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On the asymptotics of discrete order statistics

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Abstract

Let $X_1,X_2,\ldots,X_n$ be a sequence of independent, identically distributed positive integer random variables. We study the asymptotics of the likelihood that the sample maximum is achieved k times and in its spacing relative to the second highest value. Earlier and other results are discussed in context. Also, some investigation is made when the sample is Markovian. Different results emerge in this case.

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1. Introduction and results

Let $X_1,X_2,\ldots,X_n,\ldots$ be a sequence of independent, identically distributed real-valued random variables (i.i.d. r.v.'s) with common cumulative distribution function (c.d.f.) $F(x)=P(X_1 \leqslant x)$. Let $X_{(n)} \geqslant X_{(n-1)} \geqslant \cdots \geqslant X_{(1)}$ be the first n order statistics with $X_{(n)}$ being the "maximal order statistic" among $X_1,X_2,\ldots,X_n$. The study of $X_{(n)}$ has had a long history with such classical triumphs as Gnedenko's neccesary and sufficient conditions for a law of large numbers (LLN) and the central limit theorem (CLT) to hold for $X_{(n)}$ (see Galambos (1987), Leadbetter (1978) for modern accounts, and Gnedenko (1943) for the original article). We briefly recall the LLN result for context: There exists a sequence $\{u_n\}$ such that for all $\varepsilon > 0$, $\lim_{n\to\infty} P(|X_{(n)} - u_n| \geqslant \varepsilon) = 0$ if and only if (iff) for all $x > 0$, $\lim_{y\to\infty} 1 - F(x+y)/1 - F(y) = 0$. In this paper, we are particularly interested in the asymptotics of $X_{(n)}$ when the sample random variables are discrete. Motivated by a number theory problem, a previous investigation by Athreya and Fidkowski (2000) considered a case of these asymptotics. Our intention here is to extend and generalize this work and place it in relation with an earlier study on discrete valued order statistics by Anderson (1970).

The treatment of extreme value theory for discrete r.v.'s is a bit delicate. For example, note that the LLN above cannot be satisfied for integer valued samples, as for $x < 1$ the limit condition is 1. Similar difficulties arise in other considerations and perhaps, as a result, the study of discrete extreme value theory seems to

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be rather sparse. See however Baryshnikov et al. (1995) where in the context of asymptotic draws, the limit behavior of the maximum $X_{(n)}$ was addressed (Theorem 1.2 below).

Let us now specify that the r.v.'s $X_1, X_2, \dots$ take values on the positive integers. We will write $P(X_1 = i) := p_i$ and $P(X_1 \geqslant i) = \sum_{j \geqslant i} p_j := \bar{p}_i$. We require that $p_i > 0$ for infinitely many values $i \geqslant 1$. For $m \geqslant 0$, we now identify two regimes of discrete distributions by their limiting hazard rate behavior:

$(A)_m$: $\lim_{t\to\infty} (\sum_{j=0}^{m} p_{t+j})/\bar{p}_t = 1$,

(B) : $\lim_{t\to\infty} p_t/\bar{p}_t = 0$.

Also useful will be the class,

$(C)_m$: $\lim_{t\to\infty} p_{t+m}/\bar{p}_t = 0$.

Note that $(C)_0 = (B)$, but that, for $m \geqslant 1$, $(C)_m$ includes both $(A)_{m-1}$ and (B) and other distributions, as $p_{t+m}/\bar{p}_t = [p_{t+m}/\bar{p}_{t+m}][1 - (\sum_{j=0}^{m-1} p_{t+j})/\bar{p}_t]$. Also, we have the inclusion $(A)_m \subset (A)_{m+1}$ for $m \geqslant 0$. As examples, the Poisson distribution falls into $(A)_0$ while the Zeta distribution belongs to (B). An example in $(A)_1$ which is not in $(A)_0$ is the Poisson distribution on the even integers, $p_{2t} = \mathrm{e}^{-\lambda}\lambda^t/t!$ and $p_{2t+1} = 0$ for $t \geqslant 1$.

A clear difference between the first two classes is given in Anderson (1970). Under the first type, we have an "almost-LLN" while under the second type we have a sort of "strong diffuseness" result.

Theorem 1.1. *There exists a sequence* $\{u_n\}$ *such that* $\lim_{n\to\infty} P(X_{(n)} = u_n$ *or* $u_n + 1) = 1$ *iff* $\{p_t\}$ *satisfies* $(A)_0$. *Also*, $\lim_{n\to\infty} P(|X_{(n)} - v_n| < y) = 0$ *for any sequence of constants* $\{v_n\}$ *and any* $y > 0$ *iff* $\{p_t\}$ *satisfies* (B).

Note that condition $(A)_0$ is the Gnedenko condition for only $x = 1$ (which should give a weaker result).

The problem we are interested in this note is the following. Let $\rho_n^{k,m}$, for integers $k \geqslant 1$ and $m \geqslant 0$, be the probability that exactly k of the n samples, $X_1, X_2, \dots, X_n$, are maximal and are strictly m units more than the next ordered value. That is,

$$\rho_n^{k,m} = P(X_{(n)} = X_{(n-1)} = X_{(n-2)} = \cdots = X_{(n-k+1)} > X_{(n-k)} + m).$$

What are the asymptotic behaviors of these quantities? This question is relevant only for discrete valued samples as the maximum is unique with probability 1 for continuous distributions.

The objects $\rho_n^{k,m}$ can also be thought of intuitively. Let n balls be thrown independently into an infinite number of boxes, numbered $1, 2, 3, \dots$ according to the distribution $\{p_t,\ t \geqslant 1\}$. Then $\rho_n^{k,m}$ is the probability that the highest non-empty box has exactly k balls in it and that the previous m boxes are empty. We can calculate $\rho_n^{k,m}$ through multinomial probabilities.

$$\rho_n^{k,m} = \sum_{t \geqslant 1} \binom{n}{k} p_t^k (p_1 + p_2 + \cdots + p_{t-m-1})^{n-k}$$

$$= \sum_{t \geqslant 1} \binom{n}{k} p_t^k (1 - \bar{p}_{t-m})^{n-k}.$$

In fact, the asymptotic behavior of $\rho_n^{1,0}$ was first addressed in Baryshnikov et al. (1995) and then later rederived unbeknown in another context in Athreya and Fidkowski (2000).

Theorem 1.2. *The following are equivalent*:

1. $\lim_{n\to\infty} \rho_n^{1,0}$ *exists*;
2. $\lim_{n\to\infty} \rho_n^{1,0} = 1$;
3. $\{p_t\}$ *is in regime* (B).

We present analagous results for the limits of $\rho_n^{k,m}$ for $k \geqslant 2$ and $m \geqslant 0$. Since $\sum_{k=1}^n \rho_n^{k,0} = 1$, we would expect that, for $k \geqslant 2$, $\rho_n^{k,0} \to 0$ iff $p_t/\bar{p}_t \to 0$. This in fact turns out to be the case. However, the limits of $\rho_n^{k,m}$ for $k \geqslant 1$ and $m \geqslant 1$ are a little different. Our main result is the following:

Theorem 1.3. *For* $k \geqslant 1$, $\lim_{n\to\infty} \rho_n^{k,0}$ *exists iff* $\{p_t\}$ *satisfies* (B).
For $k = 1$ *and* $m \geqslant 1$, $\lim_{n\to\infty} \rho_n^{k,m}$ *exists iff* $\{p_t\}$ *belongs to* $(A)_{m-1}$ *or* (B).
For $k \geqslant 2$ *and* $m \geqslant 1$, $\lim_{n\to\infty} \rho_n^{k,m}$ *exists iff* $\{p_t\}$ *satisfies* $(C)_m$.
Also, the following are equivalent:

1. $\lim_{n\to\infty} \rho_n^{1,m} = 1$ *for any* $m \geqslant 0$;
2. $\lim_{n\to\infty} \rho_n^{k,0} = 0$ *for any* $k \geqslant 2$;
3. $\{p_t\}$ *belongs* to (B).

And also the following are equivalent:

1. $\lim_{n\to\infty} \rho_n^{1,m} = 0$ *for any* $m \geqslant 1$;
2. $\{p_t\}$ *belongs to* $(A)_{m-1}$.

In addition, the following are equivalent:

1. $\lim_{n\to\infty} \rho_n^{k,m} = 0$ *for any* $k \geqslant 2$ *and* $m \geqslant 1$;
2. $\{p_t\}$ *belongs to* $(C)_m$.

The intuition is clear in that if the hazard rate vanishes at infinity, then one has heavy tails and larger and larger boxes are selected as the sample number increases. Conversely, if the hazard rate goes to 1, then one might expect some sort of clustering. This reasoning reinforces and jibes well with Anderson's results, Theorem 1.1.

As a way to check the robustness of these ideas, we investigate the limits of $\rho_n^{1,0}$ when the samples are Markovian. Interestingly, different behavior sometimes emerges. Let $X_1, X_2, \ldots$ form a Markov chain on the state space $\{1, 2, \ldots\}$ and denote $P_x(\cdot)$ as the governing measure starting from $X_1 = x$. To keep some of the spirit of the i.i.d. framework and reduce technicalities, we impose that the chain is recurrent and modify the original problem. Let $\tau_x^0 = 1$, and $\tau_x^n = \min\{j > \tau_x^{n-1}: X_j = x\}$, for $n \geqslant 1$, be the time of nth return to value x. Instead of analyzing $P(X_{(n)} > X_{(n-1)})$ we consider $P_x(X_{(\tau_x^n)} > X_{(\tau_x^n - 1)})$ with size n replaced by the random quantity τ_x^n. This allows us to decompose paths into i.i.d. x-sojourns more easily. To fix ideas, we take $x = 1$ throughout.

Let $p(i,j)$ be the homogeneous transition probabilities for $i, j \geqslant 1$. To be brief, we restrict ourselves to two cases:

1. $p(k, k+1) = p(k)$, $p(k, 1) = 1 - p(k) = q(k)$, $p(1) = 1$.
2. $p(k, k+1) = p(k)$, $p(k, k-1) = q(k)$, $p(1, 2) = 1$, $p(k) + q(k) = 1$.

Note that, in case 1, the chain is recurrent iff $\sum_{k\geqslant 1} q(k) = \infty$. Also, for the birth-death chains in case 2, the process is recurrent iff $\sum_{k\geqslant 1} \prod_{j=1}^k q(j)/p(j) = \infty$.

Theorem 1.4. *For case* 1, *suppose that the chain is recurrent. Then*, *the following are equivalent*:
1. $\lim_{n\to\infty} P_1(X_{(\tau_1^n)} > X_{(\tau_1^n - 1)})$ *exists*;
2. $\lim_{n\to\infty} P_1(X_{(\tau_1^n)} > X_{(\tau_1^n - 1)}) = 1$;
3. $\lim_{k\to\infty} q(k) = 0$.

For case 2, *suppose that* $\lim_{k\to\infty} q(k)$ *exists and that the chain is recurrent. Then*, *the following are equivalent*:
1. $\lim_{n\to\infty} P_1(X_{(\tau_1^n)} > X_{(\tau_1^n-1)})$ *exists*;
2. $\lim_{n\to\infty} P_1(X_{(\tau_1^n)} > X_{(\tau_1^n-1)}) = \frac{1}{2}$;
3. $\lim_{k\to\infty} q(k) = \frac{1}{2}$.

The structure of the article is as follows. In Section 2, we prove the i.i.d. results. In Section 3, we prove the Markovian results.

2. Proof of IID results

The main idea in the analysis of $\lim_{n\to\infty} \rho_n^{k,m}$, as in Athreya and Fidkowski (2000), is Karamata's Tauberian Theorem as found in pp. 37–38, Bingham et al. (1987):

Proposition 2.1. *Let* U *be real valued and non-decreasing*, *with* $U(x)=0$ *for* $x<0$. *Let the Laplace–Stieltjes transform* $\hat{U}(s)=\int_0^\infty \mathrm{e}^{-sx}\,\mathrm{d}U(x)<\infty$ *for all large* s. *Let* l *be a slowly varying function and let* $c \geqslant 0$ *and* $\rho \geqslant 0$ *be constants. Then*

$$U(x) \sim cx^\rho l(1/x)/\Gamma(1+\rho) \text{ as } x \to 0+ \text{ iff } \hat{U}(s) \sim cs^{-\rho} l(s) \text{ as } s \to \infty.$$

If $c=0$, *this is interpreted as*

$$U(x) = o(x^\rho l(1/x)/\Gamma(1+\rho)) \text{ as } x \to 0+ \text{ iff } \hat{U}(s) = o(s^{-\rho} l(s)) \text{ as } s \to \infty.$$

Proof of Theorem 1.3. Let F be the distribution function of the random variable $Y=-\log(1-\bar{p}_X)$ where X is the discrete r.v. with law $P(X=t)=c(p_{t+m})^k$, for $t \geqslant 1$, where c is a normalizing constant. Rewrite $\rho_n^{k,m}$ as

$$\rho_n^{k,m} = \sum_{t\geqslant 1} \binom{n}{k} (p_{t+m})^k (1-\bar{p}_t)^{n-k}$$

$$= c^{-1}\binom{n}{k} E[\mathrm{e}^{-(n-k)(-\log(1-\bar{p}_X))}] = c^{-1}\binom{n}{k}\hat{F}(n-k),$$

where $\hat{F}(\cdot)$ is the Laplace–Stieltjes transform of $F(\cdot)$. Since $\hat{F}$ is monotone,

$$\lim_{n\to\infty} \rho_n^{k,m} = \lim_{n\to\infty} c^{-1}\binom{n}{k}\hat{F}(n-k) = \lim_{s\to\infty}(s^k/(ck!))\hat{F}(s),$$

when the limits exist. By Karamata's theorem, however, $(s^k/c)\hat{F}(s)$ tends to L as $s\to\infty$ iff $f(y)=F(y)/(cy^k)$ tends to the same limit L as $y \to 0^+$. In this case, of course, $\rho_n^{k,m}$ converges to $L/k!$ as $n\to\infty$.

Let us now examine y in the interval $y \in [-\log(1-\bar{p}_t), -\log(1-\bar{p}_{t-1}))$ as $t \uparrow \infty$. Some manipulation gives, for $0 \leqslant \varepsilon < \log(1-\bar{p}_t/1-\bar{p}_{t-1})$, that

$$P(Y \leqslant -\log(1-\bar{p}_t)+\varepsilon) = P(X \geqslant t) = c\sum_{j\geqslant t} p_{j+m}^k.$$

Then,

$$\frac{\sum_{j\geqslant t} p_{j+m}^k}{(-\log(1-\bar{p}_t))^k} \geqslant f(y) \geqslant \frac{\sum_{j\geqslant t} p_{j+m}^k}{(-\log(1-\bar{p}_{t-1}))^k}$$

with the left value taken on the left endpoint $y = -\log(1-\bar{p}_t)$, and the right value taken as y tends to the right endpoint $-\log(1-\bar{p}_{t-1})$. Hence, by the squeezing property, $\lim_{y\to 0} f(y) = L$ exists iff the limits on the left and right endpoint sequences both exist and equal L, which, as $\lim_{t\to\infty} \bar{p}_t/(-\log(1-\bar{p}_t)) = 1$, means that

$$\lim_{t\to\infty} \frac{\sum_{j\geqslant t} p_{j+m}^k}{\bar{p}_t^k} = \lim_{t\to\infty} \frac{\sum_{j\geqslant t} p_{j+m}^k}{\bar{p}_{t-1}^k} = L. \tag{1}$$

However, as

$$\frac{\sum_{j\geqslant t} p_{j+m}^k}{\bar{p}_{t-1}^k} = \frac{\sum_{j\geqslant t-1} p_{j+m}^k}{\bar{p}_{t-1}^k} - \frac{p_{t+m-1}^k}{\bar{p}_{t-1}^k},$$

the conditions (1) can be distilled further as

$$\lim_{t\to\infty} \frac{\sum_{j\geqslant t} p_{j+m}^k}{\bar{p}_t^k} = L \quad \text{and} \quad \lim_{t\to\infty} p_{t+m}/\bar{p}_t = 0. \tag{2}$$

Necessity: For (2) to hold, it is necessary that $\{p_t\}$ satisfies (B) for $m=0$, and $(C)_m$ for $m \geqslant 1$. In particular, when $k=1$, as

$$p_{t+m}/\bar{p}_t = [p_{t+m}/\bar{p}_{t+m}][\bar{p}_{t+m}/\bar{p}_t], \tag{3}$$

it is necessary that $L = \lim_{t\to\infty} \bar{p}_{t+m}/\bar{p}_t = 0$ or $\lim_{t\to\infty} p_t/\bar{p}_t = 0$. These options, as

$$\bar{p}_{t+m}/\bar{p}_t = 1 - \left(\sum_{j=0}^{m-1} p_{t+j}\right)/\bar{p}_t, \tag{4}$$

are equivalent to $\{p_t\}$ satisfying $(A)_{m-1}$ or (B) respectively.

Sufficiency and Evaluation: For $k \geqslant 2$ and $m \geqslant 0$, as

$$\frac{\sum_{j\geqslant t} p_{j+m}^k}{\bar{p}_t^k} \leqslant \frac{\sum_{j\geqslant t} p_{j+m}[p_{j+m}/\bar{p}_j]^{k-1}}{\bar{p}_t} \leqslant 1 \sup_{j\geqslant t} [p_{j+m}/\bar{p}_j]^{k-1},$$

we have, noting (2), that $\lim_{t\to\infty} \rho_n^{k,m} = 0$ when $\{p_t\}$ belongs to $(C)_m$.

For $k=1$ and $m \geqslant 0$, as

$$\frac{\bar{p}_{t+m}}{\bar{p}_t} = \prod_{j=1}^{m} \frac{\bar{p}_{t+j}}{\bar{p}_{t+j-1}} = \prod_{j=1}^{m} \left(1 - \frac{p_{t+j-1}}{\bar{p}_{t+j-1}}\right),$$

we have, noting (2)–(4), that $\lim_{n\to\infty} \rho_n^{1,m}$ equals 0 or 1 if $\{p_t\}$ satisfies $(A)_{m-1}$ or (B) respectively. □

3. Proof of Markov chain results

Let $X_1, X_2, \ldots,$ be a recurrent Markov chain taking values on $\Sigma = \{1,2,\ldots\}$. Consider the sojourns to site 1 which form an i.i.d. sequence. Recall that τ_x^n denotes the nth return time to x for $n \geqslant 1$. For simplicity, we will write $\tau_x = \tau_x^1$. Denote also $\tau_A = \min\{j > 1: X_j \in A\}$ as the return time to the set A. Let in addition $M_1 = X_{(\tau_1)}$ and $M_2 = X_{(\tau_1 - 1)}$ be the maximum and second highest values on a 1-sojourn, that is, among $\{X_j: 1 \leqslant j \leqslant \tau_1\}$, with $X_1 = X_{\tau_1} = 1$. Similarly, for each n, we let $X_{(\tau_1^n)}$ and $X_{(\tau_1^n - 1)}$ be the maximum and second highest values among $\{X_j: 1 \leqslant j \leqslant \tau_1^n\}$.

Proposition 3.1. *We have* $\lim_{n\to\infty} P_1(X_{(\tau_1^n)} > X_{(\tau_1^n-1)}) = L$ *iff*

$$\lim_{k\to\infty} 1 - \frac{\sum_{l\geqslant k} P_1(\tau_l^2 < \tau_1 < \tau_{[l+1,\infty)})}{\sum_{l\geqslant k} P_1(\tau_l < \tau_1 < \tau_{[l+1,\infty)})} = L \quad \textit{and} \quad \lim_{k\to\infty} \frac{P_1(\tau_k < \tau_1 < \tau_{[k+1,\infty)}, \tau_k^2 > \tau_1)}{P_1(\tau_{[k,\infty)} < \tau_1)} = 0.$$

Proof. By decomposing on the sojourns to 1, we have

$$P_1(X_{(\tau_1^n)} > X_{(\tau_1^n-1)}) = n\sum_{t\geqslant 1} P_1(M_1 = t, M_1 > M_2)(1 - P_1(M_1 \geqslant t))^{n-1}. \tag{5}$$

Let X be a discrete r.v. with distribution $P(X = t) = cP_1(M_1 = t, M_1 > M_2)$, for $t \geqslant 1$, where again c is the appropriate normalizing constant. Let also $Y = -\log(1 - P_1(M_1 \geqslant X))$. Then the RHS in (5) becomes $c^{-1}nE[\mathrm{e}^{-(n-1)Y}]$. Now, using Karamata's theorem analogously as in the i.i.d. case (c.f. (2)), we have that $\lim_{n\to\infty} P_1(X_{(\tau_1^n)} > X_{(\tau_1^n-1)}) = L$ iff both

$$\lim_{k\to\infty} \frac{P_1(M_1 \geqslant k, M_1 > M_2)}{P_1(M_1 \geqslant k)} = L \quad \text{and} \quad \lim_{k\to\infty} \frac{P_1(M_1 = k, M_1 > M_2)}{P_1(M_1 \geqslant k)} = 0. \tag{6}$$

A moment's thought now gives that, starting from 1, the event $\{M_1 = l\}$ is the same as hitting l but not $[l+1,\infty)$ before the 1-cycle ends, and also the event $\{M_1 = l, M_1 > M_2\}$ is equivalent to touching l exactly once but not $[l+1,\infty)$ before the end of the 1-cycle:

$$P_1(M_1 = l) = P_1(\tau_l < \tau_1 < \tau_{[l+1,\infty)}) \quad \text{and} \quad P_1(M_1 = l, M_1 > M_2) = P_1(\tau_l < \tau_1 < \tau_{[l+1,\infty)}, \tau_l^2 > \tau_1).$$

Also, starting from 1, the set $\{M_1 \geqslant k\}$ is the same as moving above k on a 1-cycle: $P_1(M_1 \geqslant k) = P_1(\tau_{[k,\infty)} < \tau_1)$. We can also evaluate $P_1(M_1 \geqslant k) = \sum_{l\geqslant k} P_1(M_1 = k)$.

Substituting these expressions with easy manipulation into (6) yields the conditions in the proposition. □

To evaluate these conditions further for the Markov chain in case 2, we will need the following computation. Let $\gamma(l) = \prod_{j=1}^{l} q(j)/p(j)$ for $l \geqslant 1$.

Lemma 3.1. *For the birth-death chains in case* 2, *suppose that* $\lim_{k\to\infty} q(k) = Q$. *Then*,

$$\begin{aligned}\lim_{k\to\infty} P_{k-1}(\tau_1 < \tau_k) &= 0 \quad \textit{for } Q \leqslant \tfrac{1}{2}.\\ &> 0 \quad \textit{for } Q > \tfrac{1}{2}.\end{aligned}$$

Proof. Standard theory gives, for $k > 1$, that

$$\begin{aligned}P_{k-1}(\tau_1 < \tau_k) &= \frac{\gamma(k)}{\sum_{l=1}^{k} \gamma(l)}\\ &\leqslant \frac{1}{1 + \sum_{l=0}^{r} (p(k)\cdots p(k-l))/(q(k)\cdots q(k-l))},\end{aligned}$$

where $0 \leqslant r < k$ is a fixed integer. Therefore, $\overline{\lim}_{k\to\infty} P_{k-1}(\tau_1 < \tau_k)$ is less than $[1 - (1-Q)/Q]/[1 - ((1-Q)/Q)^{r+2}]$ when $Q \neq \frac{1}{2}$, and less than $1/(r+2)$ when $Q = \frac{1}{2}$. For $Q \leqslant \frac{1}{2}$, these upperbounds vanish as r tends to infinity.

The second statement of the lemma is argued by a similar method. □

Proof of Theorem 1.4. For case 1, we compute the limits in Proposition 3.1. With respect to the first limit, under the specified dynamics, it is impossible for $\tau_k^2 < \tau_1$, and so L can only equal $L = 1$. For the second limit, the numerator is $q(k)\prod_{l=1}^{k-1} p(l)$. The denominator is $\prod_{l=1}^{k-1} p(l)$. These observations yield the evaluation, and necessity and sufficiency implications in the first part.

For case 2, straightforward computations, using the Markov property and that the motion is nearest-neighbor, give that $P_1(\tau_l < \tau_1 < \tau_{[l+1,\infty)}) = P_1(\tau_l < \tau_1)P_l(\tau_1 < \tau_{l+1})$, and $P_1(\tau_l^2 < \tau_1 < \tau_{[l+1,\infty)}) = P_1(\tau_l < \tau_1)q(l)$ $P_{l-1}(\tau_l < \tau_1)P_l(\tau_1 < \tau_{l+1})$. Also, we have that $P_1(\tau_k < \tau_1 < \tau_{[k+1,\infty)}, \tau_k^2 > \tau_1) = P_1(\tau_k < \tau_1)q(k)P_{k-1}$ $(\tau_1 < \tau_k)$. Then, the first and second limits in Proposition 3.1 become

$$\lim_{k\to\infty} 1 - \frac{\sum_{l\geqslant k} P_1(\tau_l < \tau_1)P_l(\tau_1 < \tau_{l+1})q(l)P_{l-1}(\tau_l < \tau_1)}{\sum_{l\geqslant k} P_1(\tau_l < \tau_1)P_l(\tau_1 < \tau_{l+1})} = L \quad \text{and}$$
$$\lim_{k\to\infty} q(k)P_{k-1}(\tau_1 < \tau_k) = 0. \tag{7}$$

Necessity: Now, by assumption, we have that $\lim_{k\to\infty} q(k) = Q$, say. Then, for the second limit to hold in (7), it is necessary that either $Q = 0$ or $\lim_{k\to\infty} P_{k-1}(\tau_1 < \tau_k) = 0$. The possibility $Q = 0$ or even $Q < \frac{1}{2}$ is not allowed, as then the assumption of recurrence ($\sum_{l\geqslant 1}\gamma(l) = \infty$) would be violated. However, by Lemma 3.1, $\lim_{k\to\infty} P_{k-1}(\tau_1 < \tau_k)$ vanishes iff $Q \leqslant \frac{1}{2}$. So, necessarily the only value Q allowed is $Q = \frac{1}{2}$.

Sufficiency and Evaluation: Conversely, suppose that $Q = \frac{1}{2}$. Then, the second limit in (7) holds by Lemma 3.1. Also, by Lemma 3.1, $\lim_{l\to\infty} P_{l-1}(\tau_l < \tau_1) = 1$, and so the first limit in (7) satisfies $L = 1 - Q = \frac{1}{2}$. □

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