Spin Depolarization Decay Rates in α-Symmetric Stable Fields on Cubic Lattices

SUNDER SETHURAMAN

ETH-Zentrum

AND

LIN XU

Statistics & Operations Research Program, Princeton University

Abstract

We study the asymptotic, long-time behavior of the energy function

$$E(t;\lambda;f) = \frac{1}{t}\ln E\exp\left\{-tf\left[\frac{1}{t}\sum_{x\in\mathbf{Z}^{\mathbf{d}}}\left(\lambda\int_0^t \delta_{\{x\}}(X_s)\,ds\right)^{\alpha}\right]\right\}$$

where $\{X_s : 0 \leq s < \infty\}$ is the standard random walk on the d-dimensional lattice $\mathbb{Z}^d$, $1 < \alpha \leq 2$, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is any nondecreasing concave function. In the special case $f(x) = x$, our setting represents a lattice model for the study of transverse magnetization of spins diffusing in a homogeneous, α-stable, i.i.d., random, longitudinal field $\{\lambda V(x) : x \in \mathbb{Z}^d\}$ with common marginal distribution, the standard α-symmetric stable distribution; the parameter λ describes the intensity of the field.

Using large-deviation techniques, we show that $S_c(\lambda,\alpha,f) = \lim_{t\to\infty} E(t;\lambda;f)$ exists. Moreover, we obtain a variational formula for this decay rate S_c. Finally, we analyze the behavior $S_c(\lambda,\alpha,f)$ as $\lambda \to 0$ when $f(x) = x^\beta$ for all $1 \geq \beta > 0$.

Consequently, several physical conjectures with respect to lattice models of transverse magnetization are resolved by setting $\beta = 1$ in our results. We show that $S_c(\lambda,\alpha,1) \sim \lambda^\alpha$ for $d \geq 3$, $\lambda^\alpha(\ln\frac{1}{\lambda})^{\alpha-1}$ in $d = 2$, and $\lambda^{\frac{2\alpha}{\alpha+1}}$ in $d = 1$. ©1996 John Wiley & Sons, Inc.

1 The Model and Main Results

In [11, 10], an analysis of the tranverse magnetization of spins diffusing in a random longitudinal field is proposed. These papers give convincing physical arguments when the randomness is Gaussian or derived from an α-stable symmetric distribution. In [2], these arguments are substantiated using large-deviation techniques. Mitra and Doussal in [11] also address the question of universality by developing a discrete lattice model in which similar results for the continuum case are derived.

We now describe this model: Let $\{V(x) : x \in \mathbb{Z}^d\}$ be an i.i.d. random field with common distribution, the α-stable symmetric distribution, having characteristic function

$$\langle\exp\{itV(x)\}\rangle = \exp\{-|t|^\alpha\},$$

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where $\langle\cdot\rangle$ is expectation with respect to the α-random field $\{V(x) : x \in \mathbb{Z}^d\}$. Consider the following evolution equation governing the magnetization of spins $u(t, x)$, diffusing in the α-random field:

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} \triangle u(t,x) + i\lambda V u(t,x)\,, \quad u(0,x) = 1\,, \tag{1.1}$$

where $\triangle$ is the usual discrete Laplace operator on $\mathbb{Z}^d$ and $i = \sqrt{-1}$. The quantity of interest, the tranverse magnetization, is given by the expectation

$$M(t;\lambda) = \langle u(t,x)\rangle\,,$$

which is independent of x because the α-random field is stationary. An immediate consequence of the Feynman-Kac formula is that

$$M(t;\lambda) = \exp\left\{-\sum_{x\in\mathbb{Z}^d}\left(\lambda\int_0^t \delta_x(X_s)\,ds\right)^\alpha\right\}, \tag{1.2}$$

where $X.$ is the standard continuous random walk on $\mathbb{Z}^d$ with generator the discrete Laplacian $\frac{1}{2}\triangle$. The physical results in [11] cite that $M(t;\lambda)$ exponentially decays in time and that the decay coefficients, $\lambda \to 0$, exhibit different behaviors depending on the lattice dimension d. We will make this statement more precise in a moment.

In another point of view, functionals such as $M(t;\lambda)$ in (1.2) continue to be of great interest from the perspective of random polymer models. Some work in this regard may be found in [15, 9, 1, 8, 13]. In the polymer context, $M(t;\lambda)$ is identified as the partition function for the Hamiltonian

$$H(t;\lambda) = \sum_{x\in\mathbb{Z}^d}\left(\lambda\int_0^t \delta_x(X_s)\,ds\right)^\alpha .$$

In this paper, we consider a Hamiltonian

$$H(t;\lambda;f) = tf\left(\frac{1}{t}\sum_{x\in\mathbb{Z}^d}\left(\lambda\int_0^t \delta_x(X_s)\,ds\right)^\alpha\right), \tag{1.3}$$

where $\lambda > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing concave function (in the magnetization context, $f(x) = x$). We now specify throughout this article that $1 < \alpha \le 2$.

As mentioned earlier, the magnetization is assumed to experience an exponential time decay. Therefore our first goal is to study the long-time behavior of the object

$$E(t;\lambda;f)=\frac{1}{t}\ln E[\exp\{-H(t;\lambda;f)\}] \tag{1.4}$$

where E is expectation with respect to the random walk. We prove below the existence of the limit

$$S_c(\lambda,\alpha,f)=-\lim_{t\to\infty}E(t)\,. \tag{1.5}$$

From the view of polymer and statistical physics models, this exponent is simply the pressure function whose study, as is well-known, is the first step toward understanding the Gibbs states of the dynamics. Our primary aim is to investigate the behavior of $S_c(\lambda,\alpha,f)$ as $\lambda\to 0$ in different dimensions.

To prepare for the main results, we develop some notation. Consider the space $\mathbf{\Omega}=D(\mathbb{R},\mathbb{Z}^d)$ of all right continuous functions $\omega:\mathbb{R}\to\mathbb{Z}^d$ with discontinuities of only the first kind and $\omega(0)=0$. Similarly, we may define the space $\mathbf{\Omega}_{a,b}=D([a,b],\mathbb{Z}^d)$. Place on $\mathbf{\Omega}$ the standard topology induced by the Skorohod convergence on bounded intervals of the real line. Define by $\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$ the set of the all probability measures on $\mathbf{\Omega}$ that govern processes with stationary increments. Again, $\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$ will be a Polish space when endowed with the usual weak topology. Denote by E^P expectation with respect to $P\in\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$. Let Q_0 denote the law induced by the random walk X_s on $\mathbf{\Omega}$ (extend, for instance, another independent random walk in the negative time direction). Obviously $Q_0\in\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$.

For any $P\in\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$, let $P[-t,t]$ be the restriction of P to $\mathbf{\Omega}_{-t,t}$. We may define the relative entropy of $P[-t,t]$ with respect to $Q_0[-t,t]$ by the formula

$$H_t(P\mid Q_0)=\sup_f\left\{\int f\,dP-\ln\left[\int\exp(f)\,dQ_0\right]\right\},$$

where supremum is over all bounded continuous functions on $\mathbf{\Omega}_{-t,t}$. It is not hard to see that that $H_t(P\mid Q_0)$ is superadditive in t if we note that Q_0 is an independent stationary increment process:

$$H_{t+s}(P\mid Q_0)\geq H_t(P\mid Q_0)+H_s(P\mid Q_0)\,.$$

Therefore, the following definition for the relative entropy, $H(P|Q_0)$, of P with respect to Q_0 is meaningful:

$$H(P\mid Q_0)=\lim_{t\to\infty}\frac{1}{2t}H_t(P\mid Q_0). \tag{1.6}$$

Now we are at a point to state our main results:

THEOREM 1.1 *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be concave and nondecreasing. Also let $1 < \alpha \le 2$. Then*

$$\lim_{t\to\infty} E(t;\lambda;f) = -\inf_{P\in\mathbf{P}_{si}(\boldsymbol{\Omega})}\left\{ f\left(|\lambda|^\alpha E^P\left[\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right]^{\alpha-1}\right) + H(P\mid Q_0)\right\}. \tag{1.7}$$

REMARKS ON THEOREM 1.1

1. The variational formula given above allows us to easily derive upper bounds for the decay rate $S_c(\lambda,\alpha,f)$. For instance, we may restrict the variational problem to those laws induced by random walks with different jump rates.
2. The minimizers of this variational problem themselves are interesting objects. We believe that they are Gibbs states (as far as we know, few have been known for this problem). More interestingly, it is believed that all minimizers in low dimensions ($d \le 3$) are long range correlated. Some interesting progress has been made in this connection [15, 1, 8, 13].
3. From valuable discussions with Professor E. Bolthausen, we have learned that similar variational formulae have been obtained in [9]. Our result, in any case, has broader scope because it treats the case of nonlinear f, which is seemingly beyond the methods in [9].
4. The existence of the decay rate $S_c(\lambda,\alpha,f)$ could be derived by establishing subadditivity of $E(t;\lambda;f)$ in t.

For functions of the form $f(x) = x^\beta$, set

$$S_c(\lambda,\alpha,\beta) = S_c(\lambda,\alpha,f).$$

Also, we understand the notation $g_1(\lambda) \sim g_2(\lambda)$ as $\lambda \to 0$ to mean that

$$0 < \liminf_{\lambda\to 0}\frac{g_1(\lambda)}{g_2(\lambda)} \le \limsup_{\lambda\to 0}\frac{g_1(\lambda)}{g_2(\lambda)} < \infty.$$

THEOREM 1.2 *For each $0 < \beta \le 1$ and $d \ge 3$, we have*

$$\lim_{\lambda\to 0}\frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}} = \left\{E^{Q_0}\left(\int_{-\infty}^{\infty}\delta_0[\omega(s)]\,ds\right)^{\alpha-1}\right\}^\beta < \infty. \tag{1.8}$$

For $d = 2$ as $\lambda \to 0$ we have

$$S_c(\lambda, \alpha, \beta) \sim \lambda^{\alpha\beta} \left(\ln \frac{1}{\lambda}\right)^{(\alpha-1)\beta} . \tag{1.9}$$

For $d = 1$ as $\lambda \to 0$ we have

$$S_c(\lambda, \alpha, \beta) \sim \lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}} . \tag{1.10}$$

REMARKS FOR THEOREM 1.2

1. Our results applied to the case $\beta = 1$ affirm exactly the conjectures made in [11, 10].
2. In the case $\beta = 1$ and $\alpha = 2$, the Gaussian case, further limit-type results in $d = 2$ follow easily from our arguments. We learned from our discussion with Bolthausen a direct argument for this limit result based on estimations of the variance of random walk's self-intersection time.

As noted earlier, a similar program has been carried out recently in [2] in the context of Brownian motion and Gaussian random fields. The method developed there, based on large-deviation theory and the entropy inequality, is quite flexible, and we will adopt this approach to our more general context. The main problem is that by considering α-stable random fields, our Hamiltonian lacks linearity; some tricks are therefore used to circumvent the nonlinearity.

The main idea to prove Theorem 1.1 is to use a certain large-deviation principle. It is not hard to see that those paths that contribute most in (1.4) tend to move far away with drift. These are not the typical paths for random walk; in fact, exponentially small chances are associated with such paths. This gives a heuristic indication of why large deviations might play an important role in this problem.

To prove Theorem 1.2, a modified random-walk measure is introduced to verify that a minimizer exists for the variational problem in Theorem 1.1. Then by comparing the entropy of this modified measure with that of the minimizer, the various decay rates in different dimensions are deduced.

The paper is organized as follows: In Section 2, we will establish a large-deviation principle for the stationary "derivatives" of random walks. In Section 3, we will then prove Theorem 1.1. In Section 4, we will derive that part of Theorem 1.2 dealing with $d \geq 3$. In Section 5, we will complete the proof of Theorem 1.2 for lower-dimensional cases.

A survey of earlier work on magnetic depolarization from both physical and mathematical points of view are listed in [2, 10]. [10] is an expository paper on this topic from a physics standpoint. Earlier work on random polymer models may be found in the references of [1, 13].

2 Large-Deviation Principle for Increments of Random Walk

In this section, we will establish a large-deviation principle for the "derivatives," the infinitesimal increments of standard, continuous-time random walk. Of course, to deal with the infinitesimals directly, we must work with temporal distributions, which are topologically unfriendly objects. So instead, we look at their integrals, the increments, directly. In spirit, however, one should view the following large-deviation principle as one for "white noise." In this sense, our large-deviation principle is nothing more than a direct extension of level-3 large deviations for i.i.d. random sequences (see [7] for a discussion of level-3 large deviations).

Although this large-deviation principle as we interpret it may be accepted readily by specialists, it hasn't appeared in the literature; as far as we know, a suitable formulation is given in [2] for the first time. We believe that this large-deviation principle is useful for other problems. In the present context, we are unable to fit what we require in terms of a large-deviation principle into any general framework, even though large-deviation theory is now quite developed.

As in Section 1, let $\mathbf{\Omega}$ be the space of right continuous paths $w(\cdot)$ and Q_0 the measure governing random walk on this space. $\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$ is again the space of measures with stationary increments on $\mathbf{\Omega}$.

For any $t > 0$ and a path $w(\cdot) \in \mathbf{\Omega}$, define the new trajectory

$$w_t(s) = \begin{cases} w(s) & \text{if } 0 \le s \le t \\ kw(t) + w(r) & \text{if } s = kt + r\,, \end{cases}$$

where k is a nonnegative integer and $0 \le r < t$. Essentially the formal derivative of $w_t(s)$ in s is the t-periodic extension of the formal derivative of $w(s)$.

Now define, for measurable $A \subset \mathbf{\Omega}$ and path ω, the empirical measure $L_t(d\omega', \omega)$:

$$L_t(A, \omega) = \frac{1}{t} \int_0^t \chi_A(\omega_t(s + \cdot) - \omega_t(s))\, ds\,, \tag{2.1}$$

where χ_A is the characteristic function of A. It is not hard to see that $L_t(\cdot, \omega) \in \mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$.

The following theorem, which specifies the large-deviation principle for $L_t(\cdot, \omega)$ under Q_0, is the main result of this section.

THEOREM 2.1 *A large-deviation principle holds for the laws of $L_t(\cdot, \omega)$ on $\mathbf{\Omega}$ under Q_0 as $t \to \infty$ with rate function $H(\cdot \mid Q_0)$, the relative entropy function with respect to Q_0.*

We have for closed sets $C \subset \mathbf{P}_{si}(\mathbf{\Omega})$

$$\limsup_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L_t(\cdot,\omega) \in C\} \le -\inf_{P\in C} H(P \mid Q_0). \tag{2.2}$$

For open sets $O \subset \mathbf{P}_{si}(\mathbf{\Omega})$,

$$\liminf_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L_t(\cdot,\omega) \in O\} \ge -\inf_{P\in O} H(P \mid Q_0). \tag{2.3}$$

REMARKS FOR THEOREM 2.1

1. The proof of this result follows arguments in section 2 of [2] closely except the part of exponential tightness. Therefore we only provide a proof for this part and skip the rest of the argument. One should consult [2] for details.

2. Properties of $H(P \mid Q_0)$ as a rate function are established in [5]. We note some facts that we will use: (i) $H(P \mid Q_0)$ is lower semicontinuous and *linear* in P; (ii) $\{P : H(P \mid Q_0) \le a\}$ is a compact set for any a. See section 3 in [5] for details.

LEMMA 2.2 *Consider a continuous function* F *on* $\mathbf{\Omega}$ *measurable with respect to paths up to time* T. *Also, let* F *satisfy* $E^{Q_0}e^F \le 1$. *Then for any* $t > 0$,

$$E^{Q_0} \exp\left\{\frac{1}{T}\int_0^t F(\omega(s+\cdot) - \omega(s))\, ds\right\} \le 1\,.$$

A proof of Lemma 2.2 closely follows the proof of lemma 4.1 in [5].

For any $\omega(\cdot)$, denote by $J(t,\omega)$ the number of jumps in the path over the time interval $[0,t]$. Note that $\{P \in \mathbf{P}_{\mathrm{si}}(\mathbf{\Omega}) : E^P J(1,\omega) \le a\}$ is a compact set for any a.

LEMMA 2.3 *We have the following superexponential tightness:*

$$\limsup_{t\to\infty} \frac{1}{t} \ln E^{Q_0} \exp\left\{tE^{L_t(\cdot,\omega)} J(1,\omega')\right\} < \infty\,. \tag{2.4}$$

PROOF: Note that $J(1,\omega)$ is a Poisson variable with unit intensity. A simple calculation yields

$$E^{Q_0} \exp\{J(1,\omega)\} = e^{e-1}\,.$$

The claim follows by combining the above equation and Lemma 2.2. ∎

The following limit is the one of the key steps to deriving our main results. Due to Varadhan's theorem (see [14]), we may state the following corollary to our principle; note in the specified domains of integration that $w_t(\cdot) = w(\cdot)$:

COROLLARY 2.4 *Assume f is as defined in Theorem* 1.1. *For any $T > 0$, we have*

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \ln E^{Q_0} \exp\left\{ -tf \frac{1}{t} \int_T^{t-T} |\lambda|^\alpha \left(\int_{-T}^{T} \delta_0(\omega(s+r) - \omega(s))\, dr \right)^{\alpha-1} ds \right\} \\
&\qquad = - \inf_{\{P\in \mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})\}} \left\{ f\left(|\lambda|^\alpha E^P \left(\int_{-T}^{T} \delta_0(\omega(r))\, dr \right)^{\alpha-1} \right) + H(P \mid Q_0) \right\}.
\end{aligned} \tag{2.5}$$

3 Decay Rates: Proof of Theorem 1.1

In this section we intend to prove Theorem 1.1 by applying the large-deviation principle outlined earlier. The first step is to rewrite the Hamiltonian $H(t; \lambda; f)$. It isn't hard to see that

$$\sum_{x\in\mathbb{Z}^d} \left(\int_0^t \delta_x(\omega(s))\, ds \right)^\alpha = \int_0^t \left(\int_0^t \delta_0(\omega(r) - \omega(s))\, dr \right)^{\alpha-1} ds\,.$$

We will prove Theorem 1.1 in several steps. First, we consider the upper bound:

LEMMA 3.1 *Assume that f, α, λ, and $E(t; \lambda; f)$ are as defined in Theorem* 1.1. *Then*

$$\begin{aligned}
&\limsup_{t\to\infty} E(t; \lambda; f) \\
&\qquad \le - \inf_{P\in \mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})} \left\{ f\left(\lambda^\alpha E^P \left(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds \right)^{\alpha-1} \right) + H(P \mid Q_0) \right\}.
\end{aligned} \tag{3.1}$$

PROOF: Observe that for T such that $0 < 2T < t$ we have

$$\begin{aligned}
&\left(\frac{1}{t} \sum_{x\in\mathbb{Z}^d} \left(\lambda \int_0^t \delta_x(\omega_s)\, ds \right)^\alpha \right) \\
&\quad = f\left(\frac{1}{t} \int_0^t \lambda^\alpha \left(\int_0^t \delta_0(\omega(r) - \omega(s))\, dr \right)^{\alpha-1} ds \right) \\
&\quad \ge f\left(\frac{1}{t} \int_T^{t-T} \lambda^\alpha \left(\int_{-T}^{T} \delta_0(\omega(r+s) - \omega(s))\, dr \right)^{\alpha-1} ds \right).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&E(t;\lambda;f)\\
&\le \frac{1}{t}\ln E^{Q_0}\left[\exp\left\{-tf\left(\frac{1}{t}\int_T^{t-T}\lambda^\alpha\left(\int_{-T}^{T}\delta_0(\omega(s+r)\right.\right.\right.\right.\\
&\qquad\left.\left.\left.\left.-\omega(r))\,ds\right)^{\alpha-1}dr\right)\right\}\right].
\end{aligned}
\tag{3.2}
$$

Now send $t\to\infty$. As an immediate consequence of Corollary 2.4, we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty} E(t;\lambda;f)\\
&\le -\inf_{P\in\mathbf{P}_{\mathrm{si}}(\boldsymbol{\Omega})}\left\{f\left(\lambda^\alpha E^P\left(\int_{-T}^{T}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)+H(P\mid Q_0)\right\}.
\end{aligned}
\tag{3.3}
$$

What remains is to check

$$
\begin{aligned}
&\lim_{T\to\infty}\inf_P\left\{f\left(\lambda^\alpha E^P\left(\int_{-T}^{T}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)+H(P\mid Q_0)\right\}\\
&\quad=\inf_P\left\{f\left(\lambda^\alpha E^P\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)+H(P\mid Q_0)\right\}.
\end{aligned}
$$

This is done in the next lemma. The proof is now complete. ∎

We now derive several properties for our variational problem. Let us designate the T-truncated version of the full-variational quantity $S_c(\lambda,\alpha,f)$ by

$$
S^T(\lambda,\alpha,f)=\inf_{P\in\mathbf{P}_{\mathrm{si}}(\boldsymbol{\Omega})}\left\{f\left(\lambda^\alpha E^P\left(\int_{-T}^{T}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)+H(P\mid Q_0)\right\}.
$$

LEMMA 3.2 *The following properties are valid:*

1. *There exists an ergodic minimizer P_0 such that $H(P_0\mid Q_0)<\infty$.*
2. $\lim_{T\to\infty}S^T(\lambda,\alpha,f)=S_c(\lambda,\alpha,f)$.

PROOF:

Step 1. Consider the object

$$
G(P)=f\left(\lambda^\alpha E^P\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)+H(P\mid Q_0).
$$

We note that $G(P)$ is a nonnegative concave function in P due to the linearity of $H(P \mid Q_0)$ in P and the concavity of f.

Now for each $0 < \mu < \frac{1}{d}$, define Q_μ as the law of $W_\mu(t)$, a nonsymmetric random walk on $\mathbb{Z}^d$:

$$W_\mu(t) = \sum_{i=1}^{A(t)} Y_i \,.$$

Here $A(t)$ is a Poisson process with intensity $2d$ and $\{Y_i : i = 1, 2, \ldots\}$ is a sequence of i.i.d. random vectors with common distribution $\Pr\{Y_1 = e_k\} = 1/2d$ for $2 \leq k \leq d$, $\Pr\{Y_1 = e_1\} = 1/2d + \mu/2$, and $\Pr\{Y_1 = -e_1\} = 1/2d - \mu/2$; $\{e_k : k = 1, \ldots, d\}$ is the standard basis on $\mathbb{Z}^d$.

It is not hard to check that $G(Q_\mu) < \infty$. From this fact, we get that $S_c(\lambda, \alpha, f) < \infty$.

Let P_n be a minimizing sequence selected to satisfy $G(P_n) < G(Q_\mu) + 1$. Hence $H(P_n \mid Q_0) < G(Q_\mu) + 1$. Immediately we have that $\{P_n : n = 1, 2 \ldots\}$ is tight. Assume P_0 to be a limit point of the sequence. Without trouble, we may conclude that P_0 is a minimizer. As $G(P)$ is concave in P, the set of minimizers is a convex set, and therefore we may take P_0 to be an extreme point, hence ergodic.

Step 2. To simply expression, we will drop λ, α, and f dependencies from our notation. Obviously $S^T \leq S_c(\lambda)$ for any $T > 0$. Let $T \to \infty$; we then have $\lim_{T\to\infty} S^T \leq S_c(\lambda)$.

What remains is to check the lower bound. To this end, define the truncated function

$$G_T(P) = f\left(\lambda^\alpha E^P\left(\int_{-T}^{T} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right) + H(P \mid Q_0)\,.$$

Now choose a $P_T \in \mathbf{P}_{\mathrm{si}}(\otimes)$ such that

$$S^T \geq G_T(P_T) - \frac{1}{1+T}\,.$$

This implies $\sup_T H(P_T \mid Q_0) < \infty$ and therefore the tightness of $\{P_T\}$. Let P_1 be a limit point of the set. Without loss of generality, we may assume $P_T \to P_1$. Then for any $\tau > 0$, we have

$$\begin{aligned}\liminf_{T\to\infty} S^T &\geq \liminf_{T\to\infty} G_\tau(P_T)\\ &= G_\tau(P_1)\,.\end{aligned}$$

At this point, we allow $\tau \to \infty$ to obtain $\lim_{T\to\infty} S^T \geq G(P_1) \geq S_c$, thus completing the proof. ∎

We now derive the lower bound in Theorem 1.1.

LEMMA 3.3 *Let f, α, λ, and $E(t;\lambda;f)$ be as in Theorem* 1.1; *then*

$$\liminf_{t\to\infty} E(t) \geq -\inf_{P\in\mathbf{P}_{si}(\mathbf{\Omega})}\left\{f\left(|\lambda|^\alpha E^P\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)+H(P\mid Q_0)\right\}.$$

PROOF: Let $P\in\mathbf{P}_{\rm si}(\mathbf{\Omega})$ be such that $H(P\mid Q_0)<\infty$. For such P, consider the Radon-Nikodym derivative $dP/dQ_0(t)$ with respect to σ-fields generated by $\{w(s):0\leq s\leq t\}$. We write

$$\begin{aligned}
&E(t;\lambda;f)\\
&=\frac{1}{t}\ln E^{Q_0}\exp\left\{-tf\left(\frac{1}{t}\int_0^t\lambda^\alpha\left(\int_0^t\delta_0(\omega(r)-\omega(s))\,dr\right)^{\alpha-1}ds\right)\right\}\\
&=\frac{1}{t}\ln E^{P}\exp\left\{-tf\left(\frac{1}{t}\int_0^t\lambda^\alpha\left(\int_0^t\delta_0(\omega(r)-\omega(s))\,dr\right)^{\alpha-1}ds\right)\right.\\
&\qquad\qquad\left.-\ln\frac{dP}{dQ_0}(t)\right\}\\
&\geq -E^Pf\left(\frac{1}{t}\int_0^t\lambda^\alpha\left(\int_0^t\delta_0(\omega(r)-\omega(s))\,dr\right)^{\alpha-1}ds\right)-\frac{1}{t}E^P\ln\frac{dP}{dQ_0}(t)\\
&\geq -f\left(\frac{1}{t}\int_0^t\lambda^\alpha E^P\left(\int_0^t\delta_0(\omega(r)-\omega(s))\,dr\right)^{\alpha-1}ds\right)-\frac{1}{t}E^P\ln\frac{dP}{dQ_0}(t)\\
&\geq -f\left(\frac{1}{t}\int_0^t\lambda^\alpha E^P\left(\int_{-\infty}^\infty\delta_0(\omega(r)-\omega(s))\,dr\right)^{\alpha-1}ds\right)-\frac{1}{t}E^P\ln\frac{dP}{dQ_0}(t)\\
&= -f\left(\lambda^\alpha E^P\left(\int_{-\infty}^\infty\delta_0(\omega(r))\,dr\right)^{\alpha-1}\right)-\frac{1}{t}E^P\ln\frac{dP}{dQ_0}(t)\,.
\end{aligned}$$

In the above series, we've applied the property that f is concave in the fourth step, that f increases in the fifth step, and that P has stationary increments in the last step.

To finish the argument, let $t\to\infty$. As a consequence,

$$E^P[\ln dP/dQ_0(t)]\to H(P\mid Q_0),$$

and we have

$$\liminf_{t\to\infty}E(t;\lambda;f)\geq -f(\lambda^\alpha E^P(\int_{-\infty}^\infty\delta_0(\omega_r)\,dr)^{\alpha-1})-H(P\mid Q_0)\,.$$

This completes the proof. ∎

4 Asymptotic Behavior of S_c in $d \geq 3$

In this section, we will address the part of Theorem 1.2 concerning transient dimensions $d \geq 3$.

Step 1. We establish the upper bound first. First, we trivially note that $H(Q_0 \mid Q_0) = 0$. Also on account of transience, the number of visits to 0 is almost surely finite in Q_0: $E^{Q_0} \int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds < \infty$. Therefore by Holder's inequality, as $0 < \alpha - 1 \leq 1$,

$$E^{Q_0}\left[\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds\right]^{\alpha-1} < \infty .$$

It is now straightforward to test Q_0 for its infimum, thus obtaining

$$\begin{aligned} S_c(\lambda, \alpha, x^\beta) &\leq \left(\lambda^\alpha E^{Q_0}\left(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right)^\beta + H(Q_0 \mid Q_0) \\ &\leq \lambda^{\alpha\beta}\left(E^{Q_0}\left(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right)^\beta . \end{aligned}$$

Hence $\limsup_{\lambda \to 0} S_c(\lambda, \alpha, \beta)/\lambda^{\alpha\beta} \leq (E^{Q_0}(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1})^\beta$.

Step 2. The lower bound now follows. For the variational quantity

$$\inf_P \left\{\left[\lambda^\alpha E^P\left(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right]^\beta + H(P \mid Q_0)\right\},$$

let P_λ be a minimizer. Recall $G(P)$ from the proof of Lemma 3.2. Then $G(P_\lambda) = S_c(\lambda, \alpha, \beta)$.

It follows from Step 1 that $\lim_{\lambda \to 0} G(P_\lambda) = 0$. Hence $\lim_{\lambda \to 0} H(P_\lambda \mid Q_0) = 0$ implies that P_λ converges weakly to Q_0. Observe now that for any $T > 0$

$$\begin{aligned} S_c(\lambda, \alpha, \beta) &= G(P_\lambda) \\ &\geq \lambda^{\alpha\beta}\left(E^{P_\lambda}\left(\int_{-T}^{T} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right)^\beta . \end{aligned}$$

Consequently,

$$\begin{aligned} \liminf_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\alpha\beta}} &\geq \lim_{\lambda \to 0} \left(E^{P_\lambda}\left(\int_{-T}^{T} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right)^\beta \\ &= \left(E^{Q_0}\left(\int_{-T}^{T} \delta_0(\omega(s))\, ds\right)^{\alpha-1}\right)^\beta . \end{aligned}$$

At this stage, we send $T \to \infty$, obtaining

$$\liminf_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\alpha\beta}} \geq \left(E^{Q_0} \left(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds \right)^{\alpha - 1} \right)^{\beta}$$

to complete the proof.

5 Asymptotic Behavior of S_c in $d = 1, 2$

In this section, we complete the proof of Theorem 1.2 by giving arguments for the second half of the theorem, which concerns the recurrent dimensions $d = 1, 2$. Our proof will consist of two parts. The first is to check the upper bound; the second is to derive the lower bound. The upper estimate is simpler than the lower because we need only select a suitable test measure P to substitute into our variational formula derived earlier for S_c in Theorem 1.1. Calculating the lower bound requires careful use of the entropy inequality.

5.1 Upper Bounds

We begin with the following lemma:

LEMMA 5.1 *Let Q_μ be as defined in the proof of Theorem 3.2. Then*

$$\lim_{\mu \downarrow 0} \frac{E^{Q_\mu} \int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds}{1/\mu} = 1 \tag{5.1}$$

when $d = 1$, and

$$\lim_{\mu \downarrow 0} \frac{E^{Q_\mu} \int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds}{\ln(1/\mu)} = \frac{2}{\pi} \tag{5.2}$$

when $d = 2$.

The proof is a straightforward calculation by using Fourier integral representations for the transition probabilities of random walk in [12]. We skip the details.

LEMMA 5.2 *We have in $d = 2$ that*

$$\limsup_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\alpha\beta} (\ln \frac{1}{\lambda})^{(\alpha - 1)\beta}} < \infty\,. \tag{5.3}$$

In $d = 1$ we have

$$\limsup_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\frac{2\alpha\beta}{2 + \beta(\alpha - 1)}}} < \infty\,. \tag{5.4}$$

PROOF: Recall the object $G(P)$ from Section 3:

$$G(P)=\left(\lambda^{\alpha}E^{P}\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right)^{\beta}+H(P\mid Q_0)\,.$$

Clearly we have that $S_c(\lambda,\alpha,\beta)\le G(P)$ for all $P\in\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$.

Consider for $d=2$ the measure

$$P=Q_{\lambda^{\frac{\alpha\beta}{2}}}\,,$$

and for $d=1$ the related

$$P=Q_{\lambda^{\frac{\alpha\beta}{2+\beta(\alpha-1)}}}\,.$$

In view of Lemma 5.1, by using the well-known fact

$$\lim_{\mu\downarrow 0}\frac{H(Q_\mu\mid Q_0)}{\mu^2}=\frac{d}{2}$$

and Holder's inequality $0<\alpha-1\le 1$, we obtain the upper estimates (5.3) and (5.4), respectively. This completes the proof. ∎

REMARK. In the case $d=2$, the above argument gives exactly

$$\limsup_{\lambda\to 0}\frac{S_c(\lambda,2,1)}{\lambda^2\ln\frac{1}{\lambda}}=\frac{2}{\pi}\,.$$

5.2 Lower Bounds

The following lemma is the crucial step in our argument:

LEMMA 5.3 *For any $t>0$,*

$$\begin{aligned}&S_c(\lambda,\alpha,\beta)\\ &\quad\ge -\frac{1}{2t}\ln E^{Q_0}\exp\left\{-2t\lambda^{\alpha\beta}\left[\int_{-t}^{t}\delta_0(\omega(s))\,ds\right]^{(\alpha-1)\beta}\right\}.\end{aligned}\tag{5.5}$$

PROOF: Observe that for each $t>0$ and $P\in\mathbf{P}_{\mathrm{si}}(\mathbf{\Omega})$

$$\begin{aligned}&\left[|\lambda|^{\alpha}E^{P}\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right]^{\beta}\\ &\quad\ge \lambda^{\beta\alpha}E^{P}\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{(\alpha-1)\beta}\\ &\quad\ge \lambda^{\beta\alpha}E^{P}\left(\int_{-t}^{t}\delta_0(\omega(s))\,ds\right)^{(\alpha-1)\beta}.\end{aligned}$$

At this point, we apply the entropy inequality to the object

$$-2t\lambda^{\alpha\beta}\left(\int_{-t}^{t}\delta_0(\omega(s))\,ds\right)^{(\alpha-1)\beta}.$$

Consequently, we obtain

$$\begin{aligned}&-E^P 2t\lambda^{\alpha\beta}\left(\int_{-t}^{t}\delta_0(\omega(s))\,ds\right)^{(\alpha-1)\beta}\\&\quad\le \ln E^{Q_0}\exp\left\{-2t\lambda^{\alpha\beta}\left[\int_{-t}^{t}\delta_0(\omega(s))\,ds\right]^{(\alpha-1)\beta}\right\}+H_t(P\mid Q_0).\end{aligned}$$

Hence

$$\begin{aligned}&\left[\lambda^{\alpha}E^P\left(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds\right)^{\alpha-1}\right]^{\beta}+\frac{1}{2t}H_t(P\mid Q_0)\\&\quad\ge -\frac{1}{2t}\ln E^{Q_0}\exp\left\{-2t\lambda^{\alpha\beta}\left[\int_{-t}^{t}\delta_0(\omega(s))\,ds\right]^{(\alpha-1)\beta}\right\}.\end{aligned}$$

Now use the fact that

$$\frac{1}{2t}H_t(P\mid Q_0)\uparrow H(P\mid Q_0).$$

to complete the proof. ∎

The next lemma, a beautiful classical limit theorem (often called the Kallianpur-Robbins law) concerning occupation times of random walk, is helpful to us. We only state it here; for a proof of the result and an interesting summary of its early history, see [3] (especially theorem 1).

LEMMA 5.4 *For $d=1$*

$$\lim_{r\to\infty}Q_0\left\{\omega:\frac{2}{\sqrt{r}}\int_0^r\delta_0(\omega(s))\,ds<a\right\}=(\pi)^{-\frac{1}{2}}\int_0^a e^{-\frac{y^2}{4}}\,dy. \tag{5.6}$$

For $d=2$

$$\lim_{r\to\infty}Q_0\left\{\omega:\frac{2\pi}{\ln r}\int_0^r\delta_0(\omega(s))\,ds<a\right\}=1-e^{-a}. \tag{5.7}$$

In the following lemma, we finish the proof of the lower estimates and, in so doing, the proof of Theorem 1.2.

LEMMA 5.5 *For $d = 2$ we have*

$$\liminf_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\alpha\beta} \left(\ln \frac{1}{\lambda}\right)^{(\alpha-1)\beta}} > 0 .$$

For $d = 1$ we have

$$\liminf_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}} > 0 .$$

PROOF: From Lemma 5.3, we have

$$\begin{aligned} &S_c(\lambda, \alpha, \beta) \\ &\geq -\frac{1}{2t} \ln E^{Q_0} \exp\left\{ -2t\lambda^{\alpha\beta}\left[\int_{-t}^{t} \delta_0(\omega(s))\, ds\right]^{(\alpha-1)\beta}\right\}. \end{aligned} \tag{5.8}$$

In the case $d = 2$ select

$$t = \frac{1}{\lambda^{\alpha\beta} \left(\ln \frac{1}{\lambda}\right)^{(\alpha-1)\beta}} .$$

Substituting now into (5.8), we have

$$\begin{aligned} &\frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\alpha\beta}(\ln \frac{1}{\lambda})^{(\alpha-1)\beta}} \\ &\geq -\frac{1}{2} \ln E^{Q_0} \exp\left\{ -\left[\frac{2}{\ln \frac{1}{\lambda}} \int_{-\frac{1}{\lambda^{\alpha\beta}(\ln \frac{1}{\lambda})^{(\alpha-1)\beta}}}^{\frac{1}{\lambda^{\alpha\beta}(\ln \frac{1}{\lambda})^{(\alpha-1)\beta}}} \delta_0(\omega(s))\, ds\right]^{(\alpha-1)\beta}\right\}. \end{aligned}$$

Understanding the convergence in (5.7) gives the result

$$\liminf_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\alpha\beta}(\ln \frac{1}{\lambda})^{(\alpha-1)\beta}} > 0 .$$

For the case $d = 1$, choose

$$t = \frac{1}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}} .$$

The same argument used for the case $d = 2$ still applies except we use (5.6), thus obtaining

$$\liminf_{\lambda \to 0} \frac{S_c(\lambda, \alpha, \beta)}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}} > 0 .$$

This completes our proof. ∎

REMARK. When $d = 2$, the above argument gives exactly

$$\liminf_{\lambda \to 0} \frac{S_c(\lambda, 2, 1)}{\lambda^2 \ln \frac{1}{\lambda}} = \frac{2}{\pi}.$$

We state as a consequence of the two remarks after Lemmas 5.2 and 5.5, respectively, the following limit:

COROLLARY 5.6 *In the case $d = 2$*

$$\lim_{\lambda \to 0} \frac{S_c(\lambda, 2, 1)}{\lambda^2 \ln \frac{1}{\lambda}} = \frac{2}{\pi}.$$

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SUNDER SETHURAMAN
Forschungsinstitut
fur Mathematik
ETH-Zentrum
CH-8092 Zürich
SWITZERLAND
E-mail:
sethuram@math.ethz.ch

LIN XU
Statistics & Operations
Research Program, CEOR
Princeton University
Princeton, NJ 08544
E-mail:
lin@chelsea.princeton.edu

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