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A CLUSTERING LAW FOR SOME DISCRETE ORDER STATISTICS

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Abstract

Let $X_1, X_2, \dots, X_n$ be a sequence of independent, identically distributed positive integer random variables with distribution function F. Anderson (1970) proved a variant of the law of large numbers by showing that the sample maximum moves asymptotically on two values if and only if F satisfies a 'clustering' condition, $\lim_{n\to\infty}[1 - F(n + 1)]/[1 - F(n)] = 0$. In this article, we generalize Anderson's result and show that it is robust by proving that, for any $r \geq 0$, the sample maximum and other extremes asymptotically cluster on $r + 2$ values if and only if $\lim_{n\to\infty}[1 - F(n + r + 1)]/[1 - F(n)] = 0$. Together with previous work which considered other asymptotic properties of these sample extremes, a more detailed asymptotic clustering structure for discrete order statistics is presented.

Keywords: Order statistics; law of large numbers; extremes; asymptotics; clustering

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1. Introduction and results

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent, identically distributed (i.i.d.) real-valued random variables with common cumulative distribution function $F(x) = \mathrm{P}(X_1 \leq x) < 1$ for all $x < \infty$. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics in a sample of size n. The 'maximal' order statistic is $X_{(n)}$ and, for $k \geq 0$, $X_{(n-k)}$ is called the 'extreme' order statistic.

The classical asymptotic theory of the sequence $\{X_{(n)}\}$ includes Gnedenko's necessary and sufficient condition for a law of large numbers to hold (see Gnedenko (1943) or Galambos (1987) for a modern account).

Proposition 1.1. *There exists a sequence $\{v_n\}$ such that, for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathrm{P}(|X_{(n)} - v_n| \geq \varepsilon) = 0$$

if and only if, for all $x > 0$,

$$\lim_{y\to\infty} \frac{1 - F(x + y)}{1 - F(y)} = 0. \tag{1.1}$$

When the random variables X_n are discrete, say integer valued, then of course Gnedenko's theorem cannot hold since, for $x < 1$, the limit appearing in (1.1) is 1. However, something can still be said in this case.

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Henceforth, let us specify that the variables $X_1, X_2, \ldots$ take values on the positive integers. We define $p_t = \mathrm{P}(X_1 = t)$ and set $\bar{p}_t = 1 - F(t-1) = \sum_{j \geq t} p_j$ for $t \geq 1$. We will assume throughout, unless specified otherwise, that $p_t > 0$ for infinitely many $t \geq 1$ (so that $F(t) < 1$ for all $t \geq 1$). There will be occasion, however, in Sections 1.2 and 1.3, when we will assume the stronger condition that p_t is positive eventually, namely when F belongs to

$$\mathbb{S} = \{F : p_t > 0 \text{ for all } t \geq t_0, \text{ some } t_0 \geq 1\}.$$

Anderson (1970) proved the following result under the assumption that $F \in \mathbb{S}$, although virtually the same proof holds under our more general assumption that $F(t) < 1$ for all $t \geq 1$.

Proposition 1.2. *There exists an integer sequence* $\{v_n\}$ *such that*

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)} = v_n \text{ or } v_n + 1) = 1$$

if and only if

$$\lim_{n\to\infty} \frac{1 - F(n+1)}{1 - F(n)} = 0. \tag{1.2}$$

Note that Anderson's condition (1.2) is Gnedenko's condition (1.1) only for $x = 1$, and would therefore give the weaker result. Anderson's condition (1.2) can be equivalently stated as $\lim_{t\to\infty} p_t/\bar{p}_t = 1$, or that the hazard rate converges to 1.

In this paper, motivated by a previous investigation of discrete order statistics, we extend Anderson's result to other extreme order statistics and also to more general but related discrete distributions. These extensions clarify further the asymptotic structure of the extreme statistics under these distributions in terms of some 'clustering' interpretations.

Consider the following classes of discrete distributions identified by their limiting hazard rate behavior. For integers $r \geq 0$, define

$$\mathbb{A}_r = \left\{F : \lim_{n\to\infty} \frac{1 - F(n+r+1)}{1 - F(n)} = 0\right\}.$$

Note that Anderson's condition (1.2) reduces to $F \in \mathbb{A}_0$. It is easy to see that $\mathbb{A}_r \subset \mathbb{A}_s$ for $r \leq s$ (see Lemma 1.1 below), and so $\mathbb{A}_r$ can be thought of as a generalization of Anderson's distributions. The distributions in $\mathbb{A}_r$ are roughly characterized by 'light' tails. For instance, the Poisson distribution lies in $\mathbb{A}_0$, but the geometric distribution, with positive limiting hazard rate, belongs to no $\mathbb{A}_r$ for $r \geq 0$.

The following was stated in the penultimate part of Theorem 1.3 of Athreya and Sethuraman (2001). (We point out that the proof of Theorem 1.3 in Athreya and Sethuraman (2001) in fact assumes the condition $F \in \mathbb{S}$ but this is not noted explicitly.)

Proposition 1.3. *Consider a distribution* $F \in \mathbb{S}$*. Then, for* $r \geq 0$*,* $F \in \mathbb{A}_r$ *if and only if*

$$\lim_{n\to\infty} \mathrm{P}(|X_{(n)} - X_{(n-1)}| > r + 1) = 0.$$

This result suggests a sort of 'clustering' of extreme values $X_{(n)}$ and $X_{(n-1)}$ under distribution $F \in \mathbb{A}_r$ on sets of cardinality $r + 2$.

Our main results generalize Proposition 1.2 and the 'if' part of Proposition 1.3, and show that a certain 'clustering' does hold, not only for the first two extreme statistics, but for the collections $\{X_{(n)}, \ldots, X_{(n-k)}\}$ for $k \geq 0$ and go on to specify a set of clustering values.

Theorem 1.1. *For $r \geq 0$, there exists an integer sequence $v_{n,r} = v_{n,r}(F)$ for $n \geq 1$ such that*

$$\lim_{n\to\infty} \mathrm{P}(X_{(n-k)} \in \{v_{n,r}, v_{n,r}+1, \ldots, v_{n,r}+r+1\}) = 1 \quad \textit{for each } k \geq 0$$

if and only if F belongs to $\mathbb{A}_r$.

We remark that the sequence $v_{n,r}$ can be taken explicitly as $v_{n,r} = \lfloor u_{n,r+1} + \frac{1}{2} \rfloor$, where $u_{n,r+1} = \sup\{z : 1 - F_{r+1}(z) \geq 1/n\}$, F_{r+1} is defined in Section 2 and $\lfloor \cdot \rfloor$ denotes the integer-part function. Define $\mathbb{V}_{n,r}$ to be the set of clustering values,

$$\mathbb{V}_{n,r} = \{v_{n,r}, v_{n,r}+1, \ldots, v_{n,r}+r+1\}.$$

An immediate corollary of Theorem 1.1 which specifies asymptotic clustering on $\mathbb{V}_{n,r}$ is the following.

Corollary 1.1. *For $r \geq 0$, there exists an integer sequence $v_{n,r} = v_{n,r}(F)$ for $n \geq 1$ such that*

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)}, \ldots, X_{(n-k)} \in \mathbb{V}_{n,r}) = 1 \quad \textit{for all } k \geq 0$$

if and only if F belongs to $\mathbb{A}_r$.

Given now that the extremes hover on the set $\mathbb{V}_{n,r}$ asymptotically, it is natural to ask how the extreme values move inside $\mathbb{V}_{n,r}$. With respect to a large class of distributions in $\mathbb{A}_0$ and $\mathbb{A}_r \setminus \mathbb{A}_{r-1}$ for $r \geq 1$, we show in Section 1.2 that the extremes move back and forth on the endpoints $v_{n,r}$ and $v_{n,r}+r+1$ infinitely often (i.o.) with probability 1 (see Proposition 1.6). With additional assumptions, we also show that the extremes move on all points in $\mathbb{V}_{n,r}$ infinitely often with probability 1 (see Proposition 1.7). It is interesting to note, on the other hand, that intermediate movement (that is, movement on $\mathbb{V}_{n,r} \setminus \{v_{n,r}, v_{n,r}+r+1\}$) may not be possible for some examples (see Example 1.1). Any more precise determination of the boundaries separating these phenomena is open however.

We remark now on the plan of the paper. In Section 1.1 we make some remarks on the structure of the distribution sets $\mathbb{A}_r$ for $r \geq 0$. Then, in Section 1.2, we prove the fine clustering properties alluded to above. In Section 1.3, we comment on some complements to Theorem 1.1. Finally, in Section 2, we prove Theorem 1.1.

1.1. Classes $\mathbb{A}_r$

An equivalent characterization of the set $\mathbb{A}_r$ is that $\mathbb{A}_r$ contains those distributions such that the probability of expiring by time $n+r$ given that one has lived up to time n tends to 1,

$$\mathbb{A}_r = \left\{ F : \lim_{n\to\infty} \frac{1}{\bar{p}_n} \sum_{l=0}^{r} p_{n+l} = 1 \right\}.$$

As mentioned before, $\mathbb{A}_r$ loosely characterizes distributions which 'cluster' in a certain way on low values or have light tails. As r grows, these tails grow heavier. More precisely, there is a natural 'grading' of the classes $\mathbb{A}_r$.

Lemma 1.1. *For $r \geq 0$,*

$$\mathbb{A}_r \subset \mathbb{A}_{r+1}.$$

Proof. This inclusion follows easily from the identity

$$\frac{1 - F(n+r+2)}{1 - F(n)} = \frac{1 - F(n+r+2)}{1 - F(n+1)} \frac{1 - F(n+1)}{1 - F(n)}.$$

Later, in Proposition 1.4, we specify distributions belonging to $\mathbb{A}_{r+1} \setminus \mathbb{A}_r$ for $r \geq 0$.

It will be useful now to compare and contrast the sets $\mathbb{A}_r$ with the following classes of distributions. Define, for $r \geq 0$,

$$\mathbb{B} = \left\{F : \lim_{n\to\infty} \frac{p_n}{\bar{p}_n} = 0\right\},$$
$$\mathbb{C}_r = \left\{F : \lim_{n\to\infty} \frac{p_{n+r}}{\bar{p}_n} = 0\right\}.$$

The set $\mathbb{B}$, in contrast to the sets $\mathbb{A}_r$, consists of distributions with 'heavy' tails. For instance, the zeta distributions belong to $\mathbb{B}$. In fact, $\mathbb{B}$ and the sets $\mathbb{A}_r$ for $r \geq 0$ are disjoint.

Lemma 1.2. *For each $r \geq 0$,*

$$\mathbb{A}_r \cap \mathbb{B} = \varnothing.$$

Proof. We have

$$\frac{1 - F(n+r+1)}{1 - F(n)} = \prod_{l=0}^{r} \frac{1 - F(n+l+1)}{1 - F(n+l)} = \prod_{l=0}^{r} \left[1 - \frac{p_{n+l}}{\bar{p}_{n+l}}\right].$$

Then, for a distribution $\{p_n\} \in \mathbb{B}$, we have that $(1 - F(n+r+1))/(1 - F(n)) \to 1$ as $n \to \infty$. Hence, $F \notin \mathbb{A}_r$ for any $r \geq 0$.

With respect to the sets $\mathbb{C}_r$, it is immediate that $\mathbb{C}_0 = \mathbb{B}$. But, on the other hand, $\mathbb{C}_r$ contains both $\mathbb{B}$ and $\mathbb{A}_{r-1}$ for $r \geq 1$.

Lemma 1.3. *For $r \geq 1$,*

$$\mathbb{C}_r \supset \mathbb{B} \cup \mathbb{A}_{r-1}.$$

Proof. Write

$$\frac{p_{n+r}}{\bar{p}_n} = \frac{p_{n+r}}{\bar{p}_{n+r}} \frac{p_{n+r}}{\bar{p}_n}.$$

Observe that both factors on the right-hand side are bounded by 1. Moreover, as $n \uparrow \infty$, $p_{n+r}/\bar{p}_{n+r}$ vanishes when $\{p_n\} \in \mathbb{B}$ and $p_{n+r}/\bar{p}_n$ vanishes when $\{p_n\} \in \mathbb{A}_{r-1}$. These observations imply the lemma.

In general, we can construct distributions in $\mathbb{C}_r \setminus (\mathbb{B} \cup \mathbb{A}_{r-1})$ for $r \geq 1$. However, the sets $\mathbb{C}_r$ and $\mathbb{A}_r \setminus \mathbb{A}_{r-1}$ are disjoint for $r \geq 1$.

Lemma 1.4. *For $r \geq 1$,*

$$\mathbb{C}_r \cap (\mathbb{A}_r \setminus \mathbb{A}_{r-1}) = \varnothing.$$

Proof. We have

$$\frac{\sum_{l=0}^{r-1} p_{n+l}}{\bar{p}_n} = \frac{\sum_{l=0}^{r} p_{n+l}}{\bar{p}_n} - \frac{p_{n+r}}{\bar{p}_n}.$$

If $\{p_n\} \in \mathbb{A}_r \cap \mathbb{C}_r$, the right-hand side above vanishes as $n \uparrow \infty$, and so $\{p_n\} \in \mathbb{A}_{r-1}$.

At this point, we specify a class of 'cyclic' distributions which belong to $\mathbb{A}_r \setminus \mathbb{A}_{r-1}$ for $r \geq 1$. Let now $\mathbb{Z}_m = \{1, 2, \ldots, m\}$ for $m \geq 1$.

Proposition 1.4. *Let Y be a random variable whose distribution function F_Y belongs to $\mathbb{A}_0$. Let also $r \geq 1$, and suppose that $\{Z_n\}$ is a family of random variables on $\mathbb{Z}_{r+1}$, independent of Y, such that* $\liminf_{n\to\infty} \mathrm{P}(Z_n = r+1) > 0$. *Then*

$$\text{the law of } X = (r+1)(Y-1) + Z_Y \text{ belongs to } \mathbb{A}_r \setminus \mathbb{A}_{r-1}. \tag{1.3}$$

The variable X has the interpretation that Y fixes a certain level and Z_Y determines, through an $(r+1)$-sided die roll, how much more to assign to X. We will call variables X satisfying (1.3) 'full $(r+1)$-cyclic random variables with level Y and increment Z_Y'. We point out also that the 'full $(r+1)$-cyclic' variables include those distributions where the increment does not depend on the level, $Z_n \equiv Z$. In this 'homogeneous' case, X is generated more simply by level Y and an independent $(r+1)$-sided die roll.

In addition, we remark that it is tempting to think that all distributions in $\mathbb{A}_r \setminus \mathbb{A}_{r-1}$ are generalizations of the full $(r+1)$-cyclic laws identified in (1.3). However, the rate at which the hazard rate of Y vanishes seems to be involved in proving a converse, and the issue is not further pursued here.

The following will be of help in proving Proposition 1.4.

Lemma 1.5. *Let $\{p_t\} \in \mathbb{A}_0$. Let also $\{c_1(t)\}$ and $\{c_2(t)\}$ be sequences of numbers* $0 \leq c_1(t)$, $c_2(t) \leq 1$ *such that* $\liminf_{t\to\infty} c_1(t) > 0$. *Then*

$$\lim_{t\to\infty} \frac{c_1(t)p_t}{c_1(t)p_t + \bar{p}_{t+1}} = 1 \tag{1.4}$$

and

$$\lim_{t\to\infty} \frac{c_2(t)p_{t+1}}{c_1(t)p_t + \bar{p}_{t+1}} = 0. \tag{1.5}$$

Proof. As $\bar{p}_{t+1} = \bar{p}_t - p_t$, we may rewrite the denominator in (1.4) as

$$c_1(t)p_t + \bar{p}_{t+1} = \bar{p}_t\left[1 + (c_1(t) - 1)\frac{p_t}{\bar{p}_t}\right].$$

But also, since

$$\frac{p_n}{\bar{p}_{n+1}} = \frac{p_n}{\bar{p}_n(1 - p_n/\bar{p}_n)},$$

we can write the denominator in (1.5) as

$$\begin{aligned} c_1(t)p_t + \bar{p}_{t+1} &= \bar{p}_{t+1}\left[1 + c_1(t)\frac{p_t}{\bar{p}_{t+1}}\right] \\ &= \bar{p}_{t+1}\left[1 + c_1(t)\frac{p_t}{\bar{p}_t(1 - p_t/\bar{p}_t)}\right]. \end{aligned}$$

From these calculations, as $\lim_{t\to\infty} p_t/\bar{p}_t = 1$, the lemma follows straightforwardly.

We now prove Proposition 1.4 for $r = 1$. A generalized argument for $r \geq 1$ is sketched in Appendix A.

Proof of Proposition 1.4 for $r = 1$. Let $\{p_t\} \in \mathbb{A}_0$ be the distribution of Y and let $(\alpha_t, 1-\alpha_t)$ be the distribution of Z_t on $\mathbb{Z}_2$ where $\liminf_{t\to\infty} 1 - \alpha_t > 0$ and so $\limsup_{t\to\infty} \alpha_t < 1$. Then the distribution of X is the distribution $\{q_t\}$ given by

$$q_t = \begin{cases} \alpha_t p_{(t+1)/2} & \text{for } t \text{ odd}, \\ (1 - \alpha_t)p_{t/2} & \text{for } t \text{ even}. \end{cases}$$

We first show that $X \notin \mathbb{A}_0$. Indeed, we can compute that

$$\begin{aligned}\bar{q}_t &= q_t + q_{t+1} + \cdots \\ &= \begin{cases} \alpha_t p_{(t+1)/2} + (1-\alpha_t) p_{(t+1)/2} + \cdots = \bar{p}_{(t+1)/2} & \text{for } t \text{ odd,} \\ (1-\alpha_t) p_{t/2} + \alpha p_{(t+2)/2} + \cdots = (1-\alpha_t) p_{t/2} + \bar{p}_{(t+2)/2} & \text{for } t \text{ even.} \end{cases}\end{aligned}$$

Then

$$\frac{q_t}{\bar{q}_t} = \begin{cases} \dfrac{\alpha_t p_{(t+1)/2}}{\bar{p}_{(t+1)/2}} & \text{for } t \text{ odd,} \\[2ex] \dfrac{(1-\alpha_t) p_{t/2}}{(1-\alpha_t) p_{t/2} + \bar{p}_{t/2+1}} & \text{for } t \text{ even.} \end{cases}$$

As $\lim_{n\to\infty} p_n/\bar{p}_n = 1$, we conclude that, for t odd, $\limsup_{t\to\infty} q_{2t-1}/\bar{q}_{2t-1} < 1$, but that, for t even, $\lim_{t\to\infty} q_{2t}/\bar{q}_{2t} = 1$ using (1.4) in Lemma 1.5. Therefore, $\{q_t\} \notin \mathbb{A}_0$.

However, X does belong to $\mathbb{A}_1$. Indeed,

$$\frac{q_t + q_{t+1}}{\bar{q}_t} = \begin{cases} \dfrac{p_{(t+1)/2}}{\bar{p}_{(t+1)/2}} & \text{for } t \text{ odd,} \\[2ex] \dfrac{(1-\alpha_t) p_{t/2}}{(1-\alpha_t) p_{t/2} + \bar{p}_{(t+2)/2}} + \dfrac{\alpha_t p_{(t+2)/2}}{(1-\alpha_t) p_{t/2} + \bar{p}_{(t+2)/2}} & \text{for } t \text{ even.} \end{cases}$$

Clearly, for t odd, the fraction $(q_t + q_{t+1})/\bar{q}_t$ converges to 1. But, also for t even, from Lemma 1.5, this fraction tends to 1. Hence, $\lim_{t\to\infty}(q_t + q_{t+1})/\bar{q}_t = 1$ and so $\{q_t\} \in \mathbb{A}_1$.

1.2. Fine clustering

We investigate the clustering interpretation of Corollary 1.1 further. A natural starting point in this regard is to ask about the asymptotic behavior of the extremes among the clustering values. Do they move together or separate themselves from each other with high probability? The answer to the last query is in the negative on both hypotheses, and is addressed in Propositions 1.6 and 1.7 and Example 1.1.

Consider the following result from the first part of Theorem 1.3 in Athreya and Sethuraman (2001).

Proposition 1.5. *Consider a distribution $F \in \mathbb{S}$.*

(i) *For each $l \geq 1$, $F \in \mathbb{B}$ if and only if the limit*

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)} = X_{(n-1)} = \cdots = X_{(n-l+1)} > X_{(n-l)})$$

exists.

(ii) *For each $m \geq 1$, $F \in \mathbb{B} \cup \mathbb{A}_{m-1}$ if and only if the limit*

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)} > X_{(n-1)} + m)$$

exists.

(iii) *For each $m \geq 1$, $F \in \mathbb{C}_m$ if and only if the limit*

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)} = X_{(n-1)} = \cdots = X_{(n-l+1)} > X_{(n-l)} + m)$$

exists for each $l \geq 2$.

A consequence of this proposition which illuminates fine clustering properties of the extremes is the following. To simplify notation, define, for $r \geq 0$,

$$\mathbb{D}_r = \begin{cases} \mathbb{A}_0 & \text{for } r = 0, \\ \mathbb{A}_r \setminus \mathbb{A}_{r-1} & \text{for } r \geq 1. \end{cases}$$

Proposition 1.6. *Consider a distribution $F \in \mathbb{D}_r \cap \mathbb{S}$ for $r \geq 0$. Then, for each $k \geq 1$ and $1 \leq l \leq k$,*

$$\mathrm{P}(X_{(n)} = \cdots = X_{(n-l+1)} = v_{n,r} + r + 1 \text{ and } X_{(n-l)} = \cdots = X_{(n-k)} = v_{n,r} \text{ i.o.}) = 1$$

and

$$\mathrm{P}(X_{(n)} = X_{(n-1)} = \cdots = X_{(n-k)} = v_{n,r} \text{ i.o.}) = 1.$$

Proof. For $r = 0$, if $F \in \mathbb{D}_r$, then necessarily $F \notin \mathbb{B}$ by Lemma 1.2. Also, for $r \geq 1$, if $F \in \mathbb{D}_r$, then we must have $F \notin \mathbb{C}_r$ by Lemma 1.4, and in particular $F \notin \mathbb{B} \cup \mathbb{A}_{r-1}$ by Lemma 1.3.

Then, by Proposition 1.5, when $F \in \mathbb{D}_r$, the limit

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)} = \cdots = X_{(n-l+1)} > X_{(n-l)} + r) \tag{1.6}$$

does not exist for $l \geq 1$. However, by Corollary 1.1,

$$\lim_{n\to\infty} \mathrm{P}(X_{(n)}, X_{(n-k)} \notin \mathbb{V}_{n,r}) = 0, \tag{1.7}$$

where of course the diameter of $\mathbb{V}_{n,r}$ is $r + 1$. Hence, using (1.7) to decompose (1.6), we have that $\lim_{n\to\infty} \mathrm{P}(E_{n,l,k,r})$ does not exist where, for $n \geq k \geq l \geq 1$,

$$E_{n,l,k,r} = \{X_{(n)} = \cdots = X_{(n-l+1)} = v_{n,r} + r + 1 \text{ and } X_{(n-l)} = \cdots = X_{(n-k)} = v_{n,r}\}.$$

Therefore,

$$\mathrm{P}(E_{n,l,k,r} \text{ i.o.}) \geq \limsup_{n\to\infty} \mathrm{P}(E_{n,l,k,r}) > 0. \tag{1.8}$$

Similarly, by Theorem 1.1, as $F \notin \mathbb{A}_{r-1}$, we have that $\mathrm{P}(X_{(n)} \in \{v_{n,r}+1, \ldots, v_{n,r}+r+1\}) \not\to 1$. But, as $F \in \mathbb{A}_r$, we have $\mathrm{P}(X_{(n)} \in \mathbb{V}_{n,r}) \to 1$ from the positive part of Theorem 1.1. Therefore, $\mathrm{P}(X_{(n)} = v_{n,r}) \not\to 0$. Hence, as $\mathrm{P}(X_{(n)} \geq X_{(n-k)} \in \mathbb{V}_{n,r}) \to 1$, by Corollary 1.1 we have that

$$\mathrm{P}(X_{(n)} = \cdots = X_{(n-k)} = v_{n,r} \text{ i.o.}) \geq \limsup_{n\to\infty} \mathrm{P}(X_{(n)} = \cdots = X_{(n-k)} = v_{n,r}) > 0. \tag{1.9}$$

We now argue that the infinitely often events in (1.8) and (1.9) are with full measure. For each $j \geq 0$, let $G_j = \{\lim_{n\to\infty} X_{(n-j)} = \infty\}$. Then, as $\mathrm{P}(G_j) = 1$ for $j \geq 0$, the events $\bigcap_{j=0}^k G_j \cap \{X_{(n)} = \cdots = X_{(n-k)} = v_{n,r} \text{ i.o.}\}$ and $\bigcap_{j=0}^k G_j \cap \{E_{n,l,k,r} \text{ i.o.}\}$ belong to the tail σ-field of the i.i.d. observations $\{X_i\}$. Then, by Kolmogorov's 0–1 law and some set-algebra, we have that $\mathrm{P}(X_{(n)} = \cdots = X_{(n-k)} = v_{n,r} \text{ i.o.}) = \mathrm{P}(E_{n,l,k,r} \text{ i.o.}) = 1$.

The result indicates that the sample extremes $X_{(n)}, X_{(n-1)}, \ldots, X_{(n-k)}$, under distributions $F \in \mathbb{D}_r \cap \mathbb{S}$, behave quite irregularly, moving on top of each other and as far as the maximum of $r + 1$ units away from each other infinitely often. What is not mentioned in the proposition is if the extremes move to intermediate distances relative to each other infinitely often. It is interesting to realize that such intermediate movement may not be possible.

Example 1.1. Let $\{p_t\} \in \mathbb{A}_0$, and consider the distribution $\{q_t\}$ given by

$$q_t = \begin{cases} (1-\alpha_{t/2})p_{t/2} & \text{for } t \text{ even,} \\ \alpha_{(t+1)/2}p_{(t+1)/2} & \text{for } t \text{ odd,} \end{cases}$$

where $0 \le \alpha_t < 1$ and $\sum_t \alpha_t < \infty$ is a summable sequence of numbers. By Proposition 1.4 for $r = 1$, we have that $\{q_t\} \in \mathbb{A}_1 \setminus \mathbb{A}_0$. Moreover, the odd and even probabilities q_{2t-1} and q_{2t} can be interpreted as the chance of choosing t under $\{p_t\}$ and then obtaining heads or tails respectively in an independent coin-toss Z_t with $\mathrm{P}(Z_t = \mathrm{H}) = \alpha_t$ and $\mathrm{P}(Z_t = \mathrm{T}) = 1 - \alpha_t$. Then, for the i.i.d. sequence $\{X_i\}$ under $\{q_t\}$, for each $l \ge 0$, as $X_{(n-l)} \uparrow \infty$ almost surely,

$$\mathrm{P}(X_{(n-l)} \text{ is odd i.o.}) = \mathrm{P}(X_{(n-l)} \uparrow \infty, X_{(n-l)} \text{ is odd i.o.}) \le \mathrm{P}(Z_t = \mathrm{H} \text{ i.o.}) = 0.$$

In particular, the extreme values $X_{(n)}, \dots, X_{(n-k)}$ take even values eventually with probability 1. So, almost surely, the relative intermediate distance of 1 is not achieved infinitely often by these extremes.

On the other hand, we might believe, under some natural assumptions, that the extremes under distributions $F_X \in \mathbb{A}_r \setminus \mathbb{A}_{r-1}$ for $r \ge 1$ considered in Proposition 1.4 should achieve all possible relative intermediate distances between 0 and $r + 1$ infinitely almost surely. This is indeed the case.

Let $k, r \ge 0$, and define a set of ordered vectors,

$$C(k, r) = \{\boldsymbol{x} = \langle x_0, \dots, x_k \rangle : r + 1 \ge x_0 \ge x_1 \ge \cdots \ge x_k \ge 0\}.$$

Define also for $\boldsymbol{x} \in C(k, r)$ the shifted vector $\boldsymbol{x}_{n,r} = \langle x_0 + v_{n,r}, \dots, x_k + v_{n,r} \rangle \in \mathbb{V}_{n,r}^{k+1}$.

Proposition 1.7. *Let $r \ge 1$, and let X be a full $(r + 1)$-cyclic variable with level Y and increment Z_Y satisfying (1.3). Let $F_X \in \mathbb{A}_r \setminus \mathbb{A}_{r-1}$ and $F_Y \in \mathbb{A}_0$ be the distributions of X and Y respectively. Suppose also that $F_Y \in \mathbb{S}$ and that $\{Z_n\}$ satisfies the condition*

$$\liminf_{n\to\infty} \mathrm{P}(Z_n = l) > 0 \quad \textit{for all } l \in \mathbb{Z}_{r+1}.$$

Then

$$v_{n,r}(F_X) = (r + 1)v_{n,0}(F_Y).$$

Also, for $k \ge 0$, when $\boldsymbol{x} \in C(k, r)$,

$$\mathrm{P}(\langle X_{(n)}, X_{(n-1)}, \dots, X_{(n-k)} \rangle = \boldsymbol{x}_{n,r} \text{ i.o.}) = 1.$$

Proof. Let $u \ge 1$ and let $\{Y_{(n)}, \dots, Y_{(n-u)}\}$ be the first $u + 1$ sample extremes under F_Y in a sample of size $n > u$. From (1.8) applied to the $\{Y_i\}$ sample, we have for $1 \le m \le u$ that $\limsup_{n\to\infty} \mathrm{P}(E_{n,m,u,0}) > 0$, where

$$E_{n,m,u,0} = \{Y_{(n)} = \cdots = Y_{(n-m+1)} = Y_{(n-m)} + 1 = \cdots = Y_{(n-u)} + 1 = v_{n,0} + 1\}.$$

On the event $E_{n,m,u,0}$, there are exactly m variables which take the maximum value $v_{n,0} + 1$, and at least $u - m + 1$ variables exactly one unit less in the $\{Y_i\}$ sample. Moreover, as the $\{Z_n\}$ are $\mathbb{Z}_{r+1}$-valued variables, we have $\{X_{(n)}, \dots, X_{(n-m+1)}\} \subset (r + 1)v_{n,0} + \mathbb{Z}_{r+1}$ and $\{X_{(n-m)}, \dots, X_{(n-u)}\} \subset (r + 1)(v_{n,0} - 1) + \mathbb{Z}_{r+1}$ if and only if $E_{n,m,u,0}$ occurs.

Specify now increments $l_0, \ldots, l_{m-1}$ with $r+1 \geq l_0 \geq \cdots \geq l_{m-1} \geq 1$. As the $\{Z_{Y_i}\}$ variables are independent and $\lim_{n\to\infty} Y_{(n-j)} = \infty$ almost surely for $j \geq 0$,

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{P}(X_{(n)} &= (r+1)v_{n,0} + l_0, \ldots, X_{(n-m+1)} = (r+1)v_{n,0} + l_{m-1},\\ &\{X_{(n-m)}, \ldots, X_{(n-u)}\} \subset (r+1)(v_{n,0}-1) + \mathbb{Z}_{r+1})\\ &\geq \limsup_{n\to\infty} \mathrm{P}(E_{n,m,u,0}) \prod_{i=0}^{m-1} \liminf_{n\to\infty} \mathrm{P}(Z_n = l_i) > 0 \end{aligned} \tag{1.10}$$

since $\liminf_{n\to\infty} \mathrm{P}(Z_n = l) > 0$ for all $l \in \mathbb{Z}_{r+1}$ by assumption.

Let us choose now $l_0 = r+1$. With this choice, as $\lim_{n\to\infty} \mathrm{P}(|X_{(n)} - X_{(n-u)}| > r+1) = 0$ from Corollary 1.1, we obtain further that

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{P}(X_{(n)} &= (r+1)(v_{n,0}+1), X_{(n-1)} = (r+1)v_{n,0} + l_1,\\ &\ldots, X_{(n-m+1)} = (r+1)v_{n,0} + l_{m-1},\\ &X_{(n-m)} = \cdots = X_{(n-m-q)} = (r+1)v_{n,0}) > 0.\end{aligned}$$

Moreover, as the stronger claim, $\lim_{n\to\infty} \mathrm{P}(X_{(n)}, \ldots, X_{(n-u)} \in \mathbb{V}_{n,r}) = 1$, also holds from Corollary 1.1, we must have

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{P}(X_{(n)} &= (r+1)(v_{n,0}+1), X_{(n-1)} = (r+1)v_{n,0} + l_1,\\ &\ldots, X_{(n-m+1)} = (r+1)v_{n,0} + l_{m-1}, X_{(n-m)} = \cdots = X_{(n-u)} = (r+1)v_{n,0},\\ &X_{(n)} = v_{n,r} + r + 1, X_{(n-m)} = \cdots = X_{(n-u)} = v_{n,r}) > 0.\end{aligned}$$

This yields $(r+1)v_{n,0} = v_{n,r}$. Substituting this relation into (1.10) and noting once more that $\lim_{n\to\infty} \mathrm{P}(X_{(n)}, \ldots, X_{(n-u)} \in \mathbb{V}_{n,r}) = 1$ (from Corollary 1.1) gives that

$$\begin{aligned}\limsup_{n\to\infty} \mathrm{P}(X_{(n)} &= v_{n,r} + l_0, X_{(n-1)} = v_{n,r} + l_1, \ldots, X_{(n-m+1)} = v_{n,r} + l_{m-1},\\ &X_{(n-m)} = \cdots = X_{(n-u)} = v_{n,r}) > 0.\end{aligned} \tag{1.11}$$

On the other hand, by (1.9), as the assumptions on Y and $\{Z_n\}$ imply that $F_X \in (\mathbb{A}_r \backslash \mathbb{A}_{r-1}) \cap \mathbb{S}$, we have that

$$\limsup_{n\to\infty} \mathrm{P}(X_{(n)} = \cdots = X_{(n-u)} = v_{n,r}) > 0. \tag{1.12}$$

We now associate to each vector $\boldsymbol{x} \in C(k,r)$ one of the above limits. When $\boldsymbol{x}$ is such that $x_k > 0$, we choose (1.11) with $m = u = k+1$ and $x_i = l_i$ for $0 \leq i \leq k$. However, when $\boldsymbol{x}$ is such that $x_{i-1} > 0$ and $x_i = \cdots = x_k = 0$ for $i \leq k$, we choose (1.11) with $m = i$, $u = k$, and $x_j = l_j$ for $0 \leq j \leq i-1$. Finally, when $\boldsymbol{x}$ is such that $x_0 = \cdots = x_k = 0$, we choose (1.12). Taking account of these choices, we have, for each $\boldsymbol{x} \in C(k,r)$, that

$$\mathrm{P}((X_{(n)}, \ldots, X_{(n-k)}) = \boldsymbol{x}_{n,r} \text{ i.o.}) \geq \limsup_{n\to\infty} \mathrm{P}((X_{(n)}, \ldots, X_{(n-k)}) = \boldsymbol{x}_{n,r}) > 0.$$

The argument now is analogous to the part of the proof of Proposition 1.6 following (1.9).

We leave open, however, any more careful investigation of the boundaries in $\mathbb{A}_r \setminus \mathbb{A}_{r-1}$ for $r \geq 1$ which separate distributions leading to the two extreme value behaviors mentioned in Example 1.1 and Proposition 1.7.

1.3. Complements

We comment now on the asymptotic behavior of sample extremes when the underlying distribution F is not in $\mathbb{A}_r$ for any $r \geq 0$, the contrapositive of Theorem 1.1. In some sense, the antithesis of clustering occurs. It turns out that the sample maximum, $X_{(n)}$, is asymptotically distinct from the other extremes and strongly so.

The results are most complete when F is *a priori* restricted to $\mathbb{S}$. We list some of these results below, focusing on the class $\mathbb{B}$.

Anderson (1970) showed that no law of large numbers is possible for the maximal statistic since $\lim_{n\to\infty} \mathrm{P}(|X_{(n)} - v_n| < y) = 0$ for any sequence $\{v_n\}$ and any $y > 0$ if and only if $F \in \mathbb{B}$. Noting Proposition 2.1 below, the same statement can be made to hold for extremes $X_{(n-k)}$ for $k \geq 1$.

Baryshnikov *et al.* (1995) considered the asymptotic chance of a draw and proved that the limit $\lim_{n\to\infty} \mathrm{P}(X_{(n)} > X_{(n-1)})$ exists if and only if $F \in \mathbb{B}$, and that it equals 1 when it exists.

Athreya and Sethuraman (2001) showed that the maximum is strongly separated asymptotically from the other values, $\lim_{n\to\infty} \mathrm{P}(X_{(n)} > X_{(n-1)} + m) = 1$ for any $m \geq 0$, and no type of draw is possible with the maximum, $\lim_{n\to\infty} \mathrm{P}(X_{(n)} = \cdots = X_{(n-k+1)} > X_{(n-k)}) = 0$ for any $k \geq 2$, if and only if $F \in \mathbb{B}$, among other results.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 follows from a generalization of the method of Anderson (1970). The strategy is in two steps.

Step 1. We map each distribution function $F \in \mathbb{A}_r$ to a continuous distribution function, F_r say, which satisfies the assumptions of Gnedenko's law of large numbers.

Step 2. We then unravel the mapping to get the desired statement for F.

There is no problem in finding a map for Step 1. The difficulty, however, is in choosing this map so that Step 2 can be managed. The contribution here is in finding such a map which generalizes Anderson's for the case $r = 0$.

Let $m \geq 1$ be fixed and, for $x > 0$ and $0 \leq l \leq m-1$, define $\lfloor x \rfloor_{m,l}$ to be the largest integer of the form $km + l$ less than x for $k \geq 0$. Of course, $\lfloor x \rfloor = \lfloor x \rfloor_{1,0}$ represents as usual the greatest integer less than x.

Let F be a discrete distribution function on the positive integers. For each $n \geq 1$, let $h(n) = -\log(1 - F(n))$. Note that h is an increasing function on the positive integers and $\lim_{n\to\infty} h(n) = \infty$. Define now $h_{m,l}$ to be an 'interpolated' version of h: for $x \geq 0$,

$$h_{m,l}(x) = h(\lfloor x \rfloor_{m,l}) + \frac{x - \lfloor x \rfloor_{m,l}}{m}(h(\lfloor x \rfloor_{m,l} + m) - h(\lfloor x \rfloor_{m,l})).$$

Note that $h_{m,l}$ is continuous, increasing, and also that $\lim_{x\to\infty} h_{m,l}(x) = \infty$. Define now, for $x \geq m$, the distribution function $F_{m,l}$ by its tail, $1 - F_{m,l}(x) = \mathrm{e}^{-h_{m,l}(x)}$ (the definition for $x < m$ is not relevant for what follows). Define also the distribution function F_m by $F_m(x) = \max\{F_{m,0}(x), \ldots, F_{m,m-1}(x)\}$.

Some simple properties seen from the definitions are $F_{m,l}(\lfloor x \rfloor_{m,l}) = F(\lfloor x \rfloor_{m,l})$ and, as $\lfloor x \rfloor_{m,l} \leq x < \lfloor x \rfloor_{m,l} + m$,

$$1 - F(\lfloor x \rfloor_{m,l} + m) \leq 1 - F_{m,l}(x) \leq 1 - F(\lfloor x \rfloor_{m,l}).$$

Also, as $\lfloor x\rfloor = \lfloor x\rfloor_{m,l_0}$ for some $0 \le l_0 \le m-1$, we have

$$F(x) = F(\lfloor x\rfloor) = F_{m,l_0}(\lfloor x\rfloor_{m,l_0}) \le F_m(\lfloor x\rfloor) \le F_m(x). \tag{2.1}$$

It will turn out that this last inequality is the key to the unraveling in the second step mentioned above.

Let us also define the sequence $\{u_{n,m}\}_{n\ge 1}$ for $m \ge 1$ by

$$u_{n,m} = \sup\left\{z : 1 - F_m(z) \ge \frac{1}{n}\right\}.$$

As F_m is continuous, $1 - F_m(u_{n,m}) = 1/n$. Also, as $F_m(z) < 1$ for $z < \infty$, we have, for fixed m, that $u_{n,m}$ increases to infinity as n goes to infinity.

The following two lemmas establish Step 1.

Lemma 2.1. *Suppose that F belongs to $\mathbb{A}_r$ for $r \ge 0$. Then, for $0 \le l \le r$ and $0 < y < \frac{1}{2}$,*

$$\lim_{x\to\infty} \frac{1 - F_{r+1,l}(x+y)}{1 - F_{r+1,l}(x)} = 0.$$

Proof. By assumption, we have that $[1 - F(n + r + 1)]/[1 - F(n)]$ vanishes as n tends to infinity. This implies that $\lim_{n\to\infty}[h(n + r + 1) - h(n)] = \infty$. We now show that

$$\lim_{x\to\infty}[h_{r+1,l}(x + y) - h_{r+1,l}(x)] = \infty,$$

from which Lemma 2.1 easily follows.

We have

$$\begin{aligned} &h_{r+1,l}(x + y) - h_{r+1,l}(x)\\ &\quad= [h(\lfloor x + y\rfloor_{r+1,l}) - h(\lfloor x\rfloor_{r+1,l})]\\ &\qquad+ \frac{x + y - \lfloor x + y\rfloor_{r+1,l}}{r + 1}(h(\lfloor x + y\rfloor_{r+1,l} + r + 1) - h(\lfloor x + y\rfloor_{r+1,l}))\\ &\qquad- \frac{x - \lfloor x\rfloor_{r+1,l}}{r + 1}(h(\lfloor x\rfloor_{r+1,l} + r + 1) - h(\lfloor x\rfloor_{r+1,l})). \end{aligned} \tag{2.2}$$

Consider now the cases when

(i) $r + 1 - y \le x - \lfloor x\rfloor_{r+1,l} \le r + 1 - y/10$,

(ii) $x - \lfloor x\rfloor_{r+1,l} > r + 1 - y/10$,

(iii) $x - \lfloor x\rfloor_{r+1,l} < r + 1 - y$.

(i) Here, $\lfloor x + y\rfloor_{r+1,l} = \lfloor x\rfloor_{r+1,l} + r + 1$ and the right-hand side of (2.2) becomes

$$\begin{aligned} &[h(\lfloor x\rfloor_{r+1,l} + r + 1) - h(\lfloor x\rfloor_{r+1,l})]\\ &\qquad+ \frac{x - \lfloor x\rfloor_{r+1,l} - r - 1 + y}{r + 1}(h(\lfloor x\rfloor_{r+1,l} + 2r + 2) - h(\lfloor x\rfloor_{r+1,l} + r + 1))\\ &\qquad- \frac{x - \lfloor x\rfloor_{r+1,l}}{r + 1}(h(\lfloor x\rfloor_{r+1,l} + r + 1) - h(\lfloor x\rfloor_{r+1,l}))\\ &\quad\ge \frac{y}{10(r + 1)}[h(\lfloor x\rfloor_{r+1,l} + r + 1) - h(\lfloor x\rfloor_{r+1,l})]. \end{aligned}$$

(ii) Here, also $\lfloor x+y \rfloor_{r+1,l} = \lfloor x \rfloor_{r+1,l} + r + 1$ and the right-hand side of (2.2) becomes

$$\begin{aligned}
&[h(\lfloor x \rfloor_{r+1,l} + r + 1) - h(\lfloor x \rfloor_{r+1,l})] \\
&\qquad + \frac{x - \lfloor x \rfloor_{r+1,l} - r - 1 + y}{r+1}(h(\lfloor x \rfloor_{r+1,l} + 2r + 2) - h(\lfloor x \rfloor_{r+1,l} + r + 1)) \\
&\qquad - \frac{x - \lfloor x \rfloor_{r+1,l}}{r+1}(h(\lfloor x \rfloor_{r+1,l} + r + 1) - h(\lfloor x \rfloor_{r+1,l})) \\
&\quad \geq \frac{9y}{10(r+1)}[h(\lfloor x \rfloor_{r+1,l} + 2r + 2) - h(\lfloor x \rfloor_{r+1,l} + r + 1)].
\end{aligned}$$

(iii) Here, $\lfloor x+y \rfloor_{r+1,l} = \lfloor x \rfloor_{r+1,l}$ and the right-hand side of (2.2) becomes

$$\frac{y}{r+1}[h(\lfloor x \rfloor_{r+1,l} + r + 1) - h(\lfloor x \rfloor_{r+1,l})].$$

Putting (i)–(iii) together gives that (2.2) is greater than

$$\frac{y}{10(r+1)} \min\{h(\lfloor x \rfloor_{r+1,l} + r + 1) - h(\lfloor x \rfloor_{r+1,l}), h(\lfloor x \rfloor_{r+1,l} + 2r + 2) - h(\lfloor x \rfloor_{r+1,l} + r + 1)\},$$

which diverges with x.

Lemma 2.2. *Suppose that F belongs to $\mathbb{A}_r$ for $r \geq 0$. Then, for any y with $0 < y < \frac{1}{2}$,*

$$\lim_{x\to\infty} \frac{1 - F_{r+1}(x+y)}{1 - F_{r+1}(x)} = 0.$$

Proof. Observe that, for fixed x, $1 - F_{r+1}(x) = 1 - F_{r+1,l_0}(x)$ for some l_0 with $0 \leq l_0 \leq r$, and so

$$\begin{aligned}
\frac{1 - F_{r+1}(x+y)}{1 - F_{r+1}(x)} &= \frac{1 - F_{r+1}(x+y)}{1 - F_{r+1,l_0}(x)} \\
&\leq \frac{1 - F_{r+1,l_0}(x+y)}{1 - F_{r+1,l_0}(x)} \\
&\leq \max_l \left\{ \frac{1 - F_{r+1,l}(x+y)}{1 - F_{r+1,l}(x)} \right\}.
\end{aligned}$$

The proof now follows from Lemma 2.1.

The following result will be useful in showing that the maximum and other extremes all have the same behavior.

Proposition 2.1. *Let x_n be a sequence of numbers tending to infinity, and let $\varepsilon > 0$ be fixed. Then*

$$\lim_{n\to\infty} \mathrm{P}(|X_{(n)} - x_n| < \varepsilon) = 1$$

if and only if

$$\lim_{n\to\infty} \mathrm{P}(|X_{(n-k)} - x_n| < \varepsilon) = 1 \quad \textit{for all } k \geq 0.$$

Proof. We first recall a standard way to determine the distribution function of the sample extremes. Let $N_n(x) = \sum_{i=1}^n \mathbf{1}_{[X_i > x]}$. Then the distribution of $X_{(n-k)}$ for $k \geq 0$ satisfies

$$\begin{aligned} \mathrm{P}(X_{(n-k)} \leq x) &= 1 - \mathrm{P}(N_n(x) \geq k+1) \\ &= \sum_{i=0}^{k} \binom{n}{i} (1 - F(x))^i F^{n-i}(x). \end{aligned}$$

Let now x_n be a sequence tending to infinity. The proof now follows once we show that

$$F^n(x_n) \to 1 \quad \text{if and only if} \quad \mathrm{P}(X_{(n-k)} \leq x_n) \to 1$$

and

$$F^n(x_n) \to 0 \quad \text{if and only if} \quad \mathrm{P}(X_{(n-k)} \leq x_n) \to 0.$$

Observe now that $F^n(x_n)$ tends to 1 or 0 if and only if $n(1 - F(x_n))$ tends to 0 or ∞ respectively. Correspondingly, $\mathrm{P}(X_{(n-k)} \leq x_n)$ tends to 1 or 0 if and only if $-\log \mathrm{P}(X_{(n-k)} \leq x_n)$ tends to 0 or ∞ respectively.

We have

$$\begin{aligned} &-\log \mathrm{P}(X_{(n-k)} \leq x_n) \\ &\quad = -n \log F(x_n) - \log\left[1 + \sum_{i=1}^{k} n^i \left[\frac{\binom{n}{i}}{n^i}\right] (1 - F(x_n))^i F^{-i}(x_n)\right] \\ &\quad = n(1 - F(x_n))(1 + o(1)) - \log\left[1 + \sum_{i=1}^{k} \frac{1}{i!} (n(1 - F(x_n)))^i (1 + o(1))\right] \end{aligned}$$

as n tends to infinity. In the limit, the 'log' term is subordinate to the first term. This proves the proposition.

We can now prove Theorem 1.1.

Proof of Theorem 1.1. We prove the result for the maximum $X_{(n)}$. The result for the other extremes would then follow from Proposition 2.1.

Sufficiency. From the definition of $u_{n,r+1}$, observe that, for $0 < \varepsilon < \frac{1}{2}$,

$$n(1 - F_{r+1}(u_{n,r+1} + \varepsilon)) \leq \frac{1 - F_{r+1}(u_{n,r+1} + \varepsilon)}{1 - F_{r+1}(u_{n,r+1})}$$

and

$$\frac{1}{n(1 - F_{r+1}(u_{n,r+1} - \varepsilon))} \leq \frac{1 - F_{r+1}(u_{n,r+1})}{1 - F_{r+1}(u_{n,r+1} - \varepsilon)}.$$

Both right-hand sides vanish as n tends to infinity from Lemma 2.1. Therefore, we conclude that

$$\lim_{n\to\infty} F_{r+1}^n(u_{n,r+1} + \varepsilon) = 1 \quad \text{and} \quad \lim_{n\to\infty} F_{r+1}^n(u_{n,r+1} - \varepsilon) = 0. \tag{2.3}$$

Now observe from (2.1) that $F(u_{n,r+1}-\varepsilon) \le F_{r+1}(u_{n,r+1}-\varepsilon)$ and

$$\begin{aligned} F_{r+1}(u_{n,r+1}+\varepsilon) &= \max_l \{F_{r+1,l}(u_{n,r+1}+\varepsilon)\} \\ &\le \max_l \{F_{r+1,l}(\lfloor u_{n,r+1}+\varepsilon\rfloor_{r+1,l}+r+1)\} \\ &= \max_l \{F(\lfloor u_{n,r+1}+\varepsilon\rfloor_{r+1,l}+r+1)\} \\ &= F(\lfloor u_{n,r+1}+\varepsilon\rfloor + r+1) \\ &\le F(u_{n,r+1}+\varepsilon+r+1). \end{aligned}$$

From (2.3),

$$\lim_{n\to\infty} \mathrm{P}\biggl(|X_{(n)}-V_n| < \frac{r+1}{2}+\varepsilon\biggr) = 1$$

with $V_n = u_{n,r+1} + (r+1)/2$. As the maximum takes only integer values, sufficiency in the theorem is proved with $v_{n,r} = \lfloor u_{n,r+1} + \frac{1}{2}\rfloor$.

Necessity. Define D_n by

$$D_n = \begin{cases} v_{n,r} + \dfrac{r+1}{2} & \text{for } r \text{ even},\\[2ex] v_{n,r} + \dfrac{r+1}{2} - \dfrac{1}{4} & \text{for } r \text{ odd}. \end{cases}$$

For $\varepsilon > 0$, we then have

$$\lim_{n\to\infty} \mathrm{P}\biggl(|X_{(n)}-D_n| \le \frac{r+1}{2}+\varepsilon\biggr) = 1 \quad \text{for } r \text{ even}$$

and

$$\lim_{n\to\infty} \mathrm{P}\biggl(-\frac{r+1}{2}-\varepsilon+\frac{1}{4} \le X_{(n)}-D_n \le \frac{r+1}{2}+\varepsilon+\frac{1}{4}\biggr) = 1 \quad \text{for } r \text{ odd}.$$

Let us concentrate on the even case for the moment. In this case,

$$\lim_{n\to\infty} F^n\biggl(D_n+\frac{r+1}{2}+\varepsilon\biggr) = 1 \quad \text{and} \quad \lim_{n\to\infty} F^n\biggl(D_n-\frac{r+1}{2}-\varepsilon\biggr) = 0.$$

From the first limit, we see that D_n must diverge to infinity, and therefore

$$\lim_{n\to\infty} n\biggl(1-F\biggl(D_n+\frac{r+1}{2}+\varepsilon\biggr)\biggr) = 0 \quad \text{and} \quad \lim_{n\to\infty} n\biggl(1-F\biggl(D_n-\frac{r+1}{2}-\varepsilon\biggr)\biggr) = \infty.$$

We may take $\{D_n\}$ to be monotonically increasing, and so, for any integer $a \ge 1$, we may find n such that $D_{n-1} \le a \le D_n$. Therefore,

$$\frac{1-F(a-(r+1)/2-\varepsilon)}{1-F(a+(r+1)/2+\varepsilon)} \ge \frac{1-F(D_n-(r+1)/2-\varepsilon)}{1-F(D_{n-1}+(r+1)/2+\varepsilon)},$$

which diverges to infinity as n tends to infinity. Hence, substituting $\varepsilon = \frac{1}{4}$ into the left-hand side, we have

$$\lim_{a\to\infty} \frac{1-F(a-r/2-1)}{1-F(a+r/2)} = \infty.$$

But this last statement is the same as saying that F belongs to $\mathbb{A}_r$.

For r odd, we follow the same steps as for r even to arrive at the statement that

$$\frac{1-F(a-(r+1)/2+1/4-\varepsilon)}{1-F(a+(r+1)/2+1/4+\varepsilon)} \geq \frac{1-F(D_n-(r+1)/2+1/4-\varepsilon)}{1-F(D_{n-1}+(r+1)/2+1/4+\varepsilon)}$$

diverges to ∞ as $n \uparrow \infty$. Substituting $\varepsilon = \frac{1}{8}$ into the left-hand side, we have that

$$\lim_{a\to\infty} \frac{1-F(a-(r+1)/2)}{1-F(a+(r+1)/2)} = 0.$$

This is equivalent to F belonging to $\mathbb{A}_r$.

Appendix A. Proposition 1.4 for $r \geq 1$

For the reader's convenience, we give a sketch proof of Proposition 1.4 for $r \geq 1$.

Let $\{p_t\}$ and $\{q_t\}$ be the distributions of Y and X respectively. Let $(\alpha_t^1, \dots, \alpha_t^{r+1})$ be the distribution of Z_t where $0 \leq \alpha_t^l \leq 1$ for $0 \leq l \leq r+1$, $\sum_{l=0}^{r+1} \alpha_t^l = 1$, and $\liminf_{t\to\infty} \alpha_t^{r+1} > 0$. Let also $S = \{0, r+1, 2(r+1), \dots\}$. Then

$$q_t = \begin{cases} \alpha_t^1 p_{(t+r)/(r+1)} & \text{for } t \in S+1, \\ \vdots & \vdots \\ \alpha_t^{r+1} p_{t/(r+1)} & \text{for } t \in S+r+1, \end{cases}$$

$$\bar{q}_t = \begin{cases} \bar{p}_{(t+r)/(r+1)} & \text{for } t \in S+1, \\ \left(\sum_{l=2}^{r+1} \alpha_t^l\right) p_{(t+r-1)/(r+1)} + \bar{p}_{(t+2r)/(r+1)} & \text{for } t \in S+2, \\ \vdots & \vdots \\ \alpha_t^{r+1} p_{t/(r+1)} + \bar{p}_{(t+r+1)/(r+1)} & \text{for } t \in S+r+1 \end{cases}$$

and

$$\sum_{l=0}^{r-1} q_{t+l} = \begin{cases} \left(\sum_{l=1}^{r} \alpha_t^l\right) p_{(t+r)/(r+1)} & \text{for } t \in S+1, \\ \left(\sum_{l=2}^{r+1} \alpha_t^l\right) p_{(t+r-1)/(r+1)} & \text{for } t \in S+2, \\ \left(\sum_{l=3}^{r+1} \alpha_t^l\right) p_{(t+r-2)/(r+1)} + \alpha_t^1 p_{(t+2r-1)/(r+1)} & \text{for } t \in S+3, \\ \vdots & \vdots \\ \alpha_t^{r+1} p_{t/(r+1)} + \left(\sum_{l=1}^{r-1} \alpha_t^l\right) p_{(t+r+1)/(r+1)} & \text{for } t \in S+r+1. \end{cases}$$

We now argue that $\{q_t\} \notin \mathbb{A}_{r-1}$. We have

$$\frac{\sum_{l=0}^{r-1} q_{t+l}}{\bar{q}_t} = \begin{cases} \dfrac{(\sum_{l=1}^{r} \alpha_t^l) p_{(t+r)/(r+1)}}{\bar{p}_{(t+r)/(r+1)}} & \text{for } t \in S+1, \\ \dfrac{(\sum_{l=2}^{r+1} \alpha_t^l) p_{(t+r-1)/(r+1)}}{(\sum_{l=2}^{r+1} \alpha_t^l) p_{(t+r-1)/(r+1)} + \bar{p}_{(t+2r)/(r+1)}} & \text{for } t \in S+2, \\ \vdots & \vdots \\ \dfrac{\alpha_t^{r+1} p_{t/(r+1)} + (\sum_{l=1}^{r-1} \alpha_t^l) p_{(t+r+1)/(r+1)}}{\alpha_t^{r+1} p_{t/(r+1)} + \bar{p}_{(t+r+1)/(r+1)}} & \text{for } t \in S+r+1. \end{cases}$$

Then, $\limsup_{t\in S+1, t\uparrow\infty} (\sum_{l=0}^{r-1} q_{t+l})/\bar{q}_t \leq \limsup_{t\uparrow\infty} \sum_{l=1}^{r} \alpha_t^l = 1 - \liminf_{t\uparrow\infty} \alpha_t^{r+1} < 1$. However, on the other hand, by Lemma 1.5, $\limsup_{t\in S+k, t\uparrow\infty} (\sum_{l=0}^{r-1} q_{t+l})/\bar{q}_t = 1$ for $2 \leq k \leq r+1$ as $\liminf_{t\uparrow\infty} \sum_{l=k}^{r+1} \alpha_t^l \geq \liminf_{t\uparrow\infty} \alpha_t^{r+1} > 0$. Therefore $\{q_t\}_{t\geq 1} \notin \mathbb{A}_r$.

But $\{q_t\} \in \mathbb{A}_r$. Indeed,

$$\sum_{l=0}^{r} q_{t+l} = \begin{cases} p_{(t+r)/(r+1)} & \text{for } t \in S+1, \\ \left(\sum_{l=2}^{r+1} \alpha_t^l\right) p_{(t+r-1)/(r+1)} + \alpha_t^1 p_{(t+2r)/(r+1)} & \text{for } t \in S+2, \\ \vdots & \vdots \\ \alpha_t^{r+1} p_{t/(r+1)} + \left(\sum_{l=1}^{r} \alpha_t^l\right) p_{(t+r+1)/(r+1)} & \text{for } t \in S+r+1. \end{cases}$$

Then, using Lemma 1.5, it is not difficult to conclude that $\lim_{t\to\infty} (\sum_{l=0}^{r} q_{t+l})/\bar{q}_t = 1$, meaning that the distribution of X belongs to $\mathbb{A}_r$.

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References

Anderson, C. W. (1970). Extreme value theory for a class of discrete distributions with applications. *J. Appl. Prob.* **7,** 99–113.

Athreya, J. S. and Sethuraman, S. (2001). On the asymptotics of discrete order statistics. *Statist. Prob. Lett.* **54,** 243–249.

Baryshnikov, Y., Eisenberg, B. and Stengle, G. (1995). A necessary and sufficient condition for the existence of the limiting probability of a tie for first place. *Statist. Prob. Lett.* **23,** 203–209.

Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*. Krieger, Melbourne, FL.

Gnedenko, B. V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **44,** 423–453.