On fluctuations in interacting particle systems

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Outline

- Local statistics: Occupation time and other functionals
- Local statistics: Tagged particles
- Bulk limits: LLN of the ‘bulk’ mass and hydrodynamics

  Micro to Macro scaling

  ‘Entropy’ averaging replacement method
- Bulk limits: Fluctuations of the ‘bulk’ mass

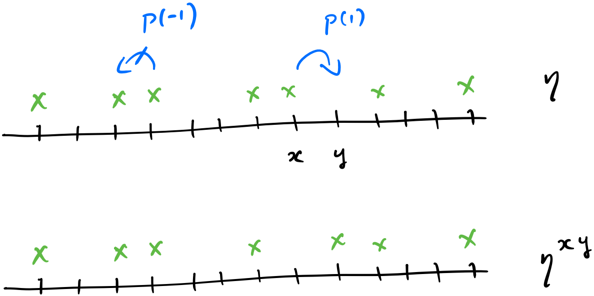
Hydrodynamics: Micro to Macro scaling

Question: How does the mass in an interacting particle system evolve starting from a given initial distribution?

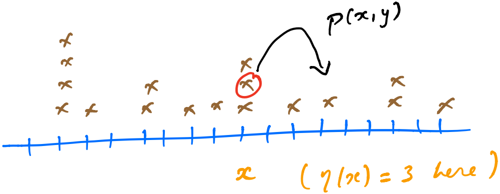
–In the following, we will focus on $S = \mathbb{T}_N^d$.

–In infinite volume on $\mathbb{Z}^d$, results also hold, but there will be extra assumptions and more machinery.

Exclusion interactions



Zero-range interactions



Initial distributions

–Deterministic states: $\eta = \eta^N$, a sequence of configurations.

–Random 'local equilibrium' states:

Let $\rho_0 : \mathbb{T}^d \to [0, 1]$ be a piecewise continuous function.
Define

$$\mu^N = \prod_x \mathrm{Bern}(\rho_0(x/N)) \ \text{ or } \ \prod_x m_{\rho_0(x/N)}$$

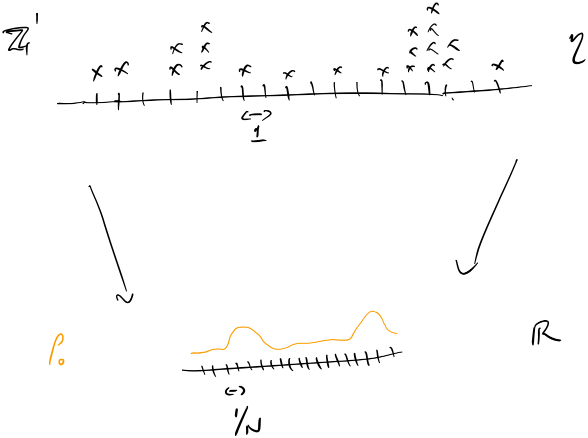
–Of course, when $\rho_0 \equiv \rho$ is a constant function, $\mu^N = \mu_\rho$ is an invariant measure.

We choose initial configurations, whether deterministic or random, so that

$$\frac{1}{N}\sum_x J(x/N)\eta^N(x) \to \int J(u)\rho_0(u)du$$

in probability, for test functions J.

Initial picture



Time scaling

In the bird's eye view, a small number of micro movements are not seen!

–Need to speed up time to correspond to the grid scaling of $1/N$.

–As suspected, the level of 'speed-up' depends on the form of the jump probability p.

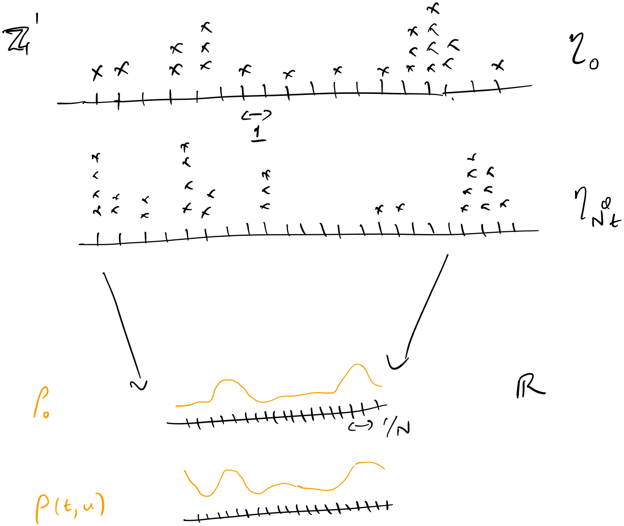
Suppose there is only one particle in the system.

Let X_t be its position at time t.

–If p has a drift, that is $v = \sum_x xp(x) \neq 0$, then in time Nt the position displaces by vNt which, in the macroscopic view, is near $vNt/N = vt$.

–If p is mean-zero (symmetric say), then $v = 0$, and in time N^2t the position displaces by $O(\sqrt{N^2t}) = O(N)$, which is macroscopically $O(1)$.

–For long-range p (not discussed) there may be other relevant scalings N^α, depending on the tails of p.



$\mathbb{Z}^1$
η_0
1
$\eta_{N^\alpha t}$
$\sim$
ρ_0
$\mathbb{R}$
1/N
$\rho(t,u)$

– In the following, we will consider the appropriately scaled evolution,

whether hyperbolic ($\theta = 1$)
or diffusive ($\theta = 2$), and

$$\eta_t^N = \eta_{N^\theta t}$$

generated by $N^\theta L$.

The mass evolution can be captured via the empirical measure.

$$\pi_t^N = \frac{1}{N^d}\sum_x \eta_t^N(x)\delta_{x/N}.$$

–Here,

$$\langle J, \pi_t^N\rangle = \frac{1}{N^d}\sum_x J(x/N)\eta_t^N(x)$$

–If $J = 1_A$, then $\langle J, \pi_t^N\rangle$ is the average number of particles in $NA \subset S$ at time $N^\theta t$.

Martingales

Let $T > 0$ be a finite time-horizon.

Recall for a Markov process Z_t
and $f : [0, T] \times S \to \mathbb{R}$ that

$$M_t^f = f(t, Z_t) - f(0, Z_0) - \int_0^t (\partial_t + L) f(s, Z_s) ds$$

and

$$(M_t^f)^2 - \int_0^t \left(Lf^2 + 2fLf \right) ds$$

are martingales.

First calculations in Exclusion

Let us make calculations in Exclusion, to make a choice of system.

To reduce notation,
let us focus on $d = 1$ and nearest-neighbor p.

Let

$$f(\eta) = \frac{1}{N} \sum_{x} J(x/N)\eta(x).$$

Discrete evolution equation

Write

$$\begin{aligned}\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle &= \int_0^t N^\theta Lf(\eta_s^N) ds + M_t^f \\ &= \int_0^t N^\theta Lf(\eta_s^N) ds + O(1/N).\end{aligned}$$

Some calculations

Recall

$$\begin{aligned}
&N^{\theta} Lf(\eta) \\
&\quad = N^{\theta} \sum_{\pm} \sum_{x} p(\pm 1)\eta(x)(1-\eta(x+1))\big(f(\eta^{xx\pm 1}) - f(\eta)\big).
\end{aligned}$$

Note that

$$\begin{aligned}
&f(\eta^{xx+1}) - f(\eta) \\
&\quad = \frac{1}{N}\Big(J((x+1)/N)\eta(x) + J(x/N)\eta(x+1) \\
&\qquad\qquad - J((x+1)/N)\eta(x+1) - J(x/N)\eta(x)\Big).
\end{aligned}$$

Then,

$$
\begin{aligned}
&= \frac{N^{\theta}}{N} \sum_{x} \big(J((x+1)/N) - J(x/N)\big) \\
&\qquad \times \big\{\eta(x)(1-\eta(x+1))p(1) - \eta(x+1)(1-\eta(x))p(-1)\big\}.
\end{aligned}
$$

To analyze further

If $p(1) = p(-1)$, then

$$\begin{aligned}
&\eta(x)(1-\eta(x+1))p(1) - \eta(x+1)(1-\eta(x))p(-1) \\
&\qquad = \big\{\eta(x) - \eta(x+1)\big\}p(1),
\end{aligned}$$

permitting another summation-by-parts.

On the other hand, if $p(1) > p(-1)$,

with $\gamma = p(1) - p(-1)$,
can write

$$\begin{aligned} p(1) &= \big(p(1) - p(-1)\big) + p(-1) \\ &= \gamma + p(-1). \end{aligned}$$

Then,

$$\begin{aligned} &\eta(x)(1-\eta(x+1))p(1) - \eta(x+1)(1-\eta(x))p(-1) \\ &\quad = p(-1)\big\{\eta(x) - \eta(x+1)\big\} + \underbrace{\gamma\eta(x)(1-\eta(x+1))}. \end{aligned}$$

In the symmetric case $p(1) = p(-1)$,
with $\theta = 2$.

Then,

$$
\begin{aligned}
N^{\theta} Lf(\eta) &= \frac{N^{\theta}}{N} \sum_{x} \big(J((x+1)/N) - J(x/N) \big) \big\{ \eta(x) - \eta(x+1) \big\} p(1) \\
&= \frac{N^{\theta}}{N} \sum_{x} \big(J((x+1)/N) - 2J(x/N) + J((x-1)/N) \big) \eta(x) p(1) \\
&\sim \frac{p(1)}{N} \frac{N^{\theta}}{N^{2}} \sum_{x} J''(x/N) \eta(x) \\
&= p(1) \langle J'', \pi_t^N \rangle .
\end{aligned}
$$

Whereas, in the asymmetric case $p(1) > p(-1)$,
with $\theta = 1$.

Then, to dominant order,

$$N^\theta Lf(\eta) \sim \frac{\gamma}{N} \frac{N^\theta}{N} \sum_x J'(x/N)\eta(x)(1 - \eta(x+1)).$$

Quadratic variation calculation

With respect to the martingale M_t^f, we compute

$$\begin{aligned}
&E\left[(M_t^f)^2\right] \\
&\quad = N^{\theta} E \int_0^t \left(Lf^2 + 2fLf\right) ds \\
&\quad = \frac{1}{N} \frac{N^{\theta}}{N^2} \int_0^t \sum_x \sum_{\pm} \left(J((x \pm 1)/N) - J(x/N)\right)^2 \\
&\qquad\qquad\qquad \times \eta_s^N(x)(1 - \eta_s^N(x \pm 1)p(\pm 1)ds \\
&\quad = O(1/N),
\end{aligned}$$

whether $\theta = 2$ or 1.

Putting together

–When p is symmetric,

$$\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle = p(1) \int_0^t \langle J'', \pi_t^N \rangle ds + O(1/N).$$

–When p is asymmetric,

$$\begin{aligned} &\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle \\ &= \int_0^t \frac{\gamma}{N} \sum_x J'(x/N) \eta_s^N(x)(1 - \eta_s^N(x+1)) ds + O(1/N). \end{aligned}$$

One can ‘see’ the equation that ‘$\rho(t, u)$’ should satisfy from these computations.

–Consider first the symmetric case. Here, the discrete equation is ‘closed’.

Ingredients to identify limit:

- Tightness/compactness of $\{\pi_t^N : t \in [0, T]\}$ in $D([0, T], \mathcal{M})$
- Absolute continuity of limit point $\pi_t^\infty(dx) = \rho(t, x)dx$
- Uniqueness of weak solution of the underlying PDE, and other properties of the solution.

With these in hand, via a subsequence of N, we have

$$\langle J, \pi_t^\infty \rangle - \langle J, \pi_0^\infty \rangle = p(1) \int_0^t \langle J'', \pi_s^\infty \rangle ds.$$

–This is a weak form of

$$\partial_t \rho = p(1) \Delta \rho.$$

–Since there is a unique bounded, weak solution, the limit trajectory

$$\pi^{\infty} = \left\{\pi_t^{\infty} = \delta_{\rho(t,u)du} : t \in [0, T]\right\}$$

is identified.

–Moreover, from continuity of $\rho(t, u)$, at time $t > 0$, we have

$$\pi_t^N \Rightarrow \delta_{\rho(t,u)du}.$$

Replacement

In the asymmetric case, the discrete equation isn't closed.

–We have to deal with the term

$$`\eta(x)\big(1-\eta(x+1)\big)'$$

–This is not restricted to the asymmetric situation.
That is, in Zero-range, even in the symmetric case, it will not be 'closed'.

In fact, making the calculation for Zero-range in $d = 1$
with nearest-neighbor symmetric p,
we get

$$\begin{aligned}&\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle \\ &\quad = p(1) \int_0^t \frac{1}{N} \sum_x J''(x/N) g(\eta_s^N(x)) ds + O(1/N).\end{aligned}$$

–Would like $g(\eta_t^N(x))$ to be approximated by a function of the empirical measure.

Main idea

Let τ_x be the shift of variables by x.

For example, $\tau_x \eta(y) = \eta(x+y)$.

–We may suggest for a function h that

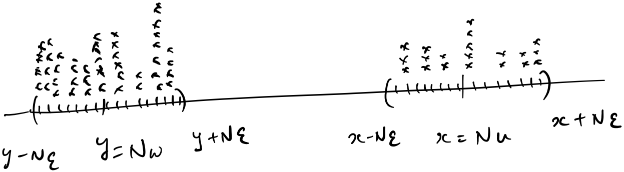
$$h\big(\tau_x \eta_s^N\big) \sim E\Big[h\big(\tau_x \eta_s^N\big) \,|\, \frac{1}{2N\epsilon + 1} \sum_{|z-x| \le N\epsilon} \eta_s^N(z)\Big].$$

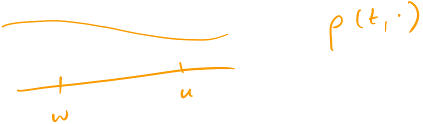
Rationale

The system has had time to mix up.

–At micro $x = Nu$, at micro time $N^\theta s$, in a micro window of width $O(N\epsilon)$, the system behaves, approximately, according to the (random) local density

$$\eta^{(N\epsilon)}_{N^\theta s}(x) := \frac{1}{2N\epsilon + 1} \sum_{|z-x|\leq N\epsilon} \eta^N_s(z).$$


$y - N\varepsilon$
$y = Nw$
$y + N\varepsilon$
$x - N\varepsilon$
$x = Nu$
$x + N\varepsilon$


$\rho(t,\cdot)$
w
u

To first order, we might think the system is approximately in an invariant state with this density:

$$\begin{aligned} E\Big[h\big(\tau_x\eta^N_s\big)\big|\eta^{(N\epsilon)}_{N^\theta s}(x)\Big] &\sim E_{\mu_{\eta^{(N\epsilon)}_{N^\theta s}(x)}}[h] \\ &= H\Big(\eta^{(N\epsilon)}_{N^\theta s}(x)\Big). \end{aligned}$$

Note that $\eta^{(N\epsilon)}_{N^\theta s}(x)$ is a function of the empirical measure:

$$\begin{aligned}\eta^{(N\epsilon)}_{N^\theta s}(x) &= \frac{1}{2N\epsilon+1}\sum_{|z-x|\le N\epsilon}\eta^N_s(z)\\ &\sim \langle i_\epsilon, \pi^N_s\rangle,\end{aligned}$$

where $i_\epsilon(u) = (2\epsilon)^{-1}1(|u|\le\epsilon)$.

–This allows us to ‘replace’ and close the equation.

With such replacement, taking $\epsilon \downarrow 0$ at the end, applied to

$$h(\eta) = \eta(0)(1 - \eta(1))$$
$$h(\eta) = g(\eta(0))$$

in the asymmetric Exclusion and
symmetric Zero-range models,

one arrives at

$$\partial_t \rho + \gamma \partial_x \big(\rho(1-\rho)\big) = 0, \ \text{ and}$$
$$\partial_t \rho = p(1)\Delta G(\rho).$$

–Here, by a calculation, $H(\rho) = \rho(1-\rho)$ and $G(\rho) = \phi(\rho)$.

Symmetric results

Theorem. Consider symmetric Exclusion or Zero-range on $\mathbb{T}_N^d$ with nearest-neighbor $p(e) = 1/(2d)$, starting from local equilibrium μ^N associated to $\rho_0(\cdot)$.

For each $t \geq 0$, we have $\pi_t^N \Rightarrow \rho(t,u)du$ where respectively

$$\partial_t \rho = p(e)\Delta\rho$$
$$\partial_t \rho = p(e)\Delta\phi(\rho)$$

such that $\rho(0,u) = \rho_0(u)$.

Comments

These results go back to Guo-Papanicolaou-Varadhan 1988 and Yau 1991.

–There are generalizations:

- p may be finite, or infinite range with different time-scalings
- Initial conditions may be deterministic or other random measures associated to ρ_0.
- S may be $\mathbb{Z}^d$, with additional growth assumptions on $\rho_0(\cdot)$

See for instance Kipnis-Landim 1999, Seppäläinen 2008,
Fritz 1990, Yau 1994,
Lu 1995, Jara 2008, SS-Shahar 2018.

Asymmetric results

Theorem. Consider asymmetric Exclusion on $\mathbb{T}_d^N$ with nearest-neighbor p, starting from local equilibrium μ^N associated to $\rho_0(\cdot)$.

For each $t \geq 0$, we have $\pi_t^N \Rightarrow \rho(t,u)du$ where $\rho(t,u)$ is the unique ‘entropy’ solution of

$$\partial_t \rho + \gamma \cdot \nabla\big(\rho(1-\rho)\big) = 0$$

such that $\rho(0,u) = \rho_0(u)$.

–Here, $\gamma = \sum zp(z)$.

Comments

Here, ‘entropy’ solution is the ‘physical’
or ‘viscosity’ solution of the equation:

Add $\epsilon\Delta\rho$ to the equation. The limit of $\rho = \rho^{\epsilon}$, as $\epsilon \downarrow 0$, is the desired solution.

–There are other ways to define the ‘entropy’ solution, e.g. via Kruzkov, Hopf-Lax, etc. formulations (not discussed).

The proof, based on Young measures and Diperna-Lions theory,

works for Exclusion, and

Zero-range models where *g* is an increasing function,

e.g. particle systems

which allow ‘monotone particle couplings’.

See Rezakhanlou (1991),
Bahadoran-Guiol-Ravishankar-Saada (2017) for instance, and also Loulakis-Stamatakis (2019).

–An open problem, however, is to show hydrodynamics for more general asymmetric interacting particle systems.

Replacement proof ideas

Consider processes in diffusive scale $\theta = 2$.

Let h be a local function,

e.g. $h(\eta) = \eta(0)\big(1 - \eta(1)\big)$ in $d = 1$, etc.

–Our goal now is to give ideas in the replacement of

$$h\big(\eta_t^N(x)\big)$$

by

$$E_{\mu_{\rho(t,u)}}[h] =: H\big(\rho(t,u)\big)$$

where $x = Nu$.

‘Entropy’ method

We discuss the ‘entropy’ method of GPV 1988. This method works well when

- the setting is translation-invariant
- the dynamics is reversible ($\theta = 2$)

–Elements of the ‘entropy’ method replacement though are also useful in the asymmetric setting as well,
e.g. the upcoming 1-block lemma will hold.

Recall τ_x denotes a shift by x.
Let J be a test function.

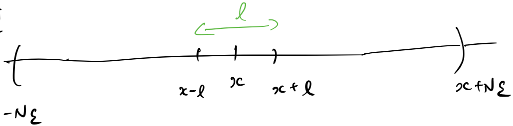
Theorem. We have

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\uparrow\infty} E_{\mu^N}\Big[\Big|\int_0^T \frac{1}{N^d}\sum_x J\big(\tfrac{x}{N}\big)\ \tau_x V_{N\epsilon}(\eta_s^N)ds\Big] = 0$$

where

$$V_m(\eta) = \Big\{h(\eta) - H\big(\eta^{(m)}(0)\big)\Big\}.$$

$d=1$



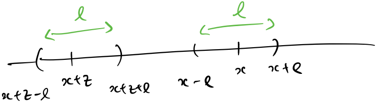
- $\tau_x h(\eta) \approx \frac{1}{2\ell+1} \sum_{|y|<\ell} \tau_{x+y} h(\eta)$

  $\approx H(\eta^{(\ell)}(x))$ "1-block"

- $H(\eta^{(\ell)}(x)) \approx H(\eta^{(N\varepsilon)}(x))$ "2-block"

  via $\eta^{(\ell)}(x) \approx \eta^{(\ell)}(x+z)$

  where $|z| < N\varepsilon$.



1-block lemma

Let P_t^N be the semigroup for the N^θ speeded up process, and let μ_ρ be a reference measure.

Denote the probability density

$$f_T^N := \frac{1}{T}\int_0^T \frac{d\mu^N P_s^N}{d\mu_\rho} ds.$$

Let also

$$W_\ell(\eta) = \Big| \frac{1}{(2\ell+1)^d} \sum_{|y|\le\ell} \tau_y h(\eta) - H(\eta^{(\ell)}(0)) \Big|.$$

Then,

$$E_{\mu^N}\Big[\int_0^T \frac{1}{N^d}\sum_x \tau_x W_\ell(\eta_s^N)ds\Big]$$

$$= T\ E_{\mu_\rho}\big[f_T^N(\eta)\ \frac{1}{N^d}\sum_x \tau_x W_\ell(\eta)\big].$$

<u>Lemma</u> (1-block) We have

$$\lim_{\ell\uparrow\infty}\lim_{N\uparrow\infty} E_{\mu_\rho}\big[f_T^N(\eta)\ \frac{1}{N^d}\sum_x \tau_x W_\ell(\eta)\big] = 0.$$

2-block lemma

We also have

Lemma (2-block) We have

$$\lim_{\ell\uparrow\infty}\lim_{\epsilon\downarrow 0}\lim_{N\uparrow\infty}\sup_{2\ell\le|y|\le 2N\epsilon} E_{\mu_\rho}\left[f_T^N(\eta)\,\frac{1}{N^d}\sum_x\left|\eta^{(\ell)}(x)-\eta^{(\ell)}(x+y)\right|\right]=0.$$

Sketch: 1-block lemma (in $d = 1$)

The idea is, when localized in a ℓ-block,
the Dirichlet form of $f = f_T^N$ vanishes in the $N \uparrow \infty$ limit.

This means f is roughly constant,
and ergodicity w.r.t. μ_ρ now applies.

Measuring f_T^N

1. Consider the relative entropy of μ^N with respect to μ_ρ:

$$H(\mu^N;\mu_\rho) = E_{\mu^N}\Big[\log\frac{d\mu^N}{d\mu_\rho}\Big]$$

$$= \sum_x \Big[\rho_0(x/N)\log\rho_0(x/N)/\rho + (1-\rho_0(x/N))\log\frac{1-\rho_0(x/N)}{1-\rho}\Big]$$

This is $O(N^d)$ as we are on the torus $\mathbb{T}_N^d$.

2. At time t, for the N^2-speeded up process,
   with semigroup P_t^N,

the rate of change is

$$\frac{d}{dt} H(\mu^N P_t^N; \mu_\rho) = N^\theta E_{\mu_\rho}\Big[\frac{d\mu^N P_t^N}{d\mu_\rho} L \log \frac{d\mu^N P_t^N}{d\mu_\rho}\Big].$$

3. A calculation gives

$$E_{\mu_\rho}\Big[\frac{d\mu^N P_t}{d\mu_\rho} L \log \frac{d\mu^N P_t}{d\mu_\rho}\Big] \le -2D\Big(\sqrt{\frac{d\mu^N P_t^N}{d\mu_\rho}}\Big)$$

where, for Exclusion,

$$\begin{aligned} D(h) &= E_{\mu_\rho}\big[h(-Lh)\big] \\ &= \frac{1}{2}\sum_{x,y} p(y-x) E_{\mu_\rho}\Big[\big(h(\eta^{xy}) - h(\eta)\big)^2\Big]. \end{aligned}$$

4. Then $(**)$,

$$\frac{d}{dt}H(\mu^N P_t^N;\mu_\rho) \le -2N^2 D\Big(\sqrt{\frac{d\mu^N P_t^N}{d\mu_\rho}}\Big)$$

and

$$\begin{aligned} H(\mu^N P_T^N;\mu_\rho) + 2N^2 \int_0^T D\Big(\sqrt{\frac{d\mu^N P_s^N}{d\mu_\rho}}\Big)ds \\ \le\ H(\mu^N;\mu_\rho)\ \le\ CN^d. \end{aligned}$$

**Uses $a\log(b/a) \le \sqrt{a}\big[\sqrt{b}-\sqrt{a}\big]$ for $a, b > 0$.

5. Abbreviate and recall

$$I(h) = D(\sqrt{h}), \ \text{and} \ f_T^N = \frac{1}{T}\int_0^T \frac{d\mu^N P_s^N}{d\mu_\rho} ds.$$

–By convexity of Dirichlet form,

$$I\big(f_T^N\big) \le CTN^{d-2},$$

which vanishes in $d = 1$.

–A more involved argument can handle $d \ge 2$.

6. Write

$$\begin{aligned} E_{\mu_\rho}\big[f(\eta)\cdot\frac{1}{N^d}\sum_x \tau_x W_\ell(\eta)\big] &= E_{\mu_\rho}\big[Av(f)\cdot W_\ell(\eta)\big] \\ &= E_{\mu_\rho}\big[f_\ell(\eta)W_\ell(\eta)\big]. \end{aligned}$$

where

$$Av(f) = \frac{1}{N^d}\sum_x \tau_x f(\eta) \ \text{ and } \ f_\ell(\eta) = E_{\mu_\rho}[Av(f)|\mathcal{F}_\ell].$$

–Here, $\mathcal{F}_\ell = \sigma\{\eta(x) : |x| \leq \ell\}$.

Note, by convexity, that $I(f_\ell) \leq CN^{d-2}$.

7. Consider the 'ℓ-block' Dirichlet form

$$I_\ell(w) = \sum_{x,y:|x|,|x+y|\le\ell} I_{x,x+y}(w)$$

where

$$I_{x,x+y}(w) = \frac{1}{2}p(y-x)E_{\mu_\rho}\Big[\big(\sqrt{w}(\eta^{xy}) - \sqrt{w}(\eta)\big)^2\Big].$$

Considering limit points as $N \uparrow \infty$, need only show

$$\lim_{\ell \uparrow \infty} \sup_{I(f)=0} E_{\mu_\rho}\big[f(\eta) V_\ell(\eta)\big] = 0.$$

But, looking at the form of $I(\cdot)$,
if $I_\ell(f) = 0$,
conclude f is constant
on configurations on $\{-\ell, \cdots, \ell\}$ such that $\eta^{(\ell)}(0) = a$.

–Here, the values $0 \le a \le 1$ (in the Exclusion process).

8. So, it would be enough to show

$$\lim_{\ell\uparrow\infty}\sup_{0\leq a\leq 1} E_{\mu_\rho}\Big[\Big|\frac{1}{(2\ell+1)^d}\sum_{|x|\leq\ell}\tau_x h(\eta)-H(a)\Big|\Big|\eta^{(\ell)}(0)=a\Big]=0.$$

–Recall $H(a)=E_{\mu_a}[h]$.

–The measure μ_ρ is product.

–At this point, the last limit can be seen via local central limit theorems, for instance.

Comment on other methods

We have discussed here the ‘entropy’ method of GPV 1988.

–There are however a few other ways ‘hydrodynamics’ has been proved:

Relative entropy methods, Compensated compactness ideas, gradient flows,
use of log-Sobolev, etc.

For example Yau 1991, Fritz-Toth 2014, Fritz 2004, 2011,
Toth-Valko 2003, Fathi-Menz 2014, Fathi-Simon 2015,
Grunewald-Otto-Villani-Westdickenberg 2009

–Recent progress on ‘Hilbert’s Sixth Problem’:
Deng-Hani-Ma 2025