right\}\\ &= \inf\left\{ I_\gamma(\mu) : \int_0^\infty \mu_T(x) - \mu_0(x)\,dx = \int_0^a \mu_T(x)\,dx \right\}.\end{aligned}$$

When starting from (LEM) or (DIC) initial conditions, we sometimes distinguish the corresponding rate functions by adding a superscript.

It follows from the definitions that

$$\mathbb{I}_\gamma^{LE}(a) \le \mathbb{I}_\gamma^{DC}(a) \quad \text{and} \quad \mathbb{J}_\gamma^{LE}(a) \le \mathbb{J}_\gamma^{DC}(a). \tag{1.7}$$

We also observe that the restriction in the infimum in the definition of $\mathbb{I}$ may take different form. For instance, when $\int_0^T J(0,t)\,dt = \int_0^a \mu_T(x)\,dx$, by the relation $\int_0^T J(0,t) - J(a,t)\,dt = \int_0^a \mu_T(x) - \mu_0(x)\,dx$, one obtains the following restriction which could be used instead: $\int_0^T J(a,t)\,dt = \int_0^a \mu_0(x)\,dx$.

In addition, by translation-invariance, considering $\mu'(t,x) = \mu(t, x+a)$, $J'(x,t) = J(x+a,t)$ and $\gamma'(x) = \gamma(x+a)$, we see, starting from a (DIC) initial state, that

$$\begin{aligned} \mathbb{I}_\gamma^{DC}(a) &= \inf\left\{ I_\gamma^{DC}(\mu) : \int_0^T J(a,t)\,dt = \int_0^a \gamma(x)\,dx \right\} \\ &= \mathbb{J}_{\gamma'}^{DC}\left(\int_0^a \gamma(x)\,dx\right). \end{aligned} \tag{1.8}$$

Although one can readily see $\mathbb{J}$ is convex, given I_γ is convex and the constraint in the definition of $\mathbb{J}$ is linear in μ and a, it is not so easily seen whether $\mathbb{I}$ is convex from this sort of argument. However, as seen later in Theorems 1.6 and 1.7, near their zeroes, both $\mathbb{J}$ and $\mathbb{I}$ behave quadratically.

Also, it is perhaps curious to note that $\mathbb{J}$ and $\mathbb{I}$ can be written completely in terms of densities μ, a consequence of the enforced ordering of particles in the nearest-neighbor $d = 1$ setting. In contrast, the large deviation rate function for the "averaged" tagged particle position in [30] involves an auxiliary current in its description.

We now give some properties of $\mathbb{J}$ and $\mathbb{I}$ and state the large deviation principles.

THEOREM 1.5. *With respect to* (*DIC*) *or* (*LEM*) *initial measures*:

(i) *$\mathbb{J}$ and $\mathbb{I}$ are finite on $\mathbb{R}$, $\lim_{|a|\uparrow\infty} \mathbb{J}(a) = \lim_{|a|\uparrow\infty} \mathbb{I}(a) = \infty$, and $\mathbb{J}$ and $\mathbb{I}$ are a good rate functions. Further, $\mathbb{J}$ and $\mathbb{I}$ have unique zeroes at the LLN constants v_T and u_T, respectively.*

(ii) *The scaled quantities $\{J_{-1,0}(N^2T)/N\}$ and $\{X_{N^2T}/N\}$ satisfy LDPs in scale N with respective rate functions $\mathbb{J}$ and $\mathbb{I}$.*

A natural question at this point is to calculate the rate functions $\mathbb{J}$ and $\mathbb{I}$. Although this appears difficult, some bounds (with nonoptimal constants) are possible under various conditions.

THEOREM 1.6. *Starting under* (*DIC*) *or* (*LEM*) *initial conditions*, *there is a constant $c_1 = c_1(\gamma)$, such that*

$$\limsup_{a\to v_T} \frac{\sqrt{T}}{(a-v_T)^2}\mathbb{J}(a), \qquad \limsup_{|a|\uparrow\infty} \frac{T}{|a|^3}\mathbb{J}(a) \le c_1,$$

$$\limsup_{a\to u_T} \frac{\sqrt{T}}{(a-u_T)^2}\mathbb{I}(a), \qquad \limsup_{|a|\uparrow\infty} \frac{T}{|a|^3}\mathbb{I}(a) \le c_1.$$

Also, *starting under* (*DIC*) *initial conditions*, *there is a constant $c_2 = c_2(\gamma) > 0$, such that*

$$\liminf_{a\to v_T} \frac{\sqrt{T}}{(a-v_T)^2}\mathbb{J}(a), \qquad \liminf_{|a|\uparrow\infty} \frac{T}{|a|^3}\mathbb{J}(a) \ge c_2,$$

$$\liminf_{a \to u_T} \frac{\sqrt{T}}{(a - u_T)^2} \mathbb{I}(a), \qquad \liminf_{|a| \uparrow \infty} \frac{T}{|a|^3} \mathbb{I}(a) \geq c_2.$$

We remark the quadratic asymptotics for $\mathbb{J}(a)$ and $\mathbb{I}(a)$ near their zeroes recalls Gaussian expansions, and the CLTs in [16], [24] and [38]. On the other hand, the cubic bounds for large $|a|$ in Theorem 1.6 seem intriguing, perhaps connected with totally asymmetric nearest-neighbor exclusion (TASEP) effects. That is, for the current or tagged particle to deviate to a far level aN, order $O(|a|N)$ particles must be driven far away from their initial positions, so that perhaps the process behaves like a driven system like TASEP.

We remark on these last points that in Derrida and Gerschenfeld [10, 11], starting from a local equilibrium measure with step profile $\gamma^{\rho_1, \rho_2}(x) = \rho_l 1_{(-\infty, 0]} + \rho_r 1_{(0, \infty)}$, the large deviation "pressure" of the current $J_{0,1}(t)$ across the bond $(0, 1)$, $\lim_{t \uparrow \infty} t^{-1/2} \log E[\exp\{\lambda J_{0,1}(t)\}] = F(\rho_l, \rho_r, \lambda)$, is found. Also, formal asymptotics with F give $P(J_{0,1}(t) = a) \sim \exp[\sqrt{t}\{-\frac{\pi^2}{12} a^3 + \cdots\}]$, for large t and large $a > 0$ (cf. page 980 [11]).

In this context, the large deviation principle in Theorem 1.5 and bounds in Theorem 1.6 prove the form of this expression with respect to the dominant third order term when starting from (DIC) initial conditions: Namely, for large a and constants c_0, c_1,

$$\begin{aligned} -c_0 |a|^3 &\geq -\inf_{|x| \geq a} \mathbb{J}(x) \geq \limsup_{t \to \infty} \frac{1}{\sqrt{t}} P\big(|J_{0,1}(t)| \geq a\big) \\ &\geq \liminf_{t \to \infty} \frac{1}{\sqrt{t}} \log P\big(|J_{0,1}(t)| \geq a\big) \geq -\inf_{|x| > a} \mathbb{J}(x) \geq -c_1 |a|^3. \end{aligned}$$

This addresses, in part, a question in [10], as to whether the large $|a|$ asymptotics would extend to nonstep profiles. See also Hurtado and Garrido [15].

Also, with respect to the current and tagged particle, fluctuations in the "KPZ" class are discussed in Praehofer and Spohn [28], Ferrari and Spohn [13] and Sasamoto [33], with respect to TASEP starting initial conditions with step or constant profiles. In particular, the scaling limits of the current and tagged particle are of "Tracy–Widom" or "Airy" process types whose marginal distribution have upper tail on order $e^{-c_0 |x|^3}$ as $x \uparrow \infty$, and lower tail on order $e^{-c_1 |x|^{3/2}}$ as $x \downarrow \infty$, for some constants c_0, c_1. In our context, starting from (DIC) initial conditions, we have from Theorem 1.6 that $\mathbb{J}(a), \mathbb{I}(a)$ are on cubic order $|a|^3$ for large $|a|$. Formally, one is tempted to link this cubic order in terms of the TASEP scaling limit process exponents. It would be interesting to investigate such analogies.

We now refine the behavior of $\mathbb{J}(a)$ and $\mathbb{I}(a)$ near their zeroes $v_T = u_T = 0$ when the deterministic initial condition has constant profile $\gamma \equiv \rho$. Arratia's CLT variances σ_J^2 and σ_X^2, mentioned earlier, can be computed by adding static and dynamic contributions, due to initial configuration and later motion fluctuations, respectively. However, starting from deterministic initial configurations, only the dynamical contributions would be present, and we show later, in Proposition 4.5, that these parts of the variances are $\sigma^2_{J,dyn} = \sqrt{\pi}\rho(1-\rho)$ and $\sigma^2_{X,dyn} = \sqrt{\pi}(1-\rho)/\rho$.

THEOREM 1.7. *For $\rho \in (0,1)$, starting from (DIC) initial configurations with profile $\gamma \equiv \rho$, we have*

$$\lim_{|a|\downarrow 0} \frac{1}{a^2}\mathbb{J}(a) = \frac{1}{2\sigma^2_{J,dyn}\sqrt{T}} = \frac{\sqrt{\pi}}{2\sqrt{T}}\rho(1-\rho)$$

and

$$\lim_{|a|\downarrow 0} \frac{1}{a^2}\mathbb{I}(a) = \frac{1}{2\sigma^2_{X,dyn}\sqrt{T}} = \frac{\sqrt{\pi}}{2\sqrt{T}}\frac{\rho}{1-\rho}.$$

At this point, one might ask about the large deviation behavior starting from initial conditions with "degenerate" profiles. In this case, diffusive scaling may not always capture for the tagged particle nontrivial LLNs, as in (1.1) or large deviations as in Theorem 1.5. For instance, starting under $\xi^{\gamma,N}$ where $\gamma(x) = 1_{(-\infty,0]}(x)$ is the step profile, in Arratia [1] it is shown that $t^{-1/2}x(t) - \sqrt{\log(t)} \to 0$ a.s. which shows that the tagged particle diverges at rate $\sqrt{t\log(t)}$. With respect to large deviations, it is clear the tagged particle, initially at the origin, cannot travel to negative locations. Also, for $a \geq 0$, the condition in $\mathbb{I}(a)$ reduces to $\int_a^\infty \mu_T(x)\,dx = 0$ which, given that $\mu(t,x)$ satisfies (1.2), is impossible since the density formally becomes positive on $\mathbb{R}$ as soon as $t > 0$. Hence, starting from this step profile configuration, formally $\mathbb{I} = \infty$. However, for the current, starting from this initial condition, in diffusive scaling, $v_T < \infty$, and a corresponding CLT is proved in [24].

On the other hand, when the degenerate initial profile has a density of particles around the tagged particle, diffusive scaling would still seem appropriate to establish an LDP for the tagged particle and current. Here, as a contrast to the results in Theorem 1.6 and to argue this last sentiment, we show quadratic upper bounds for the current and tagged particle large deviations starting from the degenerate configuration $\xi^{\gamma_1,N}$ where $\xi^{\gamma_1,N}(x) = 1$ for $|x| \leq N$ and $\xi^{\gamma_1,N}(x) = 0$ otherwise. Here, $\gamma_1(x) = 1_{[-1,1]}(x)$. Note the associated LLN speeds $v_T = u_T = 0$.

THEOREM 1.8. *Starting under $\xi^{\gamma_1,N}$, there exists $c_1 = c_1(T) > 0$ such that, for $a \geq 0$,*

$$\limsup_{N\uparrow\infty} \frac{1}{N}\log P\big(|J_{-1,0}(N^2T)|/N \geq a\big) \leq \begin{cases} -c_1 a^2, & \text{for } 0 \leq a \leq 1, \\ -\infty, & \text{for } a > 1, \end{cases}$$

$$\limsup_{N\uparrow\infty} \frac{1}{N}\log P\big(|X(N^2T)|/N \geq a\big) \leq -c_1 a^2.$$

The interpretation, for instance, with respect to the tagged particle, is that in configurations $\xi^{\gamma_1,N}$, although it is trapped in the middle of a large segment of particles, to displace large distances, as there are only $O(N)$ number of particles in the system, the cost is not as great as under $\xi^{\rho,N}$, where there are an infinite number of particles. At the same time, there is a positive density of particles to the left and right of the origin, unlike for the profile $\gamma(x) = 1_{(-\infty,0]}(x)$, which slows down the tagged particle so that deviations to $a \in \mathbb{R}$ have finite cost in diffusive scale. With respect to the current, a similar explanation applies; we note, however, current levels larger than N cannot happen, and so they are given infinite cost.

Finally, we remark on some natural questions.

(1) As indicated by Theorem 1.8, different large deviation behaviors might arise when starting from degenerate initial conditions. It would be of interest to investigate these phenomena and provide estimates for the corresponding rate functions. When starting from a degenerate initial profile, with a density of mass around the initial tagged particle position, although the basic argument of Theorem 1.5(ii) in Section 3 holds, main obstacles are to extend approximation Propositions 1.3 and 2.1, energy estimate Proposition 2.4, first bounds and development of the rate functions in Section 2.4 and exponential tightness Lemma 3.2.

(2) Also, a joint large deviations principle for the current and tagged particle, with rate

$$\mathbb{K}(a,b) := \inf\left\{ I_\gamma(\mu) : \int_0^\infty \big(\mu_T(x) - \mu_0(x)\big)\,dx = \int_0^b \mu_T(x)\,dx = a \right\},$$

should hold by the methods of the article. In this case, asymptotics of the rate function $\mathbb{K}(a,b)$ for (a,b) near (v_T, u_T) might be studied.

The plan of the paper is now to develop preliminary estimates in Section 2. In Section 3, we prove Theorem 1.5. Then, in Section 4, we prove Theorems 1.6, 1.7 and 1.8. These last two sections can be read independently of each other. Finally, in Section 5, as remarked earlier, we prove Proposition 1.3, and other approximations.

2. Preliminary estimates.

We develop, in several subsections, "energy" and current estimates with respect to finite rate densities, and also prove that $\mathbb{J}$ and $\mathbb{I}$ are a finite-valued rate functions.

2.1. *Approximation and limit estimates*.

We state an approximation result derived in the course of the proof of Proposition 1.3, and also certain useful limits at infinity. Proofs of these results are given in Section 5.

PROPOSITION 2.1. *Let μ be a density such that $I_0(\mu) < \infty$. Then for all $\epsilon > 0$, there is $\mu^+ \in D([0,T]; M_1)$, such that*:

(i) $\exists 0 < \delta < 1$ *such that* $\delta \le \mu^+(t,x) \le 1 - \delta$ *for* $(t,x) \in [0,T] \times \mathbb{R}$,
(ii) $\mu^+ \in C^\infty([0,T] \times \mathbb{R})$,

(iii) $H_x^+ \in C_K^\infty([0,T]\times\mathbb{R})$ *and*

(iv) $\|\partial_x^{(k)}\partial_t^{(l)}\mu^+\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$ *for* $k, l \ge 1$.

(v) *If* $\mu_0 \equiv \gamma \in M_1(\rho_*, \rho^*)$, *then* $\mu_0^+ = \sigma_\alpha * \gamma$ *for an* $\alpha > 0$. *In particular, if* $\mu_0(x) \equiv \rho$, *then* $\mu_0^+(x) \equiv \rho$.

(vi) *Also, Skorohod distance* $d(\mu^+, \mu) < \epsilon$ *in* $D([0,T]; M_1)$,

(vii) $|I_0(\mu^+) - I_0(\mu)| < \epsilon$.

(viii) *Also, suppose* $\hat{\gamma} \in M_1(\rho_*, \rho^*)$ *is piecewise continuous, and* $0 < \hat{\gamma}(x) < 1$ *for* $x \in \mathbb{R}$. *Then, if* $h(\mu_0; \hat{\gamma}) < \infty$, *we have* $|h(\mu_0^+; \hat{\gamma}) - h(\mu_0; \hat{\gamma})| \le \epsilon$.

We remark, of course, Proposition 2.1 implies that if $I_0(\mu) < \infty$, there is a sequence of densities μ^n satisfying properties (i)–(viii) which converges to μ in $D([0,T]; M_1)$.

LEMMA 2.2. *Let* $\hat{\gamma} \in M_1(\rho_*, \rho^*)$, *and* μ *be a smooth density such that* $h(\mu_0; \hat{\gamma}) < \infty$, $I_0(\mu) < \infty$, *and which also satisfies* (i)–(iv) *in Proposition* 2.1 *Then, we have*

$$\lim_{|y|\uparrow\infty} \sup_{t\in[0,T]} |\mu(t,y) - \hat{\gamma}(y)| = 0.$$

The next lemma will be used in the proof of Theorem 1.7.

LEMMA 2.3. *Let* $\{\mu\}$ *be a smooth density such that* $\mu_0(x) \equiv \rho$, $I_0(\mu) < \infty$, *and which satisfies* (i)–(iv) *in Proposition* 2.1. *Then*

$$\sup_{0\le t\le T} \int (\mu_t - \rho)^2\, dx \le 8 I_0(\mu).$$

2.2. *"Energy" and current estimates.* We give a formula for the rate $I_0(\mu)$, bounds on the "energy" $\|\partial_x \mu\|_{L^2}$, and relations with the current.

PROPOSITION 2.4. *Let* μ *be a smooth density, with finite rate* $I_0(\mu)$, *satisfying* (i)–(iv) *in Proposition* 2.1. *Suppose also there is a smooth* $\hat{\gamma} \in M_1(\rho_*, \rho^*)$, *strictly bounded between* 0 *and* 1, *such that* $h(\mu_0; \hat{\gamma}) < \infty$. *Then,*

$$\begin{aligned} I_0(\mu) = {} & \frac{1}{8}\int_0^T \int \frac{(\partial_x \mu)^2}{\mu(1-\mu)}\, dx\, dt + \frac{1}{2}[h(\mu_T; \hat{\gamma}) - h(\mu_0; \hat{\gamma})] \\ & + \frac{1}{2}\int \frac{\partial_x \hat{\gamma}}{\hat{\gamma}(1-\hat{\gamma})} \int_0^T J\, dt\, dx + \frac{1}{2}\int_0^T \int \frac{J^2}{\mu(1-\mu)}\, dx\, dt, \end{aligned}$$

$$\frac{1}{4}\|\partial_x \mu\|_{L^2}^2 \le h(\mu_0; \hat{\gamma}) + I_0(\mu) + T\|\partial_x \hat{\gamma}/(\hat{\gamma}(1-\hat{\gamma}))\|_{L^2}^2 \tag{2.1}$$

and

$$\int_0^T J(a,t)\, dt - \int_0^T J(b,t)\, dt = \int_a^b \mu_T(x) - \mu_T(0)\, dx \qquad \text{for } a, b \in \mathbb{R}. \tag{2.2}$$

PROOF. First, as $J = -(1/2)\partial_x \mu + H_x \mu(1-\mu)$, we have

$$\begin{aligned}
I_0(\mu) &= \frac{1}{2}\int_0^T \int H_x^2 \mu(1-\mu)\,dx\,dt \\
&= \frac{1}{8}\int_0^T \int \frac{(\partial_x \mu)^2}{\mu(1-\mu)}\,dx\,dt \\
&\quad + \frac{1}{2}\int_0^T \int \frac{J\partial_x \mu}{\mu(1-\mu)}\,dx\,dt + \frac{1}{2}\int_0^T \int \frac{J^2}{\mu(1-\mu)}\,dx\,dt.
\end{aligned}$$

We now find a suitable expression for the middle term. Let G_L be a smooth, nonnegative, compactly supported function in $[-L, L]$, bounded by 1, which equals 1 on $[-L+1, L-1]$, and $\sup_L \int_{A_L} (G_L')^2/G_L\,dx < \infty$ where $A_L = [L-1, L] \cup [-L, -L+1]$. Then

$$\begin{aligned}
&\partial_t \int G_L(x) h_d(\mu_t(x); \hat{\gamma})\,dx \\
&\quad = -\frac{1}{2}\int G_L(x)\frac{(\partial_x \mu_t)^2}{\mu_t(1-\mu_t)}\,dx + \int G_L(x) H_x \partial_x \mu_t\,dx \\
&\qquad + \frac{1}{2}\int G_L(x)\frac{\partial_x \hat{\gamma}\partial_x \mu}{\hat{\gamma}(1-\hat{\gamma})}\,dx - \int G_L(x) H_x \mu(1-\mu)\frac{\partial_x \hat{\gamma}}{\hat{\gamma}(1-\hat{\gamma})}\,dx \\
&\qquad + \int_{A_L} G_L'(x)\bigl[-(1/2)\partial_x \mu_t + H_x\bigl(\mu_t(1-\mu_t)\bigr)\bigr]\log \frac{\mu_t}{1-\mu_t}\frac{1-\hat{\gamma}}{\hat{\gamma}}\,dx.
\end{aligned} \tag{2.3}$$

Hence, by Schwarz's inequality and $0 \le \mu \le 1$, we can bound, with respect to a universal constant C,

$$\begin{aligned}
&\int G_L(x) h_d(\mu_T(x); \hat{\gamma}(x))\,dx + \frac{1}{4}\int_0^T \int G_L(x)\frac{(\partial_x \mu_s)^2}{\mu_s(1-\mu_s)}\,dx\,ds \\
&\quad \le \int G_L(x) h_d(\mu_0(x); \hat{\gamma}(x))\,dx + C\int_0^T \int (H_x)^2 \mu_s(1-\mu_s)\,dx\,ds \\
&\qquad + CT\int G_L(x)\frac{(\partial_x \hat{\gamma})^2}{\hat{\gamma}^2(1-\hat{\gamma})^2}\,dx \\
&\qquad + C\int_{A_L} [(G_L')^2/G_L]\biggl[\log \frac{\mu_t}{1-\mu_t}\frac{1-\hat{\gamma}}{\hat{\gamma}}\biggr]^2\,dx.
\end{aligned}$$

We can take $L \uparrow \infty$, so that the last term vanishes by Lemma 2.2. Then, by monotone convergence, with respect to a universal constant C,

$$\begin{aligned}
&h(\mu_T; \hat{\gamma}) + \frac{1}{4}\int_0^T \int \frac{(\partial_x \mu_s)^2}{\mu_s(1-\mu_s)}\,dx\,ds \\
&\qquad \le h(\mu_0; \hat{\gamma}) + C\int_0^T \int (H_x)^2 \mu_s(1-\mu_s)\,dx\,ds + CT\|\partial_x \hat{\gamma}\|_{L^2},
\end{aligned}$$

and as $0 < \mu, \hat{\gamma} < 1$, we have $\|\partial_x \mu\|_{L^2}, \|J\|_{L^2} < \infty$.

Hence, integrating (2.3) and taking limit on L, the middle term equals

$$\int_0^T \int \frac{J\partial_x \mu}{\mu(1-\mu)}\, dx\, dt = h(\mu_T; \hat{\gamma}) - h(\mu_0; \hat{\gamma}) + \int_0^T \int \frac{J\partial_x \hat{\gamma}}{\hat{\gamma}(1-\hat{\gamma})}\, dx\, dt.$$

The desired bound on $\|\partial_x \mu\|_{L^2}$ now follows. Since $\mu(1-\mu) \le 1/4$ and $\|H_x \mu(1-\mu)\|_{L^2}^2 \le I_0(\mu)$ (cf. (1.4)), by Schwarz's inequality, we may write

$$\begin{aligned}\|\partial_x \mu\|_{L^2}^2 &\le h(\mu_0; \hat{\gamma}) + \frac{1}{2}\|J\|_{L^2}^2 + \frac{T}{2}\|\partial_x \hat{\gamma}/(\hat{\gamma}(1-\hat{\gamma}))\|_{L^2}^2 + 2I_0(\mu)\\ &\le h(\mu_0; \hat{\gamma}) + \frac{1}{4}\|\partial_x \mu\|_{L^2}^2 + \frac{5}{2} I_0(\mu) + T\|\partial_x \hat{\gamma}/(\hat{\gamma}(1-\hat{\gamma}))\|_{L^2}^2.\end{aligned}$$

Finally, (2.2) expresses that the difference of the currents across a and b up to time T is equal to the difference in the masses in the interval $[a, b]$ from times T to 0. This is obtained by integrating $\partial_x J = -\partial_t \mu$. □

COROLLARY 2.5. *Let μ be a density with finite rate $I_\gamma(\mu) < \infty$. Let also $\{\mu^n\}$ be a sequence converging to μ with properties* (i)–(viii) *in Proposition* 2.1. *Then, $\partial_x \mu^n$ and J^n are uniformly bounded in $L^2([0, T] \times \mathbb{R})$ and $\partial_x \mu^n \to \partial_x \mu$, $J^n \to J$ weakly in $L^2([0, T] \times \mathbb{R})$; consequently, $\partial_x \mu, J \in L^2([0, T] \times \mathbb{R})$.*

PROOF. Let $\hat{\gamma}$ be a smooth function in $M(\rho_*, \rho^*)$ such that $0 < \gamma_* < \hat{\gamma} < \gamma^* < 1$ for some constants γ_*, γ^*, and $h(\gamma; \hat{\gamma}) < \infty$. Then, by property (viii) Proposition 2.1, as $h(\mu_0; \hat{\gamma}) < \infty$, we have $h(\mu_0^n; \hat{\gamma}) \to h(\mu_0; \hat{\gamma})$, and, in particular, $\{h(\mu_0^n; \hat{\gamma})\}$ is uniformly bounded.

Also, as $I_0(\mu) < \infty$, by property (vii) Proposition 2.1, we have $I_0(\mu^n) \to I_0(\mu)$ and $\{I_0(\mu^n)\}$ is uniformly bounded. In particular, $\{\|H_x^n \mu^n(1-\mu^n)\|_{L^2}\}$ is uniformly bounded.

Hence, as $\partial_x \hat{\gamma}/(\hat{\gamma}(1-\hat{\gamma})) \in L^2$, and by (2.1) in Proposition 2.4, we have $\{\|\partial_x \mu^n\|_{L^2}\}$ is uniformly bounded. Also, since $J^n = (1/2)\partial_x \mu^n + H_x^n \mu^n(1-\mu^n)$, we also conclude $\{\|J^n\|_{L^2}\}$ is uniformly bounded.

We can then extract subsequences $\partial_x \mu^{n_k}$ and J^{n_k} converging weakly to ζ and ϕ, respectively. Given $\mu^{n_k} \to \mu$ in $D([0, T] \times M_1)$, for smooth, compactly supported G, we have $\int G\partial_x \mu^{n_k}\, dx\, ds = \int -G_x \mu^{n_k}\, dx\, ds$ converges to both $\int G\zeta\, dx\, ds$ and $\int -G_x \mu\, dx\, ds$. Then, $\partial_x \mu$ exists weakly in L^2 and $\partial_x \mu = \zeta$. Hence, the whole sequence $\partial_x \mu^n \to \partial_x \mu$ weakly in L^2.

Similarly, noting Skorohod convergence $\mu^n \to \mu$ implies at the endpoints that μ_0^n, μ_T^n converge to μ_0, μ_T, respectively, and $\partial_t \mu^n + \partial_x J^n = 0$, we have $\phi_x = -\partial_t \mu$ weakly in L^2. Then, $\phi_x = (-1/2)\partial_{xx}\mu + \partial_x[H_x \mu(1-\mu)]$ weakly in L^2, and so $\phi = (-1/2)\partial_x \mu + H_x \mu(1-\mu) + C(t)$ with respect to a function $C(t)$ not dependent on x. But, given $\phi, \partial_x \mu, H_x \mu(1-\mu) \in L^2([0, T] \times \mathbb{R})$, we conclude $C(t) \equiv 0$. In particular, $\phi = J = -(1/2)\partial_x \mu + H_x \mu(1-\mu) \in L^2$, and the sequence $J^n \to J$ weakly in L^2. □

2.3. *Current-mass relation.* We give some properties of the integrated current $\int_0^T J(x,t)\,dt$ and prove the current-mass relation indicated in the Introduction.

PROPOSITION 2.6. *Let μ be a density such that $I_\gamma(\mu) < \infty$. Let $\{\mu^n\}$ be a sequence converging to μ with properties* (i)–(viii) *in Proposition* 2.1. *Then, $x \mapsto \int_0^T J(x,t)\,dt$ is a Lipschitz function, $\lim_{|x|\uparrow\infty} \int_0^T J(x,t)\,dt = 0$, and pointwise for $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \int_0^T J^n(x,t)\,dt = \int_0^T J(x,t)\,dt.$$

In addition, convergence (1.5), *and the "current-mass" relation* (1.6) *hold.*

PROOF. First, from (2.2) in Proposition 2.4, we have

$$\int_0^T J^n(a,t)\,dt - \int_0^T J^n(b,t)\,dt = \int_a^b \mu_T^n(x) - \mu_T^n(0)\,dx.$$

Hence $|\int_0^T J^n(a,t)\,dt - \int_0^T J^n(b,t)\,dt| \le |b-a|$ as $0 \le \mu^n \le 1$. In particular, $\int_0^T J^n(a,t)\,dt$ is Lipschitz in a. Moreover, a subsequence, $\int_0^T J^{n_k}(\cdot,t)\,dt \to \psi(\cdot)$ converges uniformly on compact subsets to a Lipschitz function ψ. Given $J^n \to J$ weakly in $L^2([0,T]\times\mathbb{R})$ by Corollary 2.5, we conclude by a limit argument with respect to $G \in L^2(\mathbb{R})$ that $\int G(a) \int_0^T J(a,t)\,dt\,da = \int G(a)\psi(a)\,da$, and so $\psi(a) = \int_0^T J(a,t)\,dt$. In particular, the whole sequence $\int_0^T J^n(\cdot,t)\,dt \to \int_0^T J(\cdot,t)\,dt$ and the limit $\int_0^T J(\cdot,t)\,dt$ is Lipschitz.

Therefore, since

$$\int \left[\int_0^T J(x,t)\,dt\right]^2 dx \le T \int \int_0^T J^2(x,t)\,dt\,dx < \infty,$$

we obtain the pointwise limit $\int_0^T J(x,t)\,dt \to 0$ as $|x| \uparrow \infty$.

Finally, given Skorohod convergence $\mu^n \to \mu$, μ_0^n and μ_T^n converge respectively to μ_0 and μ_T. Then, by taking limits, we can write

$$\int_0^T J(0,t)\,dt - \int_0^T J(L,t)\,dt = \int_0^L \mu_T(x) - \mu_0(x)\,dx.$$

Now, since $\lim_{L\to\infty} \int_0^T J(L,t)\,dt = 0$, we obtain (1.5) and (1.6). $\square$

2.4. *First estimates on $\mathbb{J}$ and $\mathbb{I}$.* We develop some first bounds on $\mathbb{J}$ and $\mathbb{I}$, and at the end show they are rate functions.

Recall $\sigma_t(x) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}$, and consider a C^∞ smooth function, supported on $[-1,1]$, say

$$\psi_0(x) = \exp\{-1/(1-x^2)\}.$$

Define the smooth, anti-symmetric function

$$\psi(x) = \begin{cases} -\psi_0\big(2(x+1/2)\big), & \text{for } x \le 0, \\ \psi_0\big(2(x-1/2)\big), & \text{for } x \ge 0 \end{cases}$$

and also the anti-derivative $\Psi(x) = \int_{-1}^{x} \psi(y)\,dy$, both supported on $[-1, 1]$.

Let $\gamma \in M_1(\rho_*, \rho^*)$ be a profile associated to an initial (LEM) local equilibrium measure or a (DIC) deterministic configuration. Recall, when $I_\gamma(\mu) < \infty$, it has explicit representation; cf. 1.4. Recall, also that v_T and u_T are the LLN speeds associated to γ; cf. (1.1).

Since $\mathbb{J}$ and $\mathbb{I}$ are given through infima, it is natural to look for explicit densities where computations can be made. Consider the density

$$\mu(s, x) = \sigma_s * \gamma(x) + \big(\lambda\epsilon(s/T)\big)\psi(x/L),$$

where $\epsilon(t)$ is a smooth, increasing function which vanishes for $0 \le t \le 1/10$, and $\epsilon(1) = 1$, and $L \ne 0$. At time $s = T/10$, $0 < \gamma_* < \sigma_s * \gamma < \gamma^* < 1$ for some constants γ_*, γ^*. We will take $0 \le \lambda < \min\{\gamma_*, 1 - \gamma^*\}/2$, small enough so that $\gamma_*/2 \le \mu \le (1 - \gamma^*)/2$ for $T/10 \le t \le T$.

Then, as μ follows the heat equation for $[0, T/10]$, μ satisfies (1.2) with respect to H_x, supported on $[T/10, T] \times [-|L|, |L|]$, given by

$$H_x = \begin{cases} \dfrac{1}{\mu(1-\mu)}\left[\dfrac{\lambda\epsilon(s/T)}{2L}\psi'\left(\dfrac{x}{L}\right) - \dfrac{\lambda L\epsilon'(s/T)}{T}\Psi\left(\dfrac{x}{L}\right)\right], \\ \qquad \text{for } \dfrac{T}{10} \le s \le T, |x| \le |L|, \\ 0, \qquad \text{otherwise.} \end{cases}$$

Also, as $\mu_0 = \gamma$, we have $h(\mu_0; \gamma) = 0$, and

$$\begin{aligned} I_0(\mu) &= \frac{1}{2}\int_{T/10}^{T}\int \frac{1}{\mu(1-\mu)}\left[\frac{\lambda\epsilon(s/T)}{2L}\psi'\left(\frac{x}{L}\right) - \frac{\lambda L\epsilon'(s/T)}{T}\Psi\left(\frac{x}{L}\right)\right]^2 dx\,ds \\ &\le \frac{4\epsilon^*}{\gamma_*(1-\gamma^*)}\left[\frac{\lambda^2 T}{4|L|}\int_{-1}^{1}\psi'(x)^2\,dx + \frac{\lambda^2|L|^3}{T}\int_{-1}^{1}\Psi(x)^2\,dx\right], \end{aligned} \tag{2.4}$$

where $\epsilon^* = 1 + \|\epsilon'\|_{L^\infty}^2$. Compute now

$$\begin{aligned} \int_0^\infty [\mu_T(x) - \mu_0(x)]\,dx &= \lambda L\int_0^1 \psi(x)\,dx + v_T \\ &= \lambda L\int_0^1 \psi(x)\,dx + \int_0^{u_T} \sigma_T * \gamma(x)\,dx, \end{aligned}$$

and, for $\mathfrak{c} \in \mathbb{R}$,

$$\int_0^{\mathfrak{c}} \mu_T(x)\,dx = \int_0^{\mathfrak{c}} \sigma_T * \gamma(x)\,dx + \lambda L\int_0^{|\mathfrak{c}|/|L|} \psi(x)\,dx.$$

Then, the restriction specified in the definition of $\mathbb{J}(\mathfrak{c})$, $\int_0^T J(0,t)\,dt = \mathfrak{c}$, is the same as

$$\lambda L \int_0^1 \psi(x)\,dx = \mathfrak{c} - v_T, \tag{2.5}$$

and the restriction listed in $\mathbb{I}(\mathfrak{c})$,

$$\int_0^T J(0,t)\,dt = \int_0^\infty \mu_T(x) - \mu_0(x)\,dx = \int_0^{\mathfrak{c}} \mu_T(x)\,dx,$$

is equivalent to

$$\lambda L \int_{|\mathfrak{c}|/|L|}^1 \psi(x)\,dx = \int_{u_T}^{\mathfrak{c}} \sigma_T * \gamma(x)\,dx. \tag{2.6}$$

LEMMA 2.7. *For $\mathfrak{c} \in \mathbb{R}$, $\mathbb{J}(\mathfrak{c}), \mathbb{I}(\mathfrak{c}) < \infty$ and in particular $\mathbb{J}(v_T) = \mathbb{I}(u_T) = 0$. Moreover, on any interval $[a,b] \subset \mathbb{R}$, $\sup_{\mathfrak{c}\in[a,b]} \mathbb{J}(c), \sup_{\mathfrak{c}\in[a,b]} \mathbb{I}(c) < \infty$.*

PROOF. For $\mathfrak{c} \in \mathbb{R}$, given bound (2.4), we need only demonstrate that restrictions (2.5) and (2.6), with respect to $\mathbb{J}$ and $\mathbb{I}$, hold with respective choices of λ and L. If $\mathfrak{c} = v_T$ or u_T, we may take $\lambda = 0$, and so clearly $\mathbb{J}(v_T) = \mathbb{I}(u_T) = 0$.

For $\mathfrak{c} \neq v_T$, let $\lambda > 0$, and note the left-hand side of (2.5) can be made equal to the right-hand side $v_T - \mathfrak{c}$ with a proper choice of L. Similarly, when $\mathfrak{c} \neq u_T$, let $\lambda > 0$, and note that the left-hand side of (2.6) vanishes for $|L| \leq |\mathfrak{c}|$ and diverges to $\pm\infty$ as $L \to \pm\infty$. Hence, a proper choice of L allows us to verify (2.6) also.

In particular, we can see, by varying L, with respect to $\mathfrak{c} \in [a,b]$ in any finite interval, we obtain $\sup_{\mathfrak{c}\in[a,b]} \mathbb{J}(\mathfrak{c}), \sup_{\mathfrak{c}\in[a,b]} \mathbb{I}(\mathfrak{c}) < \infty$. $\square$

LEMMA 2.8. *With respect to local equilibrium measures or deterministic initial configurations, $\mathbb{J}$ and $\mathbb{I}$ are lower semi-continuous.*

PROOF. We give the proof for $\mathbb{I}$; the argument for $\mathbb{J}$ is analogous. We first consider when starting from a local equilibrium measure and $I_\gamma = I_\gamma^{LE}$. Let $\{a_n\}$ be a convergent sequence $a_n \to a$. From Proposition 2.7, we have $\sup_n \mathbb{I}(a_n) < \infty$. Then, by Propositions 2.1 and 2.6, we can find densities $\{\mu^n\}$ so that $|I_\gamma^{LE}(\mu^n) - \mathbb{I}(a^n)| < n^{-1}$ and $|\int_0^T J^n(0,t)\,dt - \int_0^{a_n} \mu_T^n(x)\,dx| \leq n^{-1}$.

As I_γ^{LE} is a good rate function and $\{I_\gamma^{LE}(\mu^n)\}$ is uniformly bounded, a subsequence can be found where μ^{n_k} converges to a density $\hat{\mu}$ in $D([0,T];M_1)$ and $\liminf \mathbb{I}(a^n) = \lim \mathbb{I}(a^{n_k}) = \lim I_\gamma^{LE}(\mu^{n_k})$.

By Proposition 2.6, we have $\int_0^T J^{n_k}(0,t)\,dt \to \int_0^T \hat{J}(0,t)\,dt$. Also, as $\mu_T^{n_k} \to \hat{\mu}_T$, and $a_n \to a$, we have $\int_0^{a_n} \mu_T^{n_k}(x)\,dx \to \int_0^a \hat{\mu}_T(x)\,dx$. Then, $\int_0^T \hat{J}(0,t)\,dt = \int_0^a \hat{\mu}_T(x)\,dx$, and hence $\hat{\mu}$ satisfies the infimum restriction in the definition of $\mathbb{I}(a)$.

By lower semi-continuity of I_γ^{LE}, the desired lower semi-continuity of $\mathbb{I}$ follows as $\liminf \mathbb{I}(a_n) = \lim I_\gamma^{LE}(\mu^{n_k}) \geq I_\gamma^{LE}(\hat{\mu}) \geq \mathbb{I}(a)$.

Starting from a deterministic configuration, we can repeat the steps with I_γ^{LE} replaced by I_0. The densities $\{\mu^{n_k}\}$, by Proposition 2.1, also are such that $\mu_0^{n_k}$ converges to γ. Hence, the limit $\hat{\mu}$ satisfies $\hat{\mu}_0 = \gamma$ and so $I_0(\hat{\mu}) = I_\gamma^{DC}(\hat{\mu})$. Therefore, $\mathbb{I}$ is also lower semi-continuous in this case. $\square$

COROLLARY 2.9. *With respect to local equilibrium measures or deterministic initial conditions*, $\mathbb{J}$ *and* $\mathbb{I}$ *are finite-valued rate functions. In addition*, $\mathbb{J}(a') = 0$ *and* $\mathbb{I}(a) = 0$ *exactly when* $a' = v_T$ *and* $a = u_T$.

PROOF. We concentrate on the proof with respect to $\mathbb{I}$, as a similar argument holds for $\mathbb{J}$. First, that $\mathbb{I} : \mathbb{R} \to \mathbb{R}$, $\mathbb{I}(u_T) = 0$, and $\mathbb{I}$ is a rate function follows from Lemmas 2.7 and 2.8. We need only show that u_T is the only zero of $\mathbb{I}$.

When $a \neq u_T$, if $\mathbb{I}(a)$ vanishes, out of a minimizing sequence of densities, through Propositions 2.1 and 2.6, one can find a subsequence converging to a minimizing μ satisfying the restriction $\int_0^T J(0,t)\,dt = \int_0^a \mu_T(x)\,dx$.

With respect to local equilibrium measures, by lower semi-continuity of $h(\cdot;\gamma)$ and $I_0(\cdot)$, we have $h(\mu_0;\gamma) = I_0(\mu) = 0$. Under deterministic initial conditions, since the subsequence at time 0 converges to γ, we have $\mu_0 = \gamma$, and by lower semi-continuity, $I_0(\mu) = 0$.

Then, in either case, $\mu_0 = \gamma$ a.s. and, noting (1.4), $H_x^2 \mu(1-\mu) = 0$ a.s. In particular, $\mu_t = \sigma_t * \gamma$ is the unique bounded solution of the weak heat equation with initial data γ. However, then $\int_0^T J(0,t)\,dt = \int_0^{u_T} \mu_T(x)\,dx$ which does not equal $\int_0^a \mu_T(x)\,dx$ since μ_T is positive and $a \neq u_T$. This is a contradiction. $\square$

3. Proof of Theorem 1.5. The proofs follow in several steps which are divided into subsections. The first step is to describe key relations between a tagged particle and the current across the bond $(-1,0)$, which will allow us later to invoke large deviations of the empirical density. Next, a super-exponential inequality is given. Then, exponential tightness is established, and weak upper and lower large deviation bounds are proved. Finally, Theorem 1.5 is shown.

3.1. *Tagged particle and current relations*. For $x \in \mathbb{Z}$ and $t \geq 0$, define $J_{x,x+1}(t)$ as the integrated current up to time t across the bond $(x, x+1)$, that is, the number of particles which crossed from x to $x+1$ up to time t minus the number of particles which moved from $x+1$ to x in time t. It is well known (cf. Liggett [23], DeMasi and Ferrari [8]) that for integers $r > 0$,

$$\{X_t \geq r\} = \left\{ J_{-1,0}(t) \geq \sum_{x=0}^{r-1} \eta_t(x) \right\}. \tag{3.1}$$

Similarly, for $r < 0$,

$$\{X_t \leq r\} = \left\{ J_{-1,0}(t) \leq -\sum_{x=r}^{-1} \eta_t(x) \right\} \tag{3.2}$$

and

$$\{X_t \le 0\} = \{J_{-1,0}(t) \le 0\}.$$

Also, from a moment's thought, we have

$$J_{x-1,x}(N^2t) - J_{x,x+1}(N^2t) = \eta_{N^2t}(x) - \eta_0(x).$$

We would like to make a summation-by-parts,

$$J_{-1,0}(N^2t) = \sum_{x\ge 0} J_{x-1,x}(N^2t) - J_{x,x+1}(N^2t) = \sum_{x\ge 0} \eta_{N^2t}(x) - \eta_0(x),$$

to write the current across the bond $(-1,0)$ in terms of the empirical process. However, the above display is only formal as the sum on the right may not converge. To treat it carefully, we introduce a "cutoff" function as in Rost and Vares [32]. For $n \ge 1$, let

$$G_n(u) = 1_{[0,n]}(u)(1 - u/n).$$

Also, denote for a function $G \in C_K^\infty(\mathbb{R})$,

$$Y_t^N(G) = \frac{1}{N}\sum_x G(x/N)\eta_{N^2t}(x).$$

Then

$$\begin{aligned} Y_t^N(G_n) - Y_0^N(G_n) &= \frac{1}{N}\sum_x G_n(x/N)\big(J_{x-1,x}(N^2t) - J_{x,x+1}(N^2t)\big) \\ &= \frac{1}{N}\sum_x \big(G_n(x/N) - G_n(x-1/N)\big)J_{x-1,x}(N^2t) \\ &= \frac{1}{N}J_{-1,0}(N^2t) - \frac{1}{N}\sum_{x=1}^{nN}\frac{1}{nN}J_{x-1,x}(N^2t). \end{aligned}$$

This implies

$$\frac{1}{N}J_{-1,0}(N^2t) = Y_t^N(G_n) - Y_0^N(G_n) + \frac{1}{N}\sum_{x=1}^{nN}\frac{1}{nN}J_{x-1,x}(N^2t).$$

Hence, for $a > 0$,

$$\begin{aligned} &\{X_{N^2t}/N \ge a\} \\ &\quad = \left\{\frac{1}{N}J_{-1,0}(N^2t) \ge \frac{1}{N}\sum_{x=0}^{\lfloor aN\rfloor}\eta_{N^2t}(x)\right\} \qquad (3.3) \\ &\quad = \left\{Y_t^N(G_n) - Y_0^N(G_n) + \frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2t) \ge \frac{1}{N}\sum_{x=0}^{\lfloor aN\rfloor}\eta_{N^2t}(x)\right\}. \end{aligned}$$

A similar statement holds for $a \le 0$, namely,

$$\{X_{N^2t}/N \le a\} = \Bigg\{Y_t^N(G_n) - Y_0^N(G_n) + \frac{1}{nN^2}\sum_{x=1}^{nN} J_{x-1,x}(N^2t) \le -\frac{1}{N}\sum_{x=\lfloor aN\rfloor}^{-1} \eta_{N^2t}(x)\Bigg\},$$

where for $a = 0$, we take $\sum_{x=0}^{-1} \eta_{N^2t}(x) = 0$.

Therefore, heuristically, the tagged particle large deviations should be given in terms of the rate for the empirical density I_γ under a certain restriction, as long as the contribution from the term $(1/nN^2)\sum_{x=1}^{nN} J_{x-1,x}(N^2t)$ is superexponentially small as $n, N \uparrow \infty$.

3.2. *Superexponential estimate*. In relation to (3.3), the superexponential estimate needed is implied by the following estimate.

PROPOSITION 3.1. *For each* $\lambda > 0$, *starting from* (*LEM*) *or* (*DIC*) *initial states*,

$$\lim_{n\uparrow\infty}\lim_{N\uparrow\infty}\frac{1}{N}\log E\exp\left|\frac{\lambda N}{nN^2}\sum_{x=1}^{nN} J_{x-1,x}(N^2t)\right| = 0.$$

PROOF. By the inequality $e^{|x|} \le e^x + e^{-x}$, we can remove the absolute value in the last display. Now, note that

$$\exp\Bigg\{\frac{\lambda N}{nN^2}\sum_{x=1}^{nN} J_{x-1,x}(N^2t) - \sum_{x=1}^{nN}(e^{\lambda/nN} - 1)\int_0^{N^2t} \eta_{x-1}(1-\eta_x)(s)\,ds - \sum_{x=1}^{nN}(e^{-\lambda/nN} - 1)\int_0^{N^2t} \eta_x(1-\eta_{x-1})(s)\,ds\Bigg\}$$

is a martingale with mean 1. Then together, the second and third terms in the exponent equal

$$\begin{aligned}
&\sum_{x=1}^{nN}\bigg[(e^{\lambda/nN} - \lambda/nN - 1)\int_0^{N^2t} \eta_{x-1}(1-\eta_x)(s)\,ds \\
&\qquad + (e^{-\lambda/nN} + \lambda/nN - 1)\int_0^{N^2t} \eta_x(1-\eta_{x-1})(s)\,ds\bigg] \\
&\quad + \frac{\lambda}{nN}\int_0^{N^2t}(\eta_0 - \eta_{nN})(s)\,ds \\
&\qquad \le \frac{2e^{\lambda/nN}\lambda^2}{n^2N^2}(nN)(N^2t) + \frac{\lambda}{nN}(N^2t) \le \frac{C(t,\lambda)N}{n},
\end{aligned}$$

which gives the result with standard manipulations. □

3.3. *Exponential tightness estimate*. We now show that the scaled tagged particle positions are exponential tight.

LEMMA 3.2. *Starting from* (*LEM*) *or* (*DIC*) *initial states*, *we have*

$$\lim_{a\uparrow\infty}\lim_{N\uparrow\infty}\frac{1}{N}\log P\{|J_{-1,0}(N^2T)|/N\geq a\}$$
$$=\lim_{a\uparrow\infty}\lim_{N\uparrow\infty}\frac{1}{N}\log P\{|X_{N^2T}|/N\geq a\}=-\infty.$$

PROOF. We give the argument for the tagged particle, as the proof for the current is similar, and somewhat easier. From (3.3), we need only super-exponentially estimate, for a positive (as a similar argument works for $a<0$) and n fixed,

$$P\left\{Y_T^N(G_n)-Y_0^N(G_n)+\frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2T)\geq Y_T^N(1_{[0,a]})\right\}.$$

We need only estimate

$$E\left[\exp\left\{N\left[Y_T^N(G_n)-Y_0^N(G_n)+\frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2T)-Y_T^N\big(1_{[0,a]}\big)\right]\right\}\right]$$
$$=E[e^{Q_1}e^{Q_2}e^{Q_3}e^{Q_4}]$$

with $Q_1=NY_T^N(G_n)$, $Q_2=-NY_0^N(G_n)$, $Q_3=(nN)^{-1}\sum_{x=1}^{nN}J_{x-1,x}(N^2t)$ and $Q_4=-\sum_{x=0}^{\lfloor aN\rfloor}\eta_{N^2T}(x)$. By Chebyshev, we can estimate the exponential terms separately. For fixed n, $\lim N^{-1}\log E[e^{4Q_3}]$ is bounded from Proposition 3.1, and as $Q_1\leq nN$ by properties of G_n, $\lim N^{-1}\log E[e^{4Q_1}]$ is also bounded. In addition, as $\exp\{4Q_2\}\leq 1$, this term can be neglected.

Finally, by Borcea, Branden and Liggett [6], Theorem 5.2, as the initial measure of type (LEM) or (DIC) is a product measure [of degenerate Bernoulli's under (DIC) initial configurations], the coordinates $\{\eta_{N^2T}(x)\}$ are negatively associated. Hence, $E[e^{4Q_4}]\leq\prod_{x=1}^{\lfloor aN\rfloor}E[e^{-4\eta_{N^2T}(x)}]$, and using $\log(1-x)\leq -x$ for $0\leq x\leq 1$, we write

$$\begin{aligned}\frac{1}{N}\log E[e^{4Q_4}]&\leq\frac{1}{N}\sum_{x=1}^{\lfloor aN\rfloor}\log E[e^{-4\eta_{N^2T}(x)}]\\&\leq\frac{1}{N}\sum_{x=1}^{\lfloor aN\rfloor}\log\big[(e^{-4}-1)P\big(\eta_{N^2T}(x)=1\big)+1\big]\\&\leq\frac{e^{-4}-1}{N}E\left[\sum_{x=1}^{\lfloor aN\rfloor}\eta_{N^2T}(x)\right]\to(e^{-4}-1)\int_0^a m(T,x)\,dx,\end{aligned}$$

where $m(T,x) = \sigma_T * \gamma(x)$ is the solution of the hydrodynamic equation (Proposition 1.1). Since $\sigma_{T/10} * \gamma(x) \geq \gamma_* > 0$ as $\gamma \in M_1(\rho_*, \rho^*)$ for $\rho_*, \rho^* > 0$, the right-hand side is bounded above by $(e^{-4} - 1)\gamma_* a \downarrow -\infty$ as $a \uparrow \infty$. □

3.4. *Weak LDP upper bounds*. The weak upper bound for the tagged particle deviations, starting from local equilibrium measures or deterministic initial configuration, follows in several steps and is stated in Step 6. As the same argument works for the current, we also state its associated weak upper bound in Step 6, below. For the convenience of the reader, we indicate the modifications needed in Step 1; the other steps involve similar changes.

Step 1. Consider an interval $[a,b]$ for $0 < a < b$; subsequent arguments carry over straightforwardly to all intervals $[a,b] \subset \mathbb{R}$ using (3.2) by splitting at the origin if necessary. Now, divide $[a,b]$ into m equal intervals $A_k = [c_k, c_{k+1}]$. Then, by the union of events estimate,

$$\limsup_{N\to\infty} \frac{1}{N} \log P(X_{N^2T}/N \in [a,b]) \leq \max_k \limsup_{N\to\infty} \frac{1}{N} \log P(X_{N^2T}/N \in A_k).$$

Then, from (3.3) and Proposition 3.1, we have that

$$\begin{aligned}
&\limsup_{N\uparrow\infty} \frac{1}{N} \log P(X_{N^2T}/N \in [a,b]) \\
&\quad\leq \limsup_{m\uparrow\infty} \limsup_{\delta\downarrow 0} \limsup_{n\uparrow\infty} \max_{1\leq k\leq m} \limsup_{N\uparrow\infty} \frac{1}{N} \\
&\qquad\times \log P\bigl(Y_T^N(G_n) - Y_0^N(G_n) \in \bigl[Y_T^N(1_{[0,c_k]}) - \delta, Y_T^N(1_{[0,c_{k+1}]}) + \delta\bigr]\bigr).
\end{aligned}$$

Since the maps $\mu \mapsto \int G(x)\mu_T\,dx, \int G(x)\mu_0\,dx, \int_0^c \mu_T\,dx$, for compactly supported G and constants c, are continuous in the Skorohod topology on $D([0,T]; M_1)$, from Corollary 1.4, we conclude, for fixed k, n and δ that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log p\bigl(Y_T^N(G_n) - Y_0^N(G_n)\,ds \in \bigl[Y_T^N(1_{[0,c_k]}) - \delta, Y_T^N(1_{[0,c_{k+1}]}) + \delta\bigr]\bigr) \\
&\quad\leq -\inf\biggl\{I_\gamma(\mu); \int G_n(x)[\mu_T(x) - \mu_0(x)]\,dx \\
&\qquad\qquad \in \biggl[\int_0^{c_k} \mu_T(x)\,dx - \delta, \int_0^{c_{k+1}} \mu_T(x)\,dx + \delta\biggr]\biggr\}.
\end{aligned} \tag{3.4}$$

We now indicate the modifications needed for the current in this step. For $0 < a < b$, from (3.3) and Proposition 3.1, we have

$$\begin{aligned}
&\limsup_{N\uparrow\infty} \frac{1}{N} \log P\bigl(J_{-1,0}(N^2T)/N \in [a,b]\bigr) \\
&\quad\leq \limsup_{m\uparrow\infty} \limsup_{\delta\downarrow 0} \limsup_{n\uparrow\infty} \max_{1\leq k\leq m} \limsup_{N\uparrow\infty} \frac{1}{N} \log P\bigl(Y_T^N(G_n) - Y_0^N(G_n) \\
&\qquad\qquad \in [c_k - \delta, c_{k+1} + \delta]\bigr).
\end{aligned}$$

From continuity of the maps $\mu \mapsto \int G\mu_T\,dx$ and $\mu \mapsto \int G\mu_0\,dx$, and Corollary 1.4, we further bound

$$\limsup_{N\to\infty} \frac{1}{N} \log P\big(Y_T^N(G_n) - Y_0^N(G_n)\,ds \in [c_k - \delta, c_{k+1} + \delta]\big)$$
$$\leq -\inf\left\{ I_\gamma(\mu); \int G_n(x)[\mu_T(x) - \mu_0(x)]\,dx \in [c_k - \delta, c_{k+1} + \delta]\right\}.$$

Step 2. Next, we give a uniform upper bound of the infimum in (3.4). We exhibit a density $\mu^{\mathfrak{c}}$ satisfying, for each $\delta > 0$ and all large n,

$$\int G_n(x)[\mu_T^{\mathfrak{c}}(x) - \mu_0^{\mathfrak{c}}(x)]\,dx \in \left[\int_0^{\mathfrak{c}} \mu_T^{\mathfrak{c}}(x)\,dx - \delta, \int_0^{\mathfrak{c}} \mu_T^{\mathfrak{c}}(x)\,dx + \delta\right]$$

and $\sup_{\mathfrak{c}\in[a,b]} I_\gamma(\mu^{\mathfrak{c}}) < B_0 < \infty$ where B_0 is independent of n and δ.

This is accomplished by the constructions in Section 2.4, namely one takes $\mu^{\mathfrak{c}} = \sigma_t * \gamma + \lambda\epsilon(t/T)\psi(x/L)$ with λ, L chosen so that $\lambda L \int_{|\mathfrak{c}/L|}^1 \psi(x)\,dx = \int_{u_T}^{\mathfrak{c}} \sigma_T * \gamma(x)\,dx$. Let $J^{\mathfrak{c}}$ be its current, and $H_x^{\mathfrak{c}}$ be the associated function with respect to (1.2).

Proposition 2.7 gives $I_\gamma(\mu^{\mathfrak{c}})$ is uniformly bounded for $\mathfrak{c} \in [a,b]$. Now compute

$$\begin{aligned}
&\int G_n(x)[\mu_T^{\mathfrak{c}}(x) - \mu_0^{\mathfrak{c}}(x)]\,dx \\
&\quad = \int_0^T \int G_n(x)[(1/2)\partial_{xx}\mu^{\mathfrak{c}} - \partial_x H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})]\,dx\,dt \\
&\quad = \int_0^T -(1/2)\partial_x\mu^{\mathfrak{c}}(t,0) + H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})(t,0)\,dt \\
&\qquad + \frac{1}{n}\int_0^T \int_0^n [(1/2)\partial_x\mu^c - H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})]\,dx\,dt.
\end{aligned} \tag{3.5}$$

Since $\int_0^{\mathfrak{c}} \mu_T^{\mathfrak{c}}(x)\,dx = \int_0^T J^{\mathfrak{c}}(0,t)\,dt$ and $J^{\mathfrak{c}}(0,t) = -(1/2)\partial_x\mu^{\mathfrak{c}}(t,0) + H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})(t,0)$, we have

$$\begin{aligned}
&\sup_{\mathfrak{c}\in[a,b]} \left| \int G_n(x)[\mu_T^{\mathfrak{c}}(x) - \mu_0^{\mathfrak{c}}(x)]\,dx - \int_0^{\mathfrak{c}} \mu_T^{\mathfrak{c}}(x)\,dx \right| \\
&\quad \leq \frac{1}{n}\left| \int_0^T \int_0^n (1/2)\partial_x\mu^{\mathfrak{c}} - H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\,dx\,dt \right| \\
&\quad \leq \sup_{\mathfrak{c}\in[a,b]} \left| \frac{1}{2n}\int_0^T \big(\mu_t^{\mathfrak{c}}(n) - \mu_t^{\mathfrak{c}}(0)\big)\,dt \right| + \frac{1}{\sqrt{n}} \|H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\|_{L^2([0,T]\times\mathbb{R})}.
\end{aligned}$$

Since $\|H_x^{\mathfrak{c}}\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\|_{L^2}^2 \leq 2I_0(\mu^{\mathfrak{c}})$, the right-hand side is $O(n^{-1/2})$ by Lemma 2.7.

Step 3. As I_γ is a good rate function, by the uniform bounds in step 2, out of minimizers $\nu^{k,n,\delta}$ over $k = k(m)$, and n, δ in the infimum in (3.4), by the uniform bound on $I_\gamma(\nu^{k,n,\delta})$, we can extract a subsequence, on which the limsup of (3.4) is attained as $\delta \downarrow 0$ and $n, m \uparrow \infty$, and which converges in $D([0,T]; M_1)$ to a $\bar{\mu}$.

By Proposition 2.1, the subsequence, labeled $\nu^{k,n,\delta}$ itself for simplicity, may be approximated by $\{\mu^{k,n,\delta}\}$ so that $\mu^{k,n,\delta}$ is smooth, strictly bounded between 0 and 1, $H_x^{k,n,\delta} \in C^\infty([0,T] \times \mathbb{R})$, Skorohod distance $d(\mu^{k,n,\delta}, \nu^{k,n,\delta}) \downarrow 0$, $|I_0(\nu^{k,n,\delta}) - I_0(\mu^{k,n,\delta})| \downarrow 0$ and when $\hat{\gamma} \in M_1(\rho_*, \rho^*)$ is piecewise continuous and $0 < \hat{\gamma}(x) < 1$ for $x \in \mathbb{R}$, $|h(\nu_0^{k,n,\delta}; \hat{\gamma}) - h(\mu_0^{k,n,\delta}; \hat{\gamma})| \downarrow 0$. Also, as $[a,b]$ is compact, the subsequence can be chosen so that c_{k+1} converges to a $\mathfrak{c} \in [a,b]$.

Given $\nu^{k,n,\delta}$ satisfies the restriction in (3.4), we may also arrange

$$\begin{aligned} \int_0^{c_k} \mu_T^{k,n,\delta}(x)\,dx - 2\delta &\le \int G_n(x)[\mu_T^{k,n,\delta}(x) - \mu_0^{k,n,\delta}(x)]\,dx \\ &\le \int_0^{c_{k+1}} \mu_T^{k,n,\delta}(x)\,dx + 2\delta. \end{aligned} \tag{3.6}$$

With these specifications, by lower semi-continuity, we have that (3.4) is less than, in the case of starting from a local equilibrium measure,

$$\lim_{m\uparrow\infty} \lim_{\delta\downarrow 0} \lim_{n\uparrow\infty} \max_k -I_\gamma^{LE}(\mu^{k,n,\delta}) \le -I_\gamma^{LE}(\bar{\mu}).$$

When starting from a deterministic configuration, noting $\nu_0^{k,n,\delta} = \bar{\mu}_0 = \gamma$, (3.4) is less than

$$\lim_{m\uparrow\infty} \lim_{\delta\downarrow 0} \lim_{n\uparrow\infty} \max_k -I_0(\mu^{k,n,\delta}) \le -I_0(\bar{\mu}) = -I_\gamma^{DC}(\bar{\mu}).$$

Step 4. We now show that $\bar{\mu}$ satisfies

$$\int_0^T \bar{J}(0,t)\,dt = \int_0^{\mathfrak{c}} \bar{\mu}_T(x)\,dx. \tag{3.7}$$

As convergence in $D([0,T]; M_1)$ implies $\mu_T^{k,n,\delta} \to \bar{\mu}_T$, $c_{k+1} - c_k = m^{-1}$ and $0 \le \mu_T^{k,n,\delta}(x) \le 1$, we have both

$$\int_0^{c_k} \mu_T^{k,n,\delta}(x)\,dx, \int_0^{c_{k+1}} \mu_T^{k,n,\delta}(x)\,dx \to \int_0^{\mathfrak{c}} \bar{\mu}_T(x)\,dx.$$

Also, following sequence (3.5),

$$\begin{aligned} &\int G_n(x)[\mu_T^{k,n,\delta}(x) - \mu_0^{k,n,\delta}(x)]\,dx \\ &\quad = \int_0^T J^{k,n,\delta}(0,t)\,dt \\ &\qquad + \frac{1}{n}\int_0^T \int_0^n [(1/2)\partial_x \mu_t^{k,n,\delta} - H_x^{k,n,\delta}\mu^{k,n,\delta}(1-\mu^{k,n,\delta})(t,x)]\,dx\,dt. \end{aligned}$$

Since $\|H_x^{k,n,\delta}\mu^{k,n,\delta}(1-\mu^{k,n,\delta})\|_{L^2}^2 \le 2I_0(\mu^{k,n,\delta})$ is uniformly bounded, the last term is bounded uniformly by $n^{-1}T + (nT)^{-1/2}\sqrt{2I_0(\mu^{k,n,\delta})}$. On the other hand, $\int_0^T J^{k,n,\delta}(0,t)\,dt \to \int_0^T \bar{J}(0,t)\,dt$ by Proposition 2.6.

Hence, noting (3.6), we obtain (3.7) immediately.

Step 5. Therefore,

$$
\begin{aligned}
&\limsup_{m\uparrow\infty}\limsup_{\delta\downarrow 0}\limsup_{n\uparrow\infty}\max_{1\le k\le m}\\
&\quad -\inf\biggl\{I_\gamma(\mu);\int G_n(x)[\mu_T(x)-\mu_0(x)]\,dx\\
&\qquad\qquad \in\biggl[\int_0^{c_k}\mu_T(x)\,dx-\delta,\int_0^{c_{k+1}}\mu_T(x)\,dx+\delta\biggr]\biggr\}\\
&\quad\le -I_\gamma(\bar\mu)\le -\min_{\mathfrak{c}\in[a,b]}\mathbb{I}(\mathfrak{c}).
\end{aligned}
$$

Step 6. The weak LDP upper bound, with respect to the tagged particle, for compact $K\subset\mathbb{R}$,

$$
\limsup_{N\uparrow\infty}\frac{1}{N}P(X_{N^2t}/N\in K)\le -\inf_{a\in K}\mathbb{I}(a), \tag{3.8}
$$

is now standard, given that $\mathbb{I}$ is lower semi-continuous (Lemma 2.8).

Similarly, we have the weak upper bound for the current

$$
\limsup_{N\uparrow\infty}\frac{1}{N}P\bigl(J_{-1,0}(N^2t)/N\in K\bigr)\le -\inf_{a\in K}\mathbb{J}(a). \tag{3.9}
$$

3.5. *LDP lower bound*. As before, we concentrate on the tagged particle deviations, as the proof for the current is analogous. For the first step, the scheme for the weak upper bound is used. Let $O\subset\mathbb{R}$ be a nonempty open set, and suppose $a\in O$. We also assume $a>0$ as a similar argument works for $a\le 0$ by focusing on a subinterval to the left of the origin. Let $\epsilon>0$ be such that $a-\epsilon>0$ and $(a-\epsilon,a+\epsilon)\subset O$.

Then, for $\theta>0$,

$$
\begin{aligned}
&\lim_{N\uparrow\infty}\frac{1}{N}\log P(X_{N^2T}/N\in O)\\
&\quad\ge\lim_{N\uparrow\infty}\frac{1}{N}P\bigl(X_{N^2T}/N\in(a-\epsilon,a+\epsilon)\bigr)\\
&\quad\ge\lim_{n\uparrow\infty}\lim_{N\to\infty}\frac{1}{N}\log P\biggl(Y_T^N\bigl(1_{[0,a-\epsilon]}\bigr)<Y_T^N(G_n)-Y_0^N(G_n)\\
&\qquad\qquad+\frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2T)<Y_T^N\bigl(1_{[0,a+\epsilon]}\bigr)\\
&\qquad\qquad\text{and } Y_T^N\bigl(1_{[a-\epsilon,a+\epsilon]}\bigr)>\theta\biggr).
\end{aligned}
\tag{3.10}
$$

From Proposition 3.1 and Corollary 1.4, the left-hand side of (3.10) is greater than

$$\lim_{\theta\downarrow 0}\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty}\lim_{N\to\infty}\frac{1}{N}\log P\bigl(Y_T^N\bigl(1_{[0,a-\epsilon]}\bigr)+\delta<Y_T^N(G_n)-Y_0^N(G_n) \\ <Y_T^N\bigl(1_{[0,a+\epsilon]}\bigr)-\delta,\ \text{and } Y_T^N\bigl(1_{[a-\epsilon,a+\epsilon]}\bigr)>\theta\bigr)$$

$$\geq \lim_{\theta\downarrow 0}\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty} \\ -\inf\biggl\{I_\gamma(\mu):\int_0^{a-\epsilon}\mu_T(x)\,dx+\delta<\int G_n(x)[\mu_T(x)-\mu_0(x)]\,dx \\ <\int_0^{a+\epsilon}\mu_T(x)\,dx-\delta,\ \text{and } \int_{a-\epsilon}^{a+\epsilon}\mu_T(x)\,dx>\theta\biggr\}. \tag{3.11}$$

Now, for $\alpha>0$, let $\bar\mu$ be a density such that $|I_\gamma(\bar\mu)-\mathbb{I}(a)|<\alpha$, and

$$\int_0^T \bar J(0,t)\,dt=\int_0^a\bar\mu_T(x)\,dx.$$

By the method used for (3.5) and (3.8) in the last section, through approximations of $\bar\mu$ with smooth μ^n by Proposition 2.1, we can show that

$$\lim_n\int_0^\infty G_n(x)[\bar\mu_T(x)-\bar\mu_0(x)]\,dx=\int_0^T\bar J(0,t)\,dt. \tag{3.12}$$

We will need now to approximate $\bar\mu$ as follows to ensure a certain positivity. Let $\chi=\sigma_s*\gamma+\lambda\epsilon(t/T)\psi(x/L)$ from Section 2.4 where λ, L are chosen so that $\int_0^T J^\chi(0,t)\,dt=\int_0^a\chi_T(x)\,dx$. Recall $I_\gamma(\chi)<\infty$, and note (3.12), with χ and J^χ replacing $\bar\mu$ and J, also holds by the explicit construction. For $0<b<1$, define $\mu^b=(1-b)\chi+b\bar\mu$. Clearly, $\lim_{b\uparrow 1}\mu^b=\bar\mu$ uniformly, and so in $D([0,T];M_1)$. In fact, $\lim_{b\uparrow 1}I_\gamma(\mu^b)=I_\gamma(\bar\mu)$: By lower semi-continuity, $\liminf I_\gamma(\mu^b)\geq I_\gamma(\bar\mu)$ and, by convexity, $\limsup I_\gamma(\mu^b)\leq I_\gamma(\bar\mu)$. Now, for given $\beta>0$, let b be such that $|I_\gamma(\mu^b)-I_\gamma(\bar\mu)|<\beta$.

With $\theta>0$, noting

$$\lim_n\int_0^\infty G_n(x)[\mu_T^b(x)-\mu_0^b(x)]\,dx=\int_0^a\mu_T^b(x)\,dx,$$

we have for $n\geq N(\theta,\bar\mu,\chi)$ that

$$\int_0^{a-\epsilon}\mu_T^b(x)\,dx+b\int_{a-\epsilon}^a\bar\mu_T(x)\,dx+(1-b)\int_{a-\epsilon}^a\chi_T(x)\,dx-\theta \\ \leq\int G_n(x)[\mu_T^b(x)-\mu_0^b(x)]\,dx \\ \leq\int_0^{a+\epsilon}\mu_T^b(x)\,dx-b\int_a^{a+\epsilon}\bar\mu_T(x)\,dx-(1-b)\int_a^{a+\epsilon}\chi_T(x)\,dx+\theta.$$

By the construction of χ, $\int_{a-\epsilon}^{a} \chi_T(x)\,dx$, $\int_a^{a+\epsilon} \chi_T(x)\,dx \geq c\epsilon$ for a constant $c > 0$. Hence, we can choose $\theta = \theta(\epsilon, b, \chi)$ so that for all small δ,

$$(1-b)\int_a^{a+\epsilon} \chi_T(x)\,dx - \theta, (1-b)\int_{a-\epsilon}^{a} \chi_T(x)\,dx - \theta > \delta.$$

Therefore, as $\bar{\mu}$ is nonnegative, μ^b satisfies the restriction in the infimum in (3.11). In particular, we have

$$\lim_{N\uparrow\infty} \frac{1}{N} P(X_{N^2t}/N \in O) \geq -I_\gamma(\mu^b) \geq -I_\gamma(\bar{\mu}) - \beta \geq -\mathbb{I}(a) - \alpha - \beta.$$

Hence,

$$\lim_{N\uparrow\infty} \frac{1}{N} P(X_{N^2t}/N \in O) \geq -\inf_{a\in O} \mathbb{I}(a). \tag{3.13}$$

Analogously, we have weak lower bound large deviations for the current

$$\lim_{N\uparrow\infty} \frac{1}{N} P\big(J_{-1,0}(N^2t)/N \in O\big) \geq -\inf_{a\in O} \mathbb{J}(a). \tag{3.14}$$

PROOF OF THEOREM 1.5. First, the functions $\mathbb{J}$ and $\mathbb{I}$ are finite-valued rate functions which vanish exactly at v_T and u_T, respectively, by Corollary 2.9.

Next, a "weak" LDP is found from (3.9) and (3.14) with respect to rate function $\mathbb{J}$, and (3.8) and (3.13) with respect to $\mathbb{I}$. Standard arguments, given exponential tightness (Lemma 3.2), extend the "weak" LDP to the full large deviation principle.

Finally, given the LDP and exponential tightness, it follows that (1) $\mathbb{J}$ and $\mathbb{I}$ are good rate functions by Lemma 1.2.18 [9], and also that (2) $\lim_{|a|\uparrow\infty} \mathbb{J}(a) = \lim_{|a|\uparrow\infty} \mathbb{I}(a) = \infty$. $\square$

4. Asymptotic evaluations. We prove Theorems 1.6, 1.7 and 1.8 in succeeding subsections.

4.1. *Proof of Theorem* 1.6. We first prove the upper bounds which are implied by the following lemma, and then the lower bounds.

LEMMA 4.1. *Starting from* (*DIC*) *or* (*LEM*) *initial conditions*, *there are constants* c_0, c_1, c_2, c_3 *depending only on* γ, *such that when*, *respectively*, $|a - v_T|/\sqrt{T} \geq c_0$ *and* $|a - u_T|/\sqrt{T} \geq c_0$, *we have*, *in turn*,

$$\mathbb{J}(a) \leq \frac{c_1|a - v_T|^3}{T} \quad \textit{and} \quad \mathbb{I}(a) \leq \frac{c_1|a - u_T|^3}{T}.$$

Also, *when*, *respectively*, $|a - v_T|/\sqrt{T} \leq c_2$ *and* $|a - u_T|/\sqrt{T} \leq c_2$, *we have*, *correspondingly*,

$$\mathbb{J}(a) \leq \frac{c_3(a - v_T)^2}{\sqrt{T}} \quad \textit{and} \quad \mathbb{I}(a) \leq \frac{c_3(a - u_T)^2}{\sqrt{T}}.$$

PROOF. We prove the estimates for the current rate function, and deduce corresponding bounds for the tagged particle rate function. Let also $a > v_T$ as the argument for $a < v_T$ is analogous. For the reader's convenience, we recall estimate (2.4) and write

$$\mathbb{J}(a) \le \frac{4\epsilon^*}{\gamma_*(1-\gamma^*)}\left[\frac{\lambda^2 T}{4|L|}\int_{-1}^{1}\psi'(x)^2\,dx + \frac{\lambda^2|L|^3}{T}\int_{-1}^{1}\Psi(x)^2\,dx\right].$$

Recall also the restriction equation (2.6) when $\mathfrak{c} = a$,

$$\lambda = \frac{a - v_T}{L\int_0^1 \psi\,dx},$$

subject to $0 < \lambda \le \min\{\gamma_*, 1-\gamma^*\}/2$. The requirement on λ holds when

$$L \ge |a - v_T| \Big/ \left[\frac{1}{2}\min\{\gamma_*, (1-\gamma^*)\}\int_0^1 \psi\,dx\right] := \kappa_0|a - v_T|.$$

Now take L in the form $L = \kappa\sqrt{T}$. Substituting into the bound for $\mathbb{J}(a)$, we obtain

$$\mathbb{J}(a) \le \frac{(a-v_T)^2}{\sqrt{T}}\frac{4\epsilon^*\kappa}{\gamma_*(1-\gamma^*)}\left[\frac{1}{4\kappa^4}\int_{-1}^{1}\psi'(x)^2\,dx + \int_{-1}^{1}\Psi(x)^2\,dx\right].$$

Hence, when a is large, say $\kappa = |a - v_T|\kappa_0/\sqrt{T} \ge 1$, we have $\mathbb{J}(a) \le c(\gamma)|a - v_T|^3/T$. Correspondingly, when a is such that $|a - v_T|\kappa_0/\sqrt{T} \le 1$, we choose $\kappa = 1$ to get $\mathbb{J}(a) \le c(\gamma)|a - v_T|^2/\sqrt{T}$.

The bounds on the tagged particle rate function $\mathbb{I}$ follow from the current rate function bounds. First, by (1.7), $\mathbb{I}_\gamma^{LE}(a) \le \mathbb{I}_\gamma^{DC}(a)$. Also, by (1.8), with $\gamma'(x) = \gamma(x+a)$, $\mathbb{I}_\gamma^{DC}(a) = \mathbb{J}_{\gamma'}^{DC}(\int_0^a \gamma\,dx)$. For fixed a, let now $v_T(\gamma')$ be the LLN integrated current through the origin starting from γ'. Then

$$\begin{aligned}\int_0^a \gamma\,dx - v_T(\gamma') &= \int_0^a \gamma\,dx - \int_0^\infty \sigma_T * \gamma' - \gamma'\,dx\\ &= \int_0^a \sigma_T * \gamma\,dx - \int_0^\infty \sigma_T * \gamma - \gamma\,dx = \int_{u_T}^a \sigma_T * \gamma\,dx.\end{aligned}$$

Hence, $\gamma_*|a - u_T| \le |\int_0^a \gamma\,dx - v_T(\gamma')| \le \gamma^*|a - u_T|$. Since γ_*, γ^* are uniform lower and upper bounds on $\sigma_{T/10} * \gamma$ (and hence on $\sigma_{T/10} * \gamma'$), the desired estimates on $\mathbb{I}(a)$ are derived from the bounds on $\mathbb{J}_{\gamma'}(\int_0^a \gamma\,dx)$. $\square$

The lower bounds in Theorem 1.6 are implied by the following two estimates.

LEMMA 4.2. *Starting from a (DIC) condition with profile γ, there are constants $c_0 = c_0(\gamma, T)$, $c_1 = c_1(\gamma)$ such that for $|a| \ge c_0$, we have*

$$\mathbb{J}(a), \mathbb{I}(a) \ge \frac{c_1|a|^3}{T}.$$

LEMMA 4.3. *Starting from a* (*DIC*) *condition with profile* γ, *there is a constant* $c_1(\gamma)$, *such that for* $a \in \mathbb{R}$, *we have*

$$\mathbb{J}(a) \geq \frac{c_1(a - v_T)^2}{\sqrt{T}} \quad \textit{and} \quad \mathbb{I}(a) \geq \frac{c_1(a - u_T)^2}{\sqrt{T}}.$$

PROOF OF LEMMA 4.2. We concentrate first on the current calculation. Suppose $a > 0$, as the argument for $a < 0$ is similar. Let $\hat{\gamma} \in M_1(\rho_*, \rho^*)$ be a smooth density, strictly bounded away from 0 and 1, such that $h(\gamma; \hat{\gamma}) < \infty$. For $\epsilon > 0$, by Proposition 2.1, let μ be a smooth density such that $\mu_0 = \sigma_\alpha * \gamma$, $|h(\mu_0; \hat{\gamma}) - h(\gamma; \hat{\gamma})| < \epsilon$ and $|I_0(\mu) - \mathbb{J}(a)| \leq \epsilon$. Noting Proposition 2.6, we can, in addition, impose on the approximating density that $|\int_0^T J(0,t)\,dt - a| \leq \epsilon$.

Now, noting (2.2) in Proposition 2.4, we have the Lipschitz bound,

$$\left|\int_0^T J(x,t)\,dt - \int_0^T J(0,t)\,dt\right| = \left|\int_0^x \mu_T(z) - \mu_0(z)\,dz\right| \leq |x|.$$

Then, for $0 \leq x \leq a - \epsilon$, we have $\int_0^T J(x,t)\,dt \geq \int_0^T J(0,t)\,dt - x \geq a - \epsilon - x$ so that

$$\begin{aligned}(a-\epsilon)^3/3 &= \int_0^{a-\epsilon} [a - \epsilon - x]^2\,dx \\ &\leq \int_0^{a-\epsilon}\left[\int_0^T J(x,t)\,dt\right]^2 dx \leq T \int\int_0^T J^2\,dt\,dx.\end{aligned}$$

Hence, as $\mu(1-\mu), \hat{\gamma}(1-\hat{\gamma}) \leq 1/4$, from the formula for $I_0(\mu)$ in Proposition 2.4 and simple computations,

$$\begin{aligned}\mathbb{J}(a) \geq I_0(\mu) - \epsilon &\geq \int\int_0^T J^2\,dt\,dx - \frac{1}{2}h(\mu_0; \hat{\gamma}) - T\int \frac{(\partial_x \hat{\gamma})^2}{\hat{\gamma}^2(1-\hat{\gamma})^2}\,dx - \epsilon \\ &\geq \frac{(a-\epsilon)^3}{3T} - \frac{1}{2}h(\gamma; \hat{\gamma}) - T\int \frac{(\partial_x \hat{\gamma})^2}{\hat{\gamma}^2(1-\hat{\gamma})^2}\,dx - \frac{3}{2}\epsilon.\end{aligned} \tag{4.1}$$

For the tagged particle rate function, from (1.8), we have $\mathbb{I}_\gamma^{DC}(a) = \mathbb{J}_{\gamma'}^{DC}(\int_0^a \gamma\,dx)$ where $\gamma'(x) = \gamma(x + a)$. Since $\gamma(x) \geq \min\{\rho_*, \rho^*\}$ for all large $|x|$, $|\int_0^a \gamma\,dx| \geq c(\gamma)|a|$ for all large $|a|$ where $c(\gamma) > 0$. Also, as $\gamma', \hat{\gamma} \in M_1(\rho_*, \rho^*)$, by calculation $h(\gamma'; \hat{\gamma}) = O(|a|)$. Hence, plugging into (4.1), we obtain the desired estimate on $\mathbb{I}(a)$. $\square$

PROOF OF LEMMA 4.3. We focus first on the current rate function computation. By Proposition 2.1 and 2.6, let μ be a smooth density with properties (i)–(viii) such that $\mu_0 = \sigma_\alpha * \gamma$, $|\mathbb{J}(a) - I_0(\mu)| < \epsilon$ and $|\int_0^T J(t,0)\,dt - a| < \epsilon$. Let $v_T(\alpha)$ be the LLN speed starting from profile $\sigma_\alpha * \gamma$, and note $\lim_{\alpha\downarrow 0} |v_T - v_T(\alpha)| = 0$.

Consider solutions of $\partial_t \rho = (1/2)\rho_{xx}$ and $\partial_t \mu = (1/2)\partial_{xx}\mu - \partial_x(H_x \mu(1-\mu))$, both with initial value $\sigma_\alpha * \gamma$. The difference $U = \rho - \mu$ satisfies equation

$\partial_t U = (1/2)\partial_{xx}U - \partial_x(H_x\mu(1-\mu))$ with $U(0,x)\equiv 0$. Integrating once in the space variable, noting properties of μ, $S(t,x)=\int_{-\infty}^x U(t,y)\,dy$ satisfies $\partial_t S = (1/2)\partial_{xx}S - H_x\mu(1-\mu)$. Hence, we have

$$\begin{aligned} S(t,x) &= \sigma_t * S(0,x) \\ &\quad + \int_0^t \int \frac{1}{\sqrt{2\pi(t-s)}} e^{-(x-y)^2/(2(t-s))}[-H_x\mu(1-\mu)](s,y)\,dy\,ds \\ &= -\int_0^t \int \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/(2t)} H_x\mu(1-\mu)(t-s,y)\,dy\,ds. \end{aligned}$$

Now, the difference in integrated macroscopic currents across x up to time t with respect to ρ and μ is $-S(t,x)$; cf. above (1.5). Therefore, by the Schwarz inequality and $0\le\mu\le 1$, when $x=0$, we have for small α that

$$\begin{aligned} &\big(v_T - a + O(\epsilon)\big)^2 \\ &\quad \le \left[\int_0^T \left(\int \sigma_t^2(y)\,dy\right)^{1/2}\left(\int H_x^2\mu(1-\mu)(y,t)\,dy\right)^{1/2} dt\right]^2. \end{aligned}$$

As $\|\sigma_t^2\|_{L^2(\mathbb{R})}^2 \le Ct^{1/2}$, a further bound of the right-hand side is $2C\sqrt{T}I_0(\mu)\le 2C\sqrt{T}(\mathbb{J}(a)+O(\epsilon))$ for some universal constant C.

We now use relations (1.8) to analyze the tagged particle rate function. Indeed, let $u_T(\alpha)$ be the corresponding tagged particle LLN speed starting from profile $\rho_0 = \sigma_\alpha * \gamma$, and note $\lim_{\alpha\downarrow 0} u_T(\alpha) = u_T$. As before, by Proposition 2.1, let μ be a smooth density such that $\mu_0 = \rho_0$, $|\mathbb{I}(a) - I_0(\mu)| < \epsilon$ and by Proposition 2.6, $|\int_0^T J(a,t)\,dt - \int_0^a \rho_0(x)\,dx| < \epsilon$.

Note, with respect to density ρ, the current across a equals $\int_a^\infty \sigma_T * \rho_0 - \rho_0\,dx$, and the current across the origin equals $\int_0^\infty \sigma_T*\rho_0 - \rho_0\,dx = \int_0^{u_T(\alpha)} \sigma_T * \rho_0\,dx$. Then, for small α, the square of the difference in integrated currents with respect to ρ and μ across a equals $(\int_{u_T}^a \sigma_T*\gamma(x)\,dx + O(\epsilon))^2 \ge \gamma_*^2(a-u_T+O(\epsilon))^2$ where $\sigma_T*\gamma\ge\gamma_*>0$. But, on the other hand, as before, $\|\sigma_t(\cdot - a)\|_{L^2(\mathbb{R})}^2 \le Ct^{1/2}$, and the square current difference is still bounded by $2C\sqrt{T}I_0(\mu)\le 2C\sqrt{T}(\mathbb{I}(a) + O(\epsilon))$. This finishes the proof. $\square$

4.2. *Proof of Theorem* 1.7. Starting from a (DIC) state, since $\gamma(x)\equiv\rho$, noting (1.8), we observe that $\gamma'(x)\equiv\rho$, $\int_0^a \gamma\,dx = a\rho$, and $\mathbb{I}(a) = \mathbb{J}(a\rho)$. Hence, we need only give the argument for the current rate function $\mathbb{J}$, as the estimate for the tagged particle rate function $\mathbb{I}$ follows directly.

We now make some useful reductions. Recall, when starting under a deterministic configuration with profile $\gamma(x)\equiv\rho$, in order for $I_\gamma^{\mathrm{DC}}(\nu)<\infty$, ν must satisfy $\nu_0(x)\equiv\rho$ and $I_\gamma^{\mathrm{DC}}(\nu) = I_0(\nu) < \infty$. By Proposition 2.1 and Proposition 2.6, for each $\epsilon>0$, we can find a smooth density μ, such that $\mu_0(x)\equiv\rho$ and

$$\mathbb{J}(a) \ge I_0(\mu) - \epsilon\frac{a^2}{\sqrt{T}}.$$

In addition, we may also impose that

$$\left|\int_0^T J(0,t)\,dt - a\right| < a\epsilon.$$

For such a density μ, by Proposition 2.4 [applied with $\gamma(x) \equiv \rho$],

$$\begin{aligned} I_0(\mu) = {} & \frac{1}{8}\int\int_0^T \frac{(\partial_x \mu)^2}{\mu(1-\mu)}\,dt\,dx + \frac{1}{2}h(\mu_T;\rho) \\ & + \frac{1}{2}\int\int_0^T \frac{J^2}{\mu(1-\mu)}\,dt\,dx. \end{aligned} \tag{4.2}$$

Consider now a sequence $\{\mu^a\}$ of such $\epsilon a^2/\sqrt{T}$-minimizers of $\mathbb{J}(a)$ as $|a| \downarrow 0$. The upper and lower bounds in Lemmas 4.1 and 4.3, as $v_T = 0$, gives $I_0(\mu^a) = O(a^2/\sqrt{T})$. Then, by Lemma 2.3, we have $\mu^a \to \rho$ in $L^2([0,T]\times\mathbb{R})$, and in fact

$$\sup_{0\le t\le T}\int \bigl(\mu^a(t,x) - \rho\bigr)^2\,dx = O(a^2).$$

We now deduce that there are functions $r(t,x)$ and $j(t,x)$ on $[0,T]\times\mathbb{R}$ such that $r(0,x)\equiv 0$, $\partial_t r + \partial_x j = 0$ weakly in $L^2([0,T]\times\mathbb{R})$, $\int_0^T j(t,0)\,dt = 1$, and

$$\begin{aligned} & \rho(1-\rho) \times \liminf_{a\downarrow 0} \mathbb{J}(a)/a^2 \\ & \quad \ge \frac{1}{8}\int_0^T\int (\partial_x r)^2\,dx\,dt + \frac{1}{4}\int |r(T,x)|^2\,dx + \frac{1}{2}\int_0^T\int j^2(t,x)\,dx\,dt. \end{aligned} \tag{4.3}$$

Consider a function $\lambda^a(t,x) = \psi(\mu^a(t,x))$ where $\psi'(x) = \min\{(x(1-x))^{-1/2}, M\}$ for some $M \ge 2(\rho(1-\rho))^{-1/2}$. Then, $\partial_x\lambda^a = \psi'(\mu^a)\partial_x\mu^a \le (\mu^a(1-\mu^a))^{-1/2}\partial_x\mu^a$, and so

$$\int_0^T\int (\partial_x\lambda^a)^2\,dx\,dt \le \int_0^T\int \frac{(\partial_x\mu^a)^2}{\mu^a(1-\mu^a)}\,dx\,dt.$$

At this point, let us take weak $L^2([0,T]\times\mathbb{R})$ limits of $a^{-1}\partial_x\lambda^a$, $a^{-1}(\mu^a - \rho)$ and $a^{-1}J^a$, and label them as u, r and j, respectively. Also, take a weak $L^2(\mathbb{R})$ limit of $a^{-1}(\mu^a(T,x) - \rho)$ and call it q. Using suitable truncations, and Fatou's Lemma, given $\mu^a \to \rho$ strongly, we have

$$\int_0^T\int u^2\,dx\,dt \le \liminf \frac{1}{a^2}\int_0^T\int (\partial_x\lambda^a)^2\,dx\,dt,$$

$$\frac{1}{2\rho(1-\rho)}\int |q(x)|^2\,dx \le \liminf\frac{1}{a^2}\int h_d(\mu_T^a(x);\rho)\,dx,$$

$$\frac{1}{\rho(1-\rho)}\int_0^T\int j^2\,dx\,dt \le \liminf\frac{1}{a^2}\int_0^T\int \frac{(J^a)^2}{\mu^a(1-\mu^a)}\,dx\,dt.$$

We may identify (a) $\partial_x r = \sqrt{\rho(1-\rho)}u$, (b) $r(T,x) = q(x)$, and (c) $\partial_t r + \partial_x j = 0$ weakly in $L^2([0,T]\times\mathbb{R})$. The last two (b), (c) follow from weak limits and properties of μ^a. However, (a) also holds given the weak limits since $\partial_x \mu^a = \psi'(\mu^a)^{-1}\partial_x\lambda^a$ and $\psi'(\mu^a)^{-1} \to \sqrt{\rho(1-\rho)}$ strongly in L^2.

Now, define

$$K(t,x) = \int_0^t j(s,x)\,ds.$$

Then, the right-hand side of (4.3) becomes

$$\begin{aligned}\mathcal{K} = {} & \frac{1}{4}\int |\partial_x K(T,x)|^2\,dx + \frac{1}{2}\int_0^T \int |\partial_t K(t,x)|^2\,dx\,dt \\ & + \frac{1}{8}\int_0^T \int |\partial_{xx} K(t,x)|^2\,dx\,dt.\end{aligned}$$

By scaling, $M(t,x) = K(tT, x\sqrt{T})$, we obtain

$$\liminf_{|a|\downarrow 0} \frac{\sqrt{T}}{a^2}\mathbb{J}(a) \ge [\rho(1-\rho)]^{-1}\inf\mathcal{M},$$

where the infimum is over $M \in C^{1,2}([0,1]\times\mathbb{R})$, such that $M(0,x)\equiv 0$ and $M(1,0)=1$, and

$$\begin{aligned}\mathcal{M} = {} & \frac{1}{4}\int |M_x(1,x)|^2\,dx + \frac{1}{2}\int_0^1 \int |M_t(t,x)|^2\,dx\,dt \\ & + \frac{1}{8}\int_0^1 \int |M_{xx}(t,x)|^2\,dx\,dt.\end{aligned}$$

On the other hand, the upper bound

$$\limsup_{|a|\downarrow 0} \frac{\sqrt{T}}{a^2}\mathbb{J}(a) \le [\rho(1-\rho)]^{-1}\inf\mathcal{M} \tag{4.4}$$

also follows by a similar strategy: In Proposition 4.4 below, we evaluate $\inf\mathcal{M}$ and find a minimizer. One can find a smooth ϵ, approximating M with bounded derivatives, and trace back to obtain the corresponding density μ^a satisfying $a^{-1}(\mu^a - \rho) = \partial_x K$, $a^{-1}J^a = \partial_t K$, $a^{-1}\partial_x\mu^a = \partial_{xx}K$ with $\int_0^T J^a(0,t)\,dt = a$ and $\mu_0^a(x)\equiv\rho$. Given $\|\partial_x M\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$, we have $\|\mu^a - \rho\|_{L^\infty([0,T]\times\mathbb{R})} \le |a|\|\partial_x K\|_{L^\infty} = (|a|/\sqrt{T})\|\partial_x M\|_{L^\infty} = O(|a|)$. The argument to derive (4.4) now follows from standard approximations with respect to (4.2).

Hence, the proof of Theorem 1.7 will follow from evaluations $\inf_M \mathcal{M} = \sqrt{\pi}/2$, $\sigma^2_{X,dyn} = (1-\rho)/(\rho\sqrt{\pi})$ and $\sigma^2_{J,dyn} = \rho(1-\rho)/\sqrt{\pi}$ in Propositions 4.4 and 4.5 below.

PROPOSITION 4.4. *We have*

$$\inf_M \mathcal{M} = \frac{\sqrt{\pi}}{2},$$

where the infimum is over $M \in C^{1,2}([0,1] \times \mathbb{R})$ *such that* $M(0,x) \equiv 0$ *and* $M(1,0) = 1$.

PROOF. The argument is in three steps. (A) We first minimize

$$\int_0^1 \int_{-\infty}^{\infty} \frac{1}{2}|M_t(t,x)|^2 + \frac{1}{8}|M_{xx}(t,x)|^2\,dx\,dt \tag{4.5}$$

when $M(0,x) \equiv 0$ and $M(1,x)$ is a given compactly supported $C^4(\mathbb{R})$ function. The Euler equation is

$$M_{tt} = \frac{1}{4} M_{xxxx} \tag{4.6}$$

with the boundary conditions at $t = 0, 1$.

One can verify the solution of (4.6), which is smooth and classical, in terms of Fourier transform with respect to the x variable but not transforming the t variable, is given by

$$\hat{M}(t,y) = \hat{M}(1,y) \frac{e^{ty^2/2} - e^{-ty^2/2}}{e^{y^2/2} - e^{-y^2/2}}, \tag{4.7}$$

where

$$\hat{M}(1,y) = \frac{1}{\sqrt{2\pi}} \int e^{iyx} M(1,x)\,dx.$$

The corresponding value of (4.5), through Plancherel's formula, is expressed as

$$\int_{-\infty}^{\infty} |\hat{M}(1,y)|^2 k(y)\,dy,$$

where

$$\begin{aligned} k(y) &= \int_0^1 \frac{y^4}{8} \frac{[e^{ty^2/2} + e^{-ty^2/2}]^2 + [e^{ty^2/2} - e^{-ty^2/2}]^2}{[e^{y^2/2} - e^{-y^2/2}]^2}\,dt \\ &= \int_0^1 \frac{y^4}{4} \frac{e^{ty^2} + e^{-ty^2}}{[e^{y^2/2} - e^{-y^2/2}]^2}\,dt \\ &= \frac{y^2}{4} \frac{e^{y^2} - e^{-y^2}}{[e^{y^2/2} - e^{-y^2/2}]^2} = \frac{y^2}{4} \frac{e^{y^2/2} + e^{-y^2/2}}{e^{y^2/2} - e^{-y^2/2}}. \end{aligned}$$

Given that the integrand in (4.5) is a strict convex function of M_t and M_{xx}, solution (4.7) is the unique minimizer of (4.5) (by say straightforward modifications of the proof of [7], Theorem 2.1).

(B) Now, we consider the term

$$\frac{1}{4}\int_{-\infty}^{\infty}|M_x(1,x)|^2\,dx = \frac{1}{4}\int_{-\infty}^{\infty} y^2|\hat{M}(1,y)|^2\,dy$$

and minimize, over $M \in L^2([0,1]\times\mathbb{R})$,

$$\begin{aligned}&\frac{1}{4}\int_{-\infty}^{\infty}|\hat{M}(1,y)|^2 y^2\left[1+\frac{e^{y^2/2}+e^{-y^2/2}}{e^{y^2/2}-e^{-y^2/2}}\right]dy\\ &\quad=\frac{1}{2}\int_{-\infty}^{\infty}|\hat{M}(1,y)|^2 y^2\frac{e^{y^2/2}}{e^{y^2/2}-e^{-y^2/2}}\,dy\end{aligned}$$

subject to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{M}(1,y)\,dy = 1.$$

Recall that the minimizer of

$$\int |g(y)|^2 K(y)\,dy$$

when $\int g(y)\,dy = a$ is given by $g(y) = cK(y)^{-1}$ and $c = a[\int K(y)^{-1}\,dy]^{-1}$, with minimum value $a^2[\int K(y)^{-1}\,dy]^{-1}$. Hence, with $a=\sqrt{2\pi}$ and

$$K(y) = \frac{y^2}{2}\frac{e^{y^2/2}}{e^{y^2/2}-e^{-y^2/2}},$$

we identify $M(1,y)$ through its transform $\hat{M}(1,y) = cK(y)^{-1}$. Denote $\widetilde{M}$ as the function in (4.7) with this choice of $M(1,y)$.

(C) Let now M^* be a compactly supported $C^{2,4}([0,1]\times\mathbb{R})$ function such that $M^*(0,x)\equiv 0$ and $M^*(1,0)=1$ whose $\mathcal{M}$-value approximates $\inf_M \mathcal{M}$. From steps (A) and (B), we obtain a lower bound of the infimum value which is actually achieved by the smooth $C^{2,4}([0,1]\times\mathbb{R})$ function $\widetilde{M}$. Therefore, $\widetilde{M}$ is a minimizer.

Finally, given

$$\begin{aligned}\int K(y)^{-1}\,dy &= 2\int\frac{1-e^{-y^2}}{y^2}\,dy\\ &= 2\int_0^1\int e^{-ty^2}\,dy\,dt = 2\int_0^1\sqrt{\frac{\pi}{t}}\,dt = 4\sqrt{\pi},\end{aligned}$$

we obtain the infimum, $\inf_M\mathcal{M} = 2\pi/(4\sqrt{\pi}) = \sqrt{\pi}/2$, as desired. □

PROPOSITION 4.5. *Starting under initial distribution ν_ρ, the dynamical parts of the limiting variances of $T^{-1/4}J_{-1,0}(T)$ and $T^{-1/4}x(T)$ under ν_ρ are*

$$\sigma^2_{J,dyn} := \lim_{T\to\infty}\frac{1}{\sqrt{T}}E_{\nu_\rho}\big[\big(J_{-1,0}(T)-E_\eta[J_{-1,0}(T)]\big)^2\big] = \frac{\rho(1-\rho)}{\sqrt{\pi}}$$

and

$$\sigma^2_{X,dyn} := \lim_{T\to\infty} \frac{1}{\sqrt{T}} E_{\nu_\rho}\big[\big(x(T) - E_\eta[x(T)]\big)^2\big] = \frac{1-\rho}{\rho}\frac{1}{\sqrt{\pi}}.$$

PROOF. First, we note the limit distribution and variance of both $T^{-1/4}x(T)$ and $\rho^{-1}T^{-1/4}J_{-1,0}(T)$ are the same, namely $N(0,\sigma^2)$ with $\sigma^2 = \sqrt{2/\pi}(1-\rho)/\rho$; cf. [1]. Moreover, $\lim_{T\uparrow\infty} E_{\nu_\rho}[(T^{-1/4}X(T) - \rho^{-1}T^{-1/4}J_{-1,0}(T))^2] = 0$, since $(T^{-1/4}X(T) - \rho^{-1}T^{-1/4}J_{-1,0}(T))^2$ vanishes in probability, and also is uniformly integrable; cf. [8], equation (28), or [1], page 368, and [26], Proposition 4.2 and proof of Lemma 3.2.

Then we need only show

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} E_{\nu_\rho}[(E_\eta[J_{-1,0}(T)])^2] = \rho(1-\rho)\frac{\sqrt{2}-1}{\sqrt{\pi}},$$

which, given the form of the limiting variance of the scaled current, and

$$E_{\nu_\rho}[(J_{-1,0}(T))^2] = E_{\nu_\rho}\big[\big(J_{-1,0}(T) - E_\eta[J_{-1,0}(T)]\big)^2\big] + E_{\nu_\rho}[(E_\eta[J_{-1,0}(T)])^2]$$

implies the desired results.

Now, the current $J_{-1,0}$ has martingale decomposition (cf. Section 2 [26]),

$$J_{-1,0}(t) = M(t) + \frac{1}{2}\int_0^t \eta_s(-1) - \eta_s(0)\,ds.$$

Also, for $x \in \mathbb{Z}$, from "duality" (cf. Liggett [23], Section VIII.1, page 363),

$$E_\eta[\eta_t(x)] = \sum_i p(t, i-x)\eta(i),$$

where $p(t,j) = P(S_t = j)$ is the probability a continuous time random walk, starting from the origin, travels to j in time t. Then,

$$\begin{aligned} E_\eta[J_{-1,0}(T)] &= \frac{1}{2}\int_0^T E_\eta[\eta_t(-1)] - E_\eta[\eta_t(0)]\,dt \\ &= \frac{1}{2}\sum_i \eta(i)\int_0^T p(t,i+1) - p(t,i)\,dt \\ &= \frac{1}{2}\sum_i \big(\eta(i) - \rho\big)\int_0^T p(t,i+1) - p(t,i)\,dt. \end{aligned}$$

Therefore, from independence of coordinates $\{\eta(i)\}$,

$$Q_0(T) := E_{\nu_\rho}[(E_\eta[J_{-1,0}(T)])^2] = \rho(1-\rho)\sum_i \left|\frac{1}{2}\int_0^T p(t,i+1) - p(t,i)\,dt\right|^2.$$

Now, as a priori the variance $Q_0(u) \le E_{\nu_\rho}[J^2_{-1,0}(u)] = O(\sqrt{u})$, we need only find the limit of

$$\begin{aligned} Q_1(T) &= \frac{\rho(1-\rho)}{\sqrt{T}} \sum_i \left| \frac{1}{2} \int_{\epsilon T}^T p(t,i+1) - p(t,i)\,dt \right|^2 \\ &= \frac{\rho(1-\rho)}{4\sqrt{T}} \sum_i \int_{[\epsilon T, T]^2} [p(t,i+1) - p(t,i)][p(s,i+1) - p(s,i)]\,ds\,dt. \end{aligned} \tag{4.8}$$

To estimate the integrand, from Doob's inequality, note

$$\begin{aligned} p(t,x) &= \mathbb{E}[P(S_{N_t} = x)] \\ &= \mathbb{E}\Big[P(S_{N_t} = x), \sup_{t \in [\epsilon T, T]} |N_t/t - 1| \le \epsilon\Big] + O(T^{-10}), \end{aligned} \tag{4.9}$$

where N_t is a Poisson process with rate 1 independent of the discrete time random walk $\{S_k\}$, $N_t/t - 1$ is a martingale and $\mathbb{E}$ refers to expectation with respect to N_t. Further (since we could not find an appropriate continuous time version), from the local limit theorem (Petrov [27], Theorem VII.13; page 205), uniformly over x, with respect to the discrete time walk, we have for $N_t \ge 1$ that

$$\begin{aligned} P(S_{N_t} = x) &= \frac{1}{\sqrt{2\pi N_t}} e^{-x^2/(2N_t)} + \frac{1}{\sqrt{2\pi}} e^{-x^2/(2N_t)} \frac{q_2(x/\sqrt{N_t})}{N_t^{3/2}} + o(N_t^{-3/2}) \\ &= \frac{1}{\sqrt{2\pi N_t}} e^{-x^2/(2N_t)} + O(N_t^{-3/2}), \end{aligned} \tag{4.10}$$

where $q_2(y) = (\gamma_4/24\theta^4)(y^4 - 6y^2 + 3)$, γ_k is the kth order cumulant and θ^2 is the variance of the symmetric Bernoulli variable. [In our case, in Petrov's formula, $q_1(y) = (\gamma_3/6\theta^3)(y^3 - 3y) \equiv 0$ as $\gamma_3 = 0$.]

Let $p^N(t,x) = P(S_{N_t} = x)$ and $p^R(s,x) = P(S_{R_s} = x)$ where R_s is an independent Poisson process also with rate 1. We now argue that only the leading terms in (4.9) and (4.10) are significant.

Since $\sum_x p(u,x), \sum_x p^N(u,x) \le 1$, the error term on order $O(T^{-10})$ in (4.9) can be neglected in estimating (4.8). Indeed,

$$\frac{O(T^{-10})}{\sqrt{T}} \int_{[\epsilon T, T]^2} \sum_i p(s,i)\,ds\,dt = \frac{O(T^{-10})}{\sqrt{T}} \int_{[\epsilon T, T]^2} \sum_i p^N(t,i)\,ds\,dt = o(1).$$

Also, note the error term of order $O(N_t^{-3/2})$ in (4.10) is not significant with respect to (4.8). Indeed,

$$\begin{aligned} &\sum_x \frac{1}{\sqrt{2\pi N_t}} \left| e^{-(x+1)^2/(2N_t)} - e^{-x^2/(2N_t)} \right| \\ &\quad = \sum_x \frac{1}{\sqrt{2\pi N_t}} \left| e^{-(2x+1)/2N_t} - 1 \right| e^{-x^2/(2N_t)} \end{aligned}$$

$$\le Ce^{-\sqrt{N_t}/4} + C \sum_{|x|\le N_t^{3/4}} \frac{1}{\sqrt{2\pi N_t}} \frac{|x|}{N_t} e^{-x^2/2N_t}$$
$$\le \frac{C}{\sqrt{N_t}}$$

for some constants C. Then, given $|N_t/t-1|, |R_s/s-1| \le \epsilon$ for $s,t \in [\epsilon T, T]$, a product of $\sum_i (2\pi N_t)^{-1/2} |e^{-(i+1)^2/2N_t} - e^{-i^2/2N_t}|$ and the error term with respect to the s-integration, for instance, leads to bounding

$$\frac{1}{\sqrt{T}} \int_{[\epsilon T,T]^2} \sum_i \frac{1}{R_s^{3/2}} \frac{1}{\sqrt{2\pi N_t}} \left| e^{-(i+1)^2/(2N_t)} - e^{-i^2/(2N_t)} \right| ds\, dt$$
$$\le \frac{1}{\sqrt{T}} \int_{[\epsilon T,T]^2} \frac{C}{R_s^{3/2}\sqrt{N_t}} ds\, dt \le O(T^{-1/2}).$$

Therefore, focusing on the leading order terms,

$$\frac{1}{4\sqrt{T}} \sum_i \int_{[\epsilon T,T]^2} [p^N(t,i+1) - p^N(t,i)][p^R(s,i+1) - p^R(s,i)]\, ds\, dt$$
$$= o(1) + \frac{1}{8\pi\sqrt{T}} \sum_i \int_{[\epsilon T,T]^2} \frac{1}{\sqrt{R_s N_t}} \left[e^{-(i+1)^2/2N_t} - e^{-i^2/2N_t} \right]$$
$$\times \left[e^{-(i+1)^2/2R_s} - e^{-i^2/2R_s} \right] ds\, dt.$$

Now, using again $|N_t/t-1|, |R_s/s-1| \le \epsilon$ for $s,t \in [\epsilon T, T]$, we further evaluate the integral on the right-hand side as

$$o(1) + \frac{1}{8\pi\sqrt{T}} \sum_{|i|\le T^{3/4}} \int_{[\epsilon T,T]^2} \frac{i^2}{R_s N_t \sqrt{R_s N_t}} e^{(-i^2/2)[1/N_t + 1/R_s]} ds\, dt$$
$$= o(1) + \frac{1}{8\pi\sqrt{T}} \int_{[\epsilon T,T]^2} \int_{-\infty}^{\infty} \frac{x^2}{R_s N_t \sqrt{R_s N_t}} e^{(-x^2/2)[1/N_t + 1/R_s]} dx\, ds\, dt$$
$$= o(1) + \frac{\sqrt{2}}{8\sqrt{\pi T}} \int_{[\epsilon T,T]^2} (N_t + R_s)^{-3/2} ds\, dt =: Q_2(T,\epsilon).$$

Finally, we have that $Q_2(T,\epsilon)$ satisfies

$$\lim_{T\uparrow\infty} \left| Q_2(T,\epsilon) - \frac{\sqrt{2}-1}{\sqrt{\pi}} \right| \le c(\epsilon),$$

where $c(\epsilon)$ vanishes as $\epsilon \downarrow 0$. □

4.3. *Proof of Theorem* 1.8. We concentrate on the argument for the tagged particle, as a similar proof holds for the current. By symmetry,

$$P\big(|X(N^2T)|/N \ge a\big) = 2P\big(X(N^2T)/N \ge a\big).$$

From (3.1), and noting $J_{-1,0}(t) - J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(t) = \sum_{x=0}^{\lfloor aN\rfloor} \eta_t(x) - \eta_0(x)$ by the development of Section 3.1, we have

$$\{X(N^2t) \geq aN\} = \left\{J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t) \geq \sum_{x=0}^{\lfloor aN\rfloor} \eta_0(x)\right\}.$$

We now rewrite currents in terms of the standard Harris stirring process $\{\xi_t^x\}$. Namely, at time $t = 0$, a particle is put at each $x \in \mathbb{Z}$. Then, to bonds $(x, x+1)$ in $\mathbb{Z}$, associate independent Poisson clocks with parameter $1/2$. When the clock rings at a bond, interchange the positions of the particles at the bond's vertices. Let ξ_t^x be the position at time t of the particle initially at x. Then the exclusion process, starting from initial configuration η, satisfies $\eta_t(x) = 1\{x \in \{\xi_t^i : \eta(i) = 1\}\}$. More details and constructions can be found in Chapter VIII [23].

Then, for $0 \leq a \leq 1$,

$$J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t) = \sum_{x\leq\lfloor aN\rfloor} \eta_0(x) 1_{[\xi^x_{N^2t} > \lfloor aN\rfloor]} - \sum_{x>\lfloor aN\rfloor} \eta_0(x) 1_{[\xi^x_{N^2t} \leq \lfloor aN\rfloor]}.$$

Write, given the initial profile η_0 is deterministic, by Chebyshev, that

$$\begin{aligned}\frac{1}{N}\log P\bigl(X(N^2t) \geq aN\bigr) &\leq \frac{1}{N}\log E\exp\left\{-\lambda\sum_{x=0}^{\lfloor aN\rfloor}\eta_0(x)\right\}\\ &\quad + \frac{1}{N}\log E\exp\bigl\{\lambda J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t)\bigr\}.\end{aligned} \tag{4.11}$$

The first term on the right-hand side tends to $-\lambda a$ as $N \uparrow \infty$. The second term is bounded, by Chebyshev and Liggett [23], Proposition VIII.1.7, noting $e^{\alpha\sum_{i=k}^{l} 1_{[x_i\in A]}}$ is positive definite for any $\alpha \in \mathbb{R}$, and $\log(1+x) \leq x$ for $x \geq 1$, by

$$\begin{aligned}&\frac{1}{2N}\log E\exp\left\{2\lambda\sum_{x\leq\lfloor aN\rfloor}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}\right\}\\ &\quad + \frac{1}{2N}\log E\exp\left\{-2\lambda\sum_{x>\lfloor aN\rfloor}\eta_0(x)1_{[\xi^x_{N^2t}\leq\lfloor aN\rfloor]}\right\}\\ &\quad \leq \frac{1}{2N}\sum_{x\leq\lfloor aN\rfloor}\bigl(e^{2\lambda\eta_0(x)}-1\bigr)P(\xi^x_{N^2t} > \lfloor aN\rfloor)\\ &\qquad + \frac{1}{2N}\sum_{x>\lfloor aN\rfloor}\bigl(e^{-2\lambda\eta_0(x)}-1\bigr)P(\xi^x_{N^2t} \leq \lfloor aN\rfloor).\end{aligned}$$

Given $\eta_0(x) = 1_{[|x|\leq N]}$ and $\xi^x_{N^2t}$ marginally is the position of a simple random walk, started at x at time N^2t; as $N \uparrow \infty$, we have

$$\frac{1}{2N}\sum_{x\leq\lfloor aN\rfloor}\bigl(e^{2\lambda\eta_0(x)}-1\bigr)P(\xi^x_{N^2t} > \lfloor aN\rfloor) \to \frac{e^{2\lambda}-1}{2}\int_{-1}^{a} P\bigl(N(0,t) > a-x\bigr)\,dx$$

and

$$\frac{1}{2N}\sum_{x>\lfloor aN\rfloor}\big(e^{-2\lambda\eta_0(x)}-1\big)P(\xi^x_{N^2t}\le\lfloor aN\rfloor)$$
$$\to\frac{e^{-2\lambda}-1}{2}\int_a^1 P\big(N(0,t)\le a-x\big)\,dx,$$

where $N(0,t)$ is a normal distribution with mean 0 and variance t.

Hence, combining the estimates, we have that (4.11) is less than

$$-\lambda a+\frac{e^{2\lambda}-1}{2}\int_{-1}^a P\big(N(0,t)>a-x\big)\,dx+\frac{e^{-2\lambda}-1}{2}\int_a^1 P\big(N(0,t)\le a-x\big)\,dx.$$

Choosing $\lambda=\epsilon a$ for small $\epsilon>0$, we obtain further that (4.11) is bounded by

$$-\epsilon a^2\bigg[1-\frac{1}{a}\int_{1-a}^{1+a}P\big(N(0,t)>y\big)\,dy\bigg]+O(\epsilon^2a^2)\le -Ca^2$$

for a constant C, noting $1>a^{-1}\int_{1-a}^{1+a}P(N(0,t)>y)\,dy$ for $0<a\le 1$.

For $a\ge 1$, we write

$$J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(t)=\sum_{|x|\le N}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}.$$

Then, as above,

$$P\big(X(N^2t)\ge aN\big)\le e^{-\lambda\sum_{x=0}^{\lfloor aN\rfloor}\eta_0(x)}E\exp\bigg\{\lambda\sum_{|x|\le N}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}\bigg\}$$
$$\le e^{-\lambda N}\prod_{|x|\le N}E\exp\big\{\lambda 1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}\big\}.$$

Taking the logarithm, dividing by N and taking the limit, we obtain

$$\limsup_{N\uparrow\infty}\frac{1}{N}\log P\big(X(N^2t)\ge aN\big)$$
$$\le-\lambda+\limsup_{N\uparrow\infty}\frac{1}{N}\sum_{|x|\le N}(e^\lambda-1)P(\xi^x_{N^2t}>\lfloor aN\rfloor)$$
$$\le-\lambda+(e^\lambda-1)\int_{-1}^1 P\big(N(0,t)\ge a-x\big)\,dx.$$

Optimizing on λ, the right-hand side of the above display is bounded by

$$\log\int_{-1}^1 P\big(N(0,t)>a-x\big)\,dx+1-\int_{-1}^1 P\big(N(0,t)>a-x\big)\,dx<0.$$

However, for a large, this expression is bounded by $-Ca^2$.

Working with the $0\le a\le 1$ and $a>1$ bounds, we obtain the desired quadratic order estimate.

5. Proofs of approximations. We give the proofs of Propositions 1.3 and 2.1, and Lemmas 2.2 and 2.3.

5.1. *Proofs of Propositions* 1.3 *and* 2.1. The proofs are through a series of lemmas inspired by the scheme in [20] (see also Oelschläger [25], and Bertini, Landim and Mourragui [5]). As several of the steps are different, we give some details.

To this end, let μ be a density such that $I_0(\mu) < \infty$. The first lemma states that finite rate densities μ, when integrated against smooth test functions, are uniformly continuous in time; cf. Lemma 4.4 [5].

LEMMA 5.1. *Let $\eta \in D([0, T]; M_1)$ be a density such that $I_0(\eta) < \infty$, and let $\mathfrak{J} \in C_K^2(\mathbb{R})$. Then, $s \mapsto \langle \eta_s, \mathfrak{J}\rangle = \int \mathfrak{J}(x)\eta_s(x)\,dx$ is a uniformly continuous function.*

PROOF. Let $G \in C_K^{1,2}([0, T] \times \mathbb{R})$. As $I_0(\eta) < \infty$, from (1.3), we infer

$$l^2(\eta; G) \le 2I_0(\eta) \int_0^T \int G_x^2(t, x)\eta_t(x)\big(1 - \eta_t(x)\big)\,dx\,dt.$$

Let F^δ be a smooth approximations of the indicator $1_{[s,t]}(u)$. Then, by applying the previous inequality with $G^\delta = F^\delta \mathfrak{J}$, we obtain

$$\begin{aligned}\left|\int \eta_t \mathfrak{J}\,dx - \int \eta_s \mathfrak{J}\,dx\right| &= \lim_{\delta\downarrow 0}\left\{l(\eta; G^\delta) + \frac{1}{2}\int_0^T \int G_{xx}^\delta \eta_u(x)\,dx\,du\right\}\\ &\le |t - s|\|\mathfrak{J}''\|_{L^1} + \sqrt{2I_0(\eta)}|t - s|^{1/2}\|\mathfrak{J}'\|_{L^2},\end{aligned}$$

completing the proof. □

For the remainder of the subsection, let $\mu \in D([0, T]; M_1)$ be a density with finite rate, $I_0(\mu) < \infty$. We now build a succession of approximating densities in the next lemmas with special properties.

LEMMA 5.2. *For each $\epsilon > 0$, there exists a density $\hat{\mu}$, smooth in the space variable, such that*: (1) *the Skorohod distance $d(\hat{\mu}; \mu) < \epsilon$*; (2) *there is $0 < \delta_\epsilon < 1$ such that $\delta_\epsilon < \hat{\mu}(t, x) < 1 - \delta_\epsilon$ for $(t, x) \in [0, T] \times \mathbb{R}$*; (3) $|I_0(\hat{\mu}) - I_0(\mu)| < \epsilon$.

In addition, (4) *if $\hat{\gamma} \in M_1(\rho_*, \rho^*)$ is piecewise continuous, $0 < \hat{\gamma}(x) < 1$ for $x \in \mathbb{R}$, and $h(\mu_0; \hat{\gamma}) < \infty$, then also $|h(\hat{\mu}_0; \hat{\gamma}) - h(\mu_0; \hat{\gamma})| < \epsilon$.*

PROOF. For $0 < \rho_*, \rho^* < 1$, let $\gamma \in M_1(\rho_*, \rho^*)$ be a function. Consider

$$\mu^{b,\alpha} = \sigma_{t+\alpha} * \gamma + b(\sigma_\alpha * \mu - \sigma_{t+\alpha} * \gamma) \tag{5.1}$$

for $0 \le b \le 1$ and $\alpha \ge 0$. Clearly, $\mu^{b,\alpha}$ is smooth in the space variable when $\alpha > 0$.

Next, for fixed $\alpha > 0$ and $0 < b < 1$, there is $0 < \delta < 1$ such that $\delta < \mu^{b,\alpha} < 1 - \delta$ as $\sigma_{t+\alpha} * \gamma$ is strictly bounded between 0 and 1 for $t \in [0, T]$.

Now, $\mu^{b,\alpha} \to \mu^{1,\alpha}$ as $b \uparrow 1$ in $D([0, T] \times \mathbb{R})$. Also, noting $\lim_{\alpha\downarrow 0} \|\sigma_\alpha * G - G\|_{L^1(\mathbb{R})} = 0$ for $G \in L^1(\mathbb{R})$, we have also have the Skorohod convergence $\mu^{1,\alpha} \to \mu$ as $\alpha \downarrow 0$.

By lower semi-continuity of I_0,

$$\liminf I_0(\mu^{b,\alpha}) \geq I_0(\mu).$$

On the other hand, by convexity of $I_0(\nu)$, we have

$$I_0(\mu^{b,\alpha}) \leq (1 - b) I_0(\sigma_{t+\alpha} * \gamma) + b I_0(\sigma_\alpha * \mu).$$

Note that $I_0(\sigma_{t+\alpha} * \gamma) = 0$, and by translation-invariance and convexity, the right-hand side in the display is less than

$$b \int \sigma_\alpha(y) I_0\big(\mu(t, x - y)\big)\, dy = b I_0(\mu) \uparrow 1 \qquad \text{as } b \uparrow 1.$$

Similarly, if $\hat{\gamma} \in M_1(\rho_*, \rho^*)$ is piecewise continuous, $0 < \hat{\gamma} < 1$ and $h(\mu_0; \hat{\gamma}) < \infty$, then, by lower semi-continuity and convexity of $h(\cdot; \hat{\gamma})$, we have

$$h(\mu_0; \hat{\gamma}) \leq \liminf_{b\uparrow 1, \alpha\downarrow 0} h(\mu_0^{b,\alpha}; \hat{\gamma})$$

and

$$h(\mu_0^{b,\alpha}; \hat{\gamma}) \leq (1 - b) h(\sigma_\alpha * \gamma; \hat{\gamma}) + b h(\sigma_\alpha * \mu_0; \hat{\gamma}).$$

Also, once more by convexity,

$$h(\sigma_\alpha * \mu_0; \hat{\gamma}) \leq \int dy \sigma_\alpha(y) \int dx h_d\big(\mu_0(x); \hat{\gamma}(x - y)\big).$$

The right-hand side, since $|h(\mu_0(\cdot); \hat{\gamma}(\cdot - y)) - h(\mu_0; \hat{\gamma})| \leq C|y|$ by properties of $\hat{\gamma}$, converges to $h(\mu_0; \hat{\gamma})$ as $\alpha \downarrow 0$. By the same argument, $\lim_{\alpha\downarrow 0} h(\sigma_\alpha * \gamma; \hat{\gamma}) = h(\gamma; \hat{\gamma})$. Hence $\lim_{b\uparrow 1, \alpha\downarrow 0} h(\mu_0^{b,\alpha}; \hat{\gamma}) = h(\mu_0; \hat{\gamma})$.

Therefore, statements (1)–(4) hold for $\hat{\mu} = \mu^{b,\alpha}$ when $b \sim 1$, $\alpha \sim 0$. $\square$

LEMMA 5.3. *Let $\hat{\mu}$ be the density constructed in Lemma 5.2. Then*: (1) *for each* $\epsilon > 0$, *there exists a smooth density $\tilde{\mu}$ such that* $\tilde{\mu}_0 = \hat{\mu}_0$; (2) *the Skorohod distance* $d(\tilde{\mu}; \hat{\mu}) < \epsilon$; (3) $|I_0(\tilde{\mu}) - I_0(\hat{\mu})| < \epsilon$. *Also*, (4) *all partial derivatives of $\tilde{\mu}$ are uniformly bounded in* $[0, T] \times \mathbb{R}$.

PROOF. To obtain a smooth density, we need only approximate $\hat{\mu}$ by smoothing in the time variable. Define for $\beta > 0$ a density which is constant in time on a short time interval.

$$\nu^\beta(t, x) = \begin{cases} \hat{\mu}_0(x), & \text{for } 0 \leq t < \beta, \\ \hat{\mu}(t - \beta, x), & \text{for } \beta \leq t \leq T + \beta. \end{cases}$$

Let $\kappa_\varepsilon \in C_K^\infty(\mathbb{R})$ be smooth approximations of the identity in $L^1(\mathbb{R})$ such that $\kappa_\varepsilon \geq 0$, $\int \kappa_\varepsilon(x)\,dx = 1$, $\mathrm{Supp}(\kappa_\varepsilon) \subset (0, \varepsilon)$ and for $f \in L^1(\mathbb{R})$, $f * \kappa_\varepsilon \to f$ as $\varepsilon \downarrow 0$ in L^1. Form the convolution, for $0 < \varepsilon \leq \beta$,

$$\nu^{\beta,\varepsilon}(t, x) = \int_0^T \nu^\beta(t + s, x)\kappa_\varepsilon(s)\,ds.$$

It is clear, by continuity of $\hat{\mu}$ in time (Lemma 5.1), that $\lim_{\beta\downarrow 0}\lim_{\varepsilon\downarrow 0} \nu^{\beta,\varepsilon} = \hat{\mu}$ in $D([0, T]; M_1)$. By construction, $\nu^{\beta,\varepsilon}$ is smooth, and also $\nu_0^{\beta,\varepsilon} = \hat{\mu}_0$.

From lower semi-continuity and convexity

$$\liminf_{\beta,\varepsilon} I_0(\nu^{\beta,\varepsilon}) \geq I_0(\hat{\mu}) \quad \text{and} \quad I_0(\nu^{\beta,\varepsilon}) \leq \int_0^T \kappa_\varepsilon(s) I_0\big(\nu^\beta(t + s, x)\big)\,ds.$$

Using the variational definition of I_0, noting $\hat{\mu}_0 = \sigma_\alpha * (\gamma + b(\mu_0 - \gamma))$, the rate of ν^β on the interval $[0, \beta]$ is bounded by

$$\sup_{G\in C_K^{1,2}} \frac{1}{2}\int_0^\beta \int G_x \partial_x \hat{\mu}_0 - G_x^2 \hat{\mu}_0(1 - \hat{\mu}_0)\,dx\,dt \leq \frac{\beta}{8}\int \frac{(\partial_x \hat{\mu}_0)^2}{\hat{\mu}_0(1 - \hat{\mu}_0)}\,dx,$$

which vanishes as $\beta \downarrow 0$. On the other hand, by formula (1.4), the rate of ν^β on the interval $[\beta, T]$ converges to $I_0(\hat{\mu})$ as $\beta \downarrow 0$. We can conclude then that $\lim_{\beta,\varepsilon\downarrow 0} I_0(\nu^{\beta,\varepsilon}) \to I_0(\hat{\mu})$.

Moreover, by differentiating the convolutions, since $\|\nu^{\beta,\varepsilon}\|_{L^\infty} \leq 1$, we have $\|\partial_x^{(k)}\partial_t^{(l)}\tilde{\mu}\|_{L^\infty} \leq \|\partial_x^{(k)}\sigma_\alpha\|_{L^1}\|\partial_t^{(l)}\kappa_\varepsilon\|_{L^1} < \infty$.

Hence, to find the desired density, we can take $\tilde{\mu} = \nu^{\beta,\varepsilon}$ for β, ε small. $\square$

We now continue to adjust the approximation so that the associated function "H_x" of the approximating density has desired properties.

LEMMA 5.4. *Let $\tilde{\mu}$ be the density constructed in Lemma 5.3, and $\tilde{H}_x$ be associated to it via* (1.2). *Then*: (1) $\tilde{H}_x \in C^\infty([0, T] \times \mathbb{R})$; (2) $\|\tilde{H}_x\|_{L^2([0,T]\times\mathbb{R})} < \infty$; (3) $\|\tilde{H}_x\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$.

PROOF. By construction, we recall, for a $\delta > 0$, that $\delta < \tilde{\mu} < 1 - \delta$, $\tilde{\mu}$ is smooth with uniformly bounded derivatives on $[0, T] \times \mathbb{R}$ of all orders, and

$$\partial_t\tilde{\mu} = \tfrac{1}{2}\partial_{xx}\tilde{\mu} - \partial_x[\tilde{H}_x\tilde{\mu}(1 - \tilde{\mu})]. \tag{5.2}$$

Then, as $2I_0(\tilde{\mu}) = \int_0^T \int (\tilde{H}_x)^2\tilde{\mu}(1 - \tilde{\mu})\,dx\,dt < \infty$, we obtain the L^2 bound on $\tilde{H}_x$, and, by solving for $\tilde{H}_x$ in (5.2), we have that $\tilde{H}_x$ is smooth.

We now deduce that $\tilde{H}_x$ is bounded in L^∞. This bound will follow from the L^2 bound on $\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})$ and $\delta < \tilde{\mu} < 1-\delta$, if we show that $\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})$ is Lipschitz in both space and time variables with uniform constant over $[0, T] \times \mathbb{R}$. However, from (5.2), $\tilde{H}_x\hat{\mu}(1-\hat{\mu})$ is Lipschtiz in the space variable with uniform constant as $\partial_t\tilde{\mu}$ and $\partial_{xx}\tilde{\mu}$ are bounded on $[0,T]\times\mathbb{R}$.

To show $\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})$ is also Lipschitz in the time variable t uniformly over $[0,T]\times\mathbb{R}$, write

$$\begin{aligned}&\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})(x,t)\\&\quad=\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})(0,t)+(1/2)\partial_x\tilde{\mu}(x,t)-(1/2)\partial_x\tilde{\mu}(0,t)-\int_0^x\partial_t\tilde{\mu}\,dy.\end{aligned}$$

The first three terms on the right-hand side are clearly uniformly Lipschitz in t as their partial derivatives in time are bounded on $[0,T]$.

To treat the last term, consider a smooth G compactly supported in $[-\epsilon,x+\epsilon]$ which equals 1 on $[0,x]$. Since $\partial_{tt}\tilde{\mu}$ is bounded, we have

$$\left|\int_0^x\partial_{tt}\tilde{\mu}(u,y)\,dy\right|\le\left|\int G(y)\partial_{tt}\tilde{\mu}(u,y)\,dy\right|+2C\epsilon.$$

Now, by construction in the proof of Lemma 5.3, $\tilde{\mu}=\kappa_\varepsilon*\nu^\beta$, and so

$$\int G(y)\partial_{tt}\tilde{\mu}(u,y)\,dy=\int_0^T\int G(y)\kappa_\varepsilon''(s)\nu^\beta(u+s,y)\,dy\,ds.$$

As $I_0(\nu^\beta)<\infty$, we can associate via (1.2) an H_x^β to the density ν^β. From the weak formulation (1.3), and $\kappa_\varepsilon'(0)=\kappa_\varepsilon'(T)=0$, the right-hand side equals

$$\begin{aligned}&-\frac{1}{2}\int_0^T\int G''(y)\kappa_\varepsilon'(s)\nu^\beta(u+s,y)\,dy\,ds,\\&-\int_0^T\int G'(y)\kappa_\varepsilon'(s)H_x^\beta\nu^\beta(1-\nu^\beta)(u+s,y)\,dy\,ds.\end{aligned}$$

The first integral, because ν^β is bounded and $G'\neq 0$ on a set of width at most 2ϵ is uniformly bounded in time u and space x. Similarly, the second integral, as $\|H_x^\beta\nu^\beta(1-\nu^\beta)\|_{L^2}\le 2I_0(\nu^\beta)<\infty$, is also both uniformly bounded in u and x. □

The function $\tilde{H}_x$ associated to $\tilde{\mu}$ in Lemma 5.4, although smooth, does not necessarily have compact support. Let $H_x^m\in C_K^\infty((0,T]\times\mathbb{R})$ be smooth approximations of $\tilde{H}_x$ with the following properties:

$$\|H_x^m-\tilde{H}_x\|_{L^2([0,T]\times\mathbb{R})}\le m^{-1}\quad\text{and}\quad\sup_{\substack{t\in[0,T]\\x\in[-m,m]}}|H_x^m-\tilde{H}_x|\le m^{-1}.$$

Denote $w^m\in D([0,T];M_1)$ as the smooth density with initial condition $w_0^m=\tilde{\mu}_0$ which satisfies the equation

$$\partial_t v=(1/2)\partial_{xx}v-\partial_x\bigl(H_x^m v(1-v)\bigr).$$

Existence, for instance, follows from the hydrodynamic limit for weakly asymmetric exclusion processes in [18] using the replacement estimates Theorem 6.1 and Claims 1, 2 [22], Section 6; see also Theorem 3.1 [20]. Uniqueness in the class of bounded solutions follows by the method of Proposition 3.5 [25].

We now show that w^m, whose associated function $H^m_x \in C^\infty_K$, is close to $\tilde{\mu}$. Hence, w^m will turn out to be a suitable candidate with respect to Proposition 2.1. In addition, we will be able to deduce that $\mu \in \mathcal{A}$ under (LEM) initial distributions.

LEMMA 5.5. *The sequence w^m converges uniformly to $\tilde{\mu}$ on compact subsets of $[0,T]\times\mathbb{R}$, and hence in $D([0,T];M_1)$. Also, $I_0(w^m)\to I_0(\tilde{\mu})$.*

PROOF. Suppose that we have proven $w^m \to \tilde{\mu}$ uniformly on compact subsets. As $\|H^m_x - \hat{H}_x\|_{L^2}\to 0$, we would then conclude $I_0(w^m)\to I_0(\tilde{\mu})$. In the following, the constant C may change line to line.

Now, given $\partial_t\sigma_t(x) = (1/2)\partial_{xx}\sigma_t(x)$, we have for $t,h>0$ that

$$\begin{aligned}\sigma_h * w^m_t(y) &- \sigma_{t+h} * w^m_0(y)\\ &= \int_0^t \int H^m_x w^m(1-w^m)(s,z)\frac{-(z-y)}{t+h-s}\sigma_{t-s+h}(z-y)\,dz\,ds.\end{aligned}$$

By properties of w^m, H^m_x and $(|z-y|/\sqrt{u})\exp(-(z-y)^2/4u)\le 1$,

$$\begin{aligned}|H^m_x w^m(1-w^m)|(s,z)&\frac{|z-y|}{t+h-s}\sigma_{t+h-s}(z-y)\\ &\le C|t-s|^{-1/2}\sigma_{2T}(z-y)\in L^1([0,t]\times\mathbb{R}).\end{aligned}$$

Hence, taking $h\downarrow 0$, we obtain

$$\begin{aligned}w^m_t(y) &= \sigma_t * w^m_0(y)\\ &\quad+\int_0^t\int H^m_x w^m(1-w^m)(s,z)\frac{-(z-y)}{t-s}\sigma_{t-s}(z-y)\,dz\,ds.\end{aligned}\tag{5.3}$$

Equation (5.3) also holds with respect to $\tilde{\mu}$.

Let now $|y|\le m/2$. We have then, using again $(|z-y|/\sqrt{u})\exp(-(z-y)^2/4u)\le 1$, and $w^m_0 = \tilde{\mu}_0$, that

$$\begin{aligned}|w^m_t(y) &- \tilde{\mu}_t(y)|\\ &\le \sigma_t * |w^m_0 - \tilde{\mu}_0|(y)\\ &\quad+\int_0^t\int |H^m_x w^m(1-w^m) - \hat{H}_x\tilde{\mu}(1-\tilde{\mu})|(s,z)\frac{|z-y|}{t-s}\sigma_{t-s}(z-y)\,dz\,ds\\ &\le C\int_0^t\int |H^m_x w^m(1-w^m) - \tilde{H}_x\tilde{\mu}(1-\tilde{\mu})|(s,z)\\ &\qquad\times (t-s)^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds.\end{aligned}\tag{5.4}$$

We now estimate the last line in two parts, noting

$$
\begin{aligned}
&|H_x^m w^m(1-w^m) - \tilde{H}_x\tilde{\mu}(1-\tilde{\mu})|(s,z)\\
&\qquad \le |H_x^m - \tilde{H}_x|(s,z) + |\tilde{H}_x||\tilde{\mu}(1-\hat{\mu}) - w^m(1-w^m)|(s,z).
\end{aligned}
$$

The first part, noting $|y| \le m/2$, by properties of H_x^m, $\|\sigma_t(x)\|_{L^2([0,T]\times\mathbb{R})} \le CT^{1/4}$, and $\sup_{t\in(0,T]} t^{-1/2}\sigma_{4t}(1) \le C$, is bounded for large m, as

$$
\begin{aligned}
&\int_0^t \int |H_x^m - \tilde{H}_x|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds\\
&\qquad \le \int_0^t \int_{|z|\ge m} |H_x^m - \tilde{H}_x|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds + m^{-1}\sqrt{t}\\
&\qquad \le C\int_0^t \int_{|z|\ge m} |H_x^m - \tilde{H}_x|(s,z)\sigma_{4(t-s)}(z-y)\,dz\,ds + m^{-1}\sqrt{t}\\
&\qquad \le Cm^{-1}T^{1/4} + m^{-1}\sqrt{T}.
\end{aligned}
$$

The second part is decomposed as the sum of three terms,

$$
\begin{aligned}
&\int_0^t \int |\tilde{H}_x||\tilde{\mu}(1-\tilde{\mu}) - w^m(1-w^m)|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds\\
&\qquad = D_1 + D_2 + D_3,
\end{aligned}
$$

where D_1, D_2, D_3 is the integral over $[0,t]\times\{|z|\ge m/2+\epsilon\}$, $[0,t]\times\{m/2 \le |z| \le m/2+\epsilon\}$ and $[0,t]\times\{|z|\le m/2\}$, respectively, for $\epsilon > 0$.

The term D_1, noting $\sup_{t\in(0,T]} t^{-1/2}\sigma_{4t}(\epsilon) \le C_\epsilon$, is bounded by

$$
\begin{aligned}
&2\int_0^t \int_{|z|\ge m/2+\epsilon} |\tilde{H}_x|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds\\
&\qquad \le C(\epsilon,T)\|\tilde{H}_x\|_{L^2([0,T]\times\{z\,:\,|z|\ge m/2\})}.
\end{aligned}
$$

The second term D_2 is bounded by

$$
\begin{aligned}
&2\int_0^t \int_{m/2\le|z|\le m/2+\epsilon} |\tilde{H}_x||t-s|^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds\\
&\qquad \le C\|\tilde{H}_x\|_{L^\infty}\int_0^t s^{-3/4}\int_0^\epsilon s^{-1/4}e^{-z^2/4s}\,dz\,ds\\
&\qquad \le C\|\tilde{H}_x\|_{L^\infty}T^{1/4}\sqrt{\epsilon}.
\end{aligned}
$$

The third term D_3 is bounded, with respect to a $\tau \ge t$, by

$$
\begin{aligned}
&2\int_0^t \int_{|z|\le m/2} |\tilde{H}_x||\tilde{\mu} - w^m|(s,z)|t-s|^{-1/2}\sigma_{2(t-s)}(z-y)\,dz\,ds\\
&\qquad \le 2\sqrt{t}\|\tilde{H}_x\|_{L^\infty}\sup_{\substack{|z|\le m/2\\ s\le\tau}} |\tilde{\mu}_s(z) - w_s^m(z)|.
\end{aligned}
\tag{5.5}
$$

Hence, for $\tau > 0$ small enough but fixed, which satisfies $2\sqrt{\tau}\|\tilde{H}_x\|_{L^\infty} = 1/2$, or $\tau = (16\|\tilde{H}_x\|_{L^\infty})^{-1}$, and $L < m/2$, we have

$$\begin{aligned}\sup_{\substack{|z|\le L\\ t\le\tau}} |\tilde{\mu}(z,t) - w^m(z,t)| &\le \sup_{\substack{|z|\le m/2\\ t\le\tau}} |\tilde{\mu}(z,t) - w^m(z,t)|\\ &\le C(T)m^{-1} + 2C(\epsilon,T)\|\tilde{H}_x\|_{L^2([0,T]\times\{z:|z|\ge m/2\})}\\ &\quad + 2C\|\tilde{H}_x\|_{L^\infty}T^{1/4}\sqrt{\epsilon}.\end{aligned}$$

Here, we absorbed the right-hand side of (5.5) into the left-hand side above.

We may repeat the same scheme, starting from time τ, where now the initial difference (5.4) is taken into account:

$$\begin{aligned}&\sup_{|y|\le m/3} \sigma_t * |w^m_\tau - \tilde{\mu}_\tau|(y)\\ &\quad\le \sup_{|z|\le m/2} |w^m_\tau - \tilde{\mu}_\tau|(z) + \sup_{|y|\le m/3}\int_{|z|>m/2} \sigma_t(y-z)\,dz\\ &\quad\le \sup_{|z|\le m/2} |w^m_\tau - \tilde{\mu}_\tau|(z) + e^{-Cm^2/T}.\end{aligned}$$

With a finite number of iterations of such type, say $N_\tau = [T/\tau] + 1$ iterations, when $L < m/N_\tau$, we obtain uniform convergence, as $m \uparrow \infty$, for $|z| \le L$ and $0 \le s \le T$. $\square$

PROOF OF PROPOSITION 2.1. The proof follows by applying Lemmas 5.2, 5.3 and 5.5 to build a density $\mu^+ = w^m$, which satisfies specifications (i)–(viii). We remark property (v) is shown as follows: When $\mu_0 = \gamma$, by construction in (5.1), we have $w^m_0 = \tilde{\mu}_0 = \hat{\mu}_0 = \mu_0^{b,\alpha} = \sigma_\alpha * \gamma$. When $\gamma(x) \equiv \rho$, this reduces to $\tilde{\mu}_0(x) \equiv \rho$. $\square$

Starting under (DIC) initial conditions, however, to prove Proposition 1.3, we will need to specify that w^m can be approximated by a suitable density with initial value equal to $\mu_0 = \gamma \in M_1(\rho_*, \rho^*)$.

LEMMA 5.6. *Recall w^m from Lemma 5.5. Suppose $\mu_0 = \gamma \in M_1(\rho_*, \rho^*)$. Then, for $\epsilon > 0$, $\exists M$ such that $\forall m \ge M$, there is a density $\bar{\chi} \in C^\infty((0,T]\times\mathbb{R})$, such that*: (1) *equation* (1.2) *is satisfied with respect to $\bar{H}_x \in C^\infty_K([0,T]\times\mathbb{R})$*; (2) *initial value $\bar{\chi}_0 = \gamma$*; (3) *the Skorohod distance $d(\bar{\chi}, w^m) < \epsilon$*; (4) $|I_0(\bar{\chi}) - I_0(w^m)| < \epsilon$.

PROOF. Consider w^m_0 from Lemma 5.5. From the assumption $\mu_0 = \gamma$, we have $w^m_0 = \tilde{\mu}_0 = \sigma_\alpha * \gamma$ from (5.1). Form the density $\bar{\chi}$ as follows:

$$\bar{\chi} = \begin{cases} \sigma_t * \gamma, & \text{for } 0 \le t \le \alpha,\\ w^m_{t-\alpha}, & \text{for } \alpha \le t \le T.\end{cases}$$

Since H_x^m is supported on a compact subset of $(0,T]\times\mathbb{R}$, $\bar\chi\in C^\infty((0,T]\times\mathbb{R})$, and satisfies (1.2) with respect to $\bar H_x\in C_K^\infty([0,T]\times\mathbb{R})$ given by

$$\bar H_x=\begin{cases}0, & \text{for } (t,x)\in[0,\alpha]\times\mathbb{R},\\ H_x^m(t-\alpha,x), & \text{for } (t,x)\in[\alpha,T]\times\mathbb{R}.\end{cases}$$

Now, $2I_0(\bar\chi)=\int_0^T\int \bar H_x^2\bar\chi(1-\bar\chi)\,dx\,dt=\int_0^{T-\alpha}\int (H_x^m)^2w^m(1-w^m)\,dx\,dt$. Then, the difference

$$2I_0(\bar\chi)-2I_0(w^m)=\int_{T-\alpha}^T\int (H_x^m)^2w^m(1-w^m)\,dx\,dt.$$

To estimate the right-hand side, recall from Lemma 5.5 that $\|H_x^m-\tilde H_x\|_{L^2}\le m^{-1}$, and $w^m\to\tilde\mu$ uniformly on compact subsets. Then

$$\begin{aligned}
&\int_{T-\alpha}^T\int (H_x^m)^2w^m(1-w^m)\,dx\,dt\\
&\quad\le 2\|H_x^m-\tilde H_x\|^2_{L^2([0,T]\times\mathbb{R})}+2\int_0^T\int_{|x|\ge L}\tilde H_x^2\,dx\,dt\\
&\qquad+4\int_0^T\int_{|x|\le L}\tilde H_x^2|w^m-\tilde\mu|\,dx\,dt+2\int_{T-\alpha}^T\int\tilde H_x^2\tilde\mu(1-\tilde\mu)\,dx\,dt\\
&\quad=B_1+B_2+B_3+B_4.
\end{aligned}$$

Choose $L=L(\tilde H_x)$ large so that $B_2\le\epsilon/4$, and take $m=m(\tilde H_x,L)$ large enough so that both $B_1,B_3\le\epsilon/4$.

The term $B_4/4$ is the rate of $\tilde\mu$ on the time interval $[T-\alpha,T]$. Since $\tilde\mu$ and $\tilde H_x$ depend on α, we bound B_4 in terms of H_x (which does not depend on α) to show that it is small when α is small. By the construction of $\tilde\mu$ in Lemma 5.3, convexity of the the rate, translation-invariance and that the rate of $\sigma_{t+\alpha}*\gamma$ vanishes, we estimate

$$\begin{aligned}
B_4&\le 2b\int\sigma_\alpha(z)\int_0^T\kappa_\varepsilon(s)\int_{T-\alpha}^T\int H_x^2\mu(1-\mu)(t+s-\beta,x-z)\,dx\,dt\,ds\,dz\\
&\le 2\int_{T-2\alpha}^T\int H_x^2\mu(1-\mu)\,dx\,dt,
\end{aligned}$$

when $\beta\le\alpha\le T-\alpha$. Then, as $I_0(\mu)<\infty$, $B_4\downarrow 0$ as $\alpha\downarrow 0$.

Hence, with α small enough, there is M so that for $m\ge M$, we have $|I_0(\bar\chi)-I_0(w^m)|<\epsilon$. Also, by Lemma 5.5, $I_0(w^m)\le I_0(\mu)+1$, and so by uniform continuity (Lemma 5.1), the Skorohod distance $d(\bar\chi;w^m)<\epsilon$. $\square$

PROOF OF PROPOSITION 1.3. Let γ be a profile associated to an (LEM) or (DIC) measure, and let μ be such that $I_\gamma(\mu)<\infty$. By successively applying Lemmas 5.2, 5.3, 5.5 and 5.6, we can approximate μ by an appropriate density μ^+ to verify $\mu\in\mathcal{A}$. Specifically, under an (LEM) initial measure, when

$I_\gamma(\mu) = I_\gamma^{LE}(\mu)$, $\mu^+ = w^m$ in Lemma 5.5 with appropriate choice of parameters $b, \alpha, \beta, \varepsilon$ and m. Under a (DIC) initial configuration, when $I_\gamma(\mu) = I_\gamma^{DC}(\mu)$, $\mu^+ = \bar{\chi}$ in Lemma 5.6 again with suitable parameters. $\square$

5.2. *Proof of Lemmas* 2.2, 2.3. We prove the lemmas in succession.

PROOF OF LEMMA 2.2. Note that

$$\begin{aligned}|\mu_t(x) - \hat{\gamma}(x)| &= \lim_{h\downarrow 0} |\sigma_h * (\mu_t - \hat{\gamma})(x)| \\ &\leq \lim_{h\downarrow 0} |\sigma_{t+h} * (\mu_0 - \hat{\gamma})(x)| \\ &\quad + \int_0^t \int |H_x|\mu(1-\mu)(s,z)|\partial_z \sigma_{t-s+h}(z-x)|\,dz\,ds.\end{aligned}$$

Since H_x has compact support in $[0, T] \times \mathbb{R}$, the second term on the right-hand side is bounded by

$$C_H \int_0^t \int_{|z|\leq M_H} \frac{|x-z|}{t-s} \sigma_{t-s}(z-x)\,dz\,ds$$

for some constants C_H, M_H. Since $(|y|/\sqrt{s})e^{-y^2/4s} \leq 1$, when $|x| \geq M_H$, we can bound it further by $4C_H\sqrt{T}e^{-(x-M_H)^2/8T}$, which vanishes as $|x| \uparrow \infty$.

The first term, however, is bounded as follows:

$$\begin{aligned}&|\sigma_{t+h} * (\mu_0 - \hat{\gamma})(x)| \\ &\quad \leq \sup_{|z|\leq l} |\mu_0 - \hat{\gamma}|(z-x) \cdot \int_{|z|\leq l} \sigma_t(z)\,dz + \sqrt{2}e^{-l^2/4T} \int_{|z|\geq l} \sigma_{2t}(z)\,dz \\ &\quad \leq \sup_{|z|\leq l} |\mu_0 - \hat{\gamma}|(z-x) + \sqrt{2}e^{-l^2/4T}.\end{aligned}$$

Now, since $h(\mu_0; \hat{\gamma}) < \infty$, $\hat{\gamma} \in M_1(\rho_*, \rho^*)$, $\|\partial_x \mu_0\|_{L^\infty} < \infty$, we conclude, for fixed l, that $\lim_{|x|\uparrow\infty} \sup_{|z|\leq l} |\mu_0 - \hat{\gamma}|(z-x) = 0$. This completes the proof. $\square$

PROOF OF LEMMA 2.3. Consider Hellinger's inequality $(\sqrt{\alpha} - \sqrt{\beta})^2 \leq h_d(\alpha; \beta)$. [Let $H(\alpha; \beta) = (\sqrt{\alpha} - \sqrt{\beta})^2 + (\sqrt{1-\alpha} - \sqrt{1-\beta})^2$. By Jensen's inequality and $\log(1-x) \leq -x$ for $0 \leq x < 1$, $h_d(\alpha; \beta) \geq -2\log[1 - (1/2)H(\alpha; \beta)] \geq H(\alpha; \beta)$.] We write then $(\mu_t(x) - \rho)^2 \leq 2(\sqrt{\mu} - \sqrt{\rho})^2 \leq 2h_d(\mu_t(x); \rho)$. Hence, by Proposition 2.4 (with respect to the density on $[0, t] \times \mathbb{R}$ with $\hat{\gamma} \equiv \rho$),

$$\begin{aligned}\int (\mu_t(x) - \rho)^2\,dx &\leq 2 \int h_d(\mu_t(x); \rho)\,dx \\ &\leq 4 \int_0^t \int (H_x^n)^2 \mu(1-\mu)\,dx\,ds \leq 8I_0(\mu)\end{aligned}$$

uniformly in $0 \leq t \leq T$. $\square$

Acknowledgments. We would like to thank Claudio Landim, K. Ravishankar and Ellen Saada for helpful conversations. Thanks also to the referee for constructive comments with respect to extensions at the end of Section 1, and in other parts.

REFERENCES

[1] ARRATIA, R. (1983). The motion of a tagged particle in the simple symmetric exclusion system on **Z**. *Ann. Probab.* **11** 362–373. MR0690134

[2] BENOIS, O., KIPNIS, C. and LANDIM, C. (1995). Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes. *Stochastic Process. Appl.* **55** 65–89. MR1312149

[3] BERTINI, L., DE SOLE, A., GABRIELLI, D., JONA-LASINIO, G. and LANDIM, C. (2006). Large deviations of the empirical current in interacting particle systems. *Teor. Veroyatn. Primen.* **51** 144–170; translation in *Theory Probab. Appl.* **51** 2–27. MR2324172

[4] BERTINI, L., DE SOLE, A., GABRIELLI, D., JONA-LASINIO, G. and LANDIM, C. (2006). Large deviation approach to non equilibrium processes in stochastic lattice gases. *Bull. Braz. Math. Soc.* (*N.S.*) **37** 611–643. MR2284891

[5] BERTINI, L., LANDIM, C. and MOURRAGUI, M. (2009). Dynamical large deviations for the boundary driven weakly asymmetric exclusion process. *Ann. Probab.* **37** 2357–2403. MR2573561

[6] BORCEA, J., BRÄNDÉN, P. and LIGGETT, T. M. (2009). Negative dependence and the geometry of polynomials. *J. Amer. Math. Soc.* **22** 521–567. MR2476782

[7] DACOROGNA, B. (2009). *Introduction to the Calculus of Variations*, 2nd ed. Imperial College Press, London. MR2480025

[8] DE MASI, A. and FERRARI, P. A. (2002). Flux fluctuations in the one dimensional nearest neighbors symmetric simple exclusion process. *J. Stat. Phys.* **107** 677–683. MR1898853

[9] DEMBO, A. and ZEITOUNI, O. (1998). *Large Deviations Techniques and Applications*, 2nd ed. *Applications of Mathematics* (*New York*) **38**. Springer, New York. MR1619036

[10] DERRIDA, B. and GERSCHENFELD, A. (2009). Current fluctuations of the one dimensional symmetric simple exclusion process with step initial condition. *J. Stat. Phys.* **136** 1–15. MR2525223

[11] DERRIDA, B. and GERSCHENFELD, A. (2009). Current fluctuations in one dimensional diffusive systems with a step initial density profile. *J. Stat. Phys.* **137** 978–1000. MR2570758

[12] FARFAN, J., LANDIM, C. and MOURRAGUI, M. (2011). Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stochastic Process. Appl.* **121** 725–758. MR2770905

[13] FERRARI, P. L. and SPOHN, H. (2006). Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process. *Comm. Math. Phys.* **265** 1–44. MR2217295

[14] GRIGORESCU, I. (2007). Large deviations for a catalytic Fleming–Viot branching system. *Comm. Pure Appl. Math.* **60** 1056–1080. MR2319055

[15] HURTADO, P. I. and GARRIDO, P. L. (2010). Large fluctuations of the macroscopic current in diffusive systems: A numerical test of the additivity principle. *Phys. Rev. E* **81** 041102.

[16] JARA, M. D. and LANDIM, C. (2006). Nonequilibrium central limit theorem for a tagged particle in symmetric simple exclusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **42** 567–577. MR2259975

[17] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314

[18] KIPNIS, C., OLLA, S. and VARADHAN, S. R. S. (1989). Hydrodynamics and large deviation for simple exclusion processes. *Comm. Pure Appl. Math.* **42** 115–137. MR0978701

[19] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. MR0834478

[20] LANDIM, C. (1992). Occupation time large deviations for the symmetric simple exclusion process. *Ann. Probab.* **20** 206–231. MR1143419

[21] LANDIM, C., OLLA, S. and VARADHAN, S. R. S. (2001). Symmetric simple exclusion process: Regularity of the self-diffusion coefficient. *Comm. Math. Phys.* **224** 307–321. MR1869001

[22] LANDIM, C. and YAU, H.-T. (1995). Large deviations of interacting particle systems in infinite volume. *Comm. Pure Appl. Math.* **48** 339–379. MR1324406

[23] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231

[24] LIGGETT, T. M. (2009). Distributional limits for the symmetric exclusion process. *Stochastic Process. Appl.* **119** 1–15. MR2485017

[25] OELSCHLÄGER, K. (1985). A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Gebiete* **69** 279–322. MR0779460

[26] PELIGRAD, M. and SETHURAMAN, S. (2008). On fractional Brownian motion limits in one dimensional nearest-neighbor symmetric simple exclusion. *ALEA Lat. Am. J. Probab. Math. Stat.* **4** 245–255. MR2448774

[27] PETROV, V. V. (1975). *Sums of Independent Random Variables. Ergebnisse der Mathematik und ihrer Grenzgebiete* **82**. Springer, New York. MR0388499

[28] PRÄHOFER, M. and SPOHN, H. (2002). Current fluctuations for the totally asymmetric simple exclusion process. In *In and Out of Equilibrium* (*Mambucaba*, 2000). *Progress in Probability* **51** 185–204. Birkhäuser, Boston, MA. MR1901953

[29] QUASTEL, J. (1995). Large deviations from a hydrodynamic scaling limit for a nongradient system. *Ann. Probab.* **23** 724–742. MR1334168

[30] QUASTEL, J., REZAKHANLOU, F. and VARADHAN, S. R. S. (1999). Large deviations for the symmetric simple exclusion process in dimensions $d \geq 3$. *Probab. Theory Related Fields* **113** 1–84. MR1670733

[31] REZAKHANLOU, F. (1994). Propagation of chaos for symmetric simple exclusions. *Comm. Pure Appl. Math.* **47** 943–957. MR1283878

[32] ROST, H. and VARES, M. E. (1985). Hydrodynamics of a one-dimensional nearest neighbor model. *Contemp. Math.* **41** 329–342. MR0814722

[33] SASAMOTO, T. (2008). Exact results for the 1D asymmetric exclusion process and KPZ fluctuations. *Eur. Phys. J. B* **64** 373–377. MR2447924

[34] SEPPÄLÄINEN, T. (2008). Translation invariant exclusion processes. Available at http://www.math.wisc.edu/~seppalai/excl-book/etusivu.html.

[35] SEPPÄLÄINEN, T. (1998). Coupling the totally asymmetric simple exclusion process with a moving interface. *Markov Process. Related Fields* **4** 593–628. MR1677061

[36] SPITZER, F. (1970). Interaction of Markov processes. *Adv. Math.* **5** 246–290. MR0268959

[37] SPOHN, H. (1991). *Large Scale Dynamics of Interacting Particles. Texts and Monographs in Physics*. Springer, Berlin.

[38] VANDENBERG-RODES, A. (2010). A limit theorem for particle current in the symmetric exclusion process. *Electron. Commun. Probab.* **15** 240–252. MR2658971

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
617 N. SANTA RITA AVE.
TUCSON, ARIZONA 85721
USA
E-MAIL: sethuram@math.arizona.edu

COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
212 MERCER ST.
NEW YORK, NEW YORK 10012
USA
E-MAIL: varadhan@cims.nyu.edu