On fluctuations in interacting particle systems

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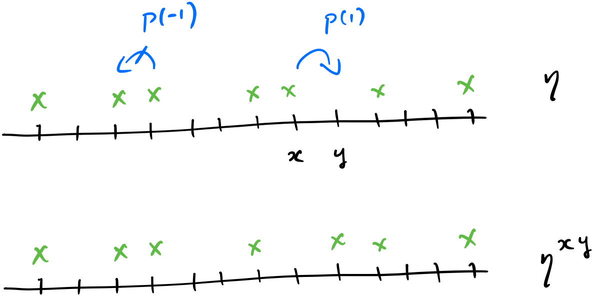
Outline

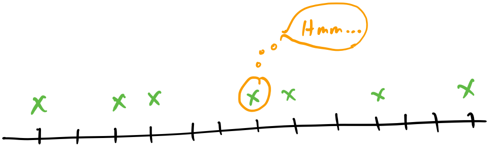
- Local statistics: Occupation times
- Local statistics: Tagged particles
  - Some history: LLNs, CLTs, and an open problem
  - H_{-1} norms and more on KV theorem
  - Extreme values limits
- Bulk limits: LLN of the ‘bulk’ mass and hydrodynamics
- Bulk limits: Fluctuations of the ‘bulk’ mass

Simple exclusion



Although the particles are not labeled, it is a natural problem to 'tag' say one of them, and to follow its motion $\{X_t\}_{t\geq 0}$.

–This motion is undoubtedly influenced by the other particles, including 'bulk' mass notions.



Notation

Recall

$$\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$$

is the configuration of the unlabeled process at time t where

$$\eta(x) = \begin{cases} 1 & \text{if } x \text{ occupied} \\ 0 & \text{otherwise.} \end{cases}$$

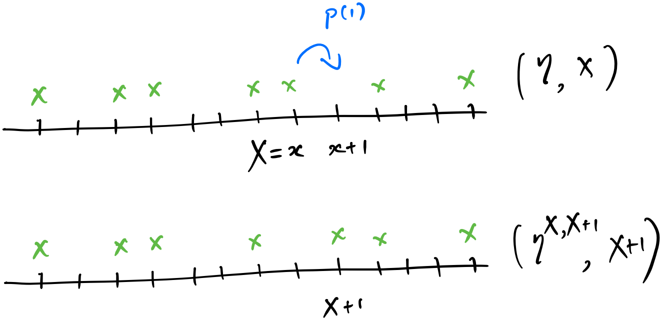
–We have noted that the process is η_t Markovian on $\{0,1\}^{\mathbb{Z}^d}$ with generator

$$Lf(\eta) = \sum_{x,i,\pm} p(\pm e_i)\eta(x)(1-\eta(x \pm e_i))\big\{f(\eta^{x,x\pm e_i}) - f(\eta)\big\}.$$

On the other hand, in general,
the process X_t is not Markovian with respect to its own history.

–But, the joint process (η_t, X_t) is Markovian on $\{0,1\}^{\mathbb{Z}^d} \times \mathbb{Z}^d$, with a corresponding generator.

Informally:



Specifically,

$$\tilde{L}f(\eta, X) = \sum_{u,v\neq X} p(v-u)\eta(u)(1-\eta(v))\big[f(\eta^{u,v}, X) - f(\eta, X)\big] + \sum_{v} p(v)(1-\eta(X+v))\big[f(\eta^{X,X+v}, X+v) - f(\eta, X)\big].$$

–One can follow also (ζ, X) where ζ is the process in the reference frame of X, namely $\zeta_t(\cdot) = \eta_t(X_t + \cdot)$.

The generator in these variables is

$$
\begin{aligned}
\hat{L}f(\zeta, X) \;&=\; \sum_{u,v\neq 0} p(v-u)\zeta(u)(1-\zeta(v))\big[f(\zeta^{u,v}, X) - f(\zeta, X)\big] \\
&\qquad + \sum_{v} p(v)(1-\zeta(v))\big[f(\theta_v\zeta, X+v) - f(\zeta, X)\big]
\end{aligned}
$$

where $\theta_v\zeta$ is the configuration which displaces the particle at the origin, namely the tagged particle, by v, and then shifts the reference frame to this new origin.

$$
(\theta_v\zeta)(y) \;=\; \begin{cases} \zeta(y+v) & \text{for } y \neq -v, 0 \\ \zeta(v) & \text{for } y = -v \\ 1 & \text{for } y = 0. \end{cases}
$$

Interestingly, the process ζ_t by itself is Markov with generator

$$Lf(\zeta) = \sum_{u,v\neq 0} p(v-u)\zeta(u)(1-\zeta(v))\big[f(\zeta^{u,v}) - f(\zeta)\big] + \sum_{v} p(v)(1-\zeta(v))\big[f(\theta_v\zeta) - f(\zeta)\big].$$

Moreover, ζ_t has much information: Here, X_t can be recovered by counting the reference frame shifts.

Invariant measures. Recall that the Bernoulli product measures μ_ρ are invariant for the η_t process.

–One may see also that $\nu_\rho = \mu_\rho(\cdot|\eta(0) = 0)$ are invariant, and extremal for ζ_t.

Martingale decomposition

Take the function $f(\zeta, X) = X$, and note

$$Lf(\zeta, X) = \sum vp(v)(1 - \zeta(v)).$$

Recall also that

$$\hat{M}(t) = f(\zeta_t, X_t) - f(\zeta_0, X_0) - \int_0^t Lf(\zeta_s, X_s) ds$$

and

$$\left(\hat{M}(t) \cdot \ell\right)^2 - \int_0^t L\left(f(\zeta_s, X_s) \cdot \ell)\right)^2 - 2(f(\zeta_s, X_s) \cdot \ell) L\left(f(\zeta_s, X_s) \cdot \ell\right) ds$$

are martingales.

We see then that

$$X_t \;=\; \hat{M}(t) + \int_0^t \underbrace{\sum vp(v)(1-\zeta_s(v))}\, ds$$

and the quadratic variation

$$\langle \hat{M}\cdot\ell\rangle_t \;=\; \int_0^t \sum_v (v\cdot\ell)^2 p(v)(1-\eta_s(v))ds.$$

Some tagged particle literature

Let us start from ν_ρ,
tagging the particle at the origin.

We will concentrate on the finite-range setting.

–Let $\gamma = \sum_z zp(z)$
be the drift of the jump probabilities.

–Note: Other initial conditions can be considered.
We will discuss such a problem in the second part of the talk.

In the $d = 1$ nearest-neighbor setting, recall the models are as follows.

TASEP $p(1) = 1$, $p(-1) = 0$.

ASEP $p = p(1) > p(-1) = 1 - p = q$.

SSEP $p(1) = p(-1) = 1/2$.

LLN: In $d \geq 1$, introducing a scaling factor N,

$$\frac{1}{N} X_{Nt} \to \gamma[1-\rho]t,$$

as $N \uparrow \infty$.

–implicit in the more general works, which treat more general initial conditions,

Saada '87, Rezakhanlou '94.

CLT: in 1D, with respect to ASEP,

$$\frac{X_N t - \gamma(1-\rho)Nt}{\sqrt{N}} \Rightarrow \sqrt{\gamma(1-\rho)t}B(t).$$

Kipnis '87, Ferrari-Fontes '96.

Tracy-Widom '07-'09 (wrt step profiles, a nonstationary setting).

In contrast, in 1D, with respect to SSEP,

$$\frac{1}{N^{1/4}} X_{Nt} \Rightarrow \left(\tfrac{2}{\pi}\right)^{1/4} \left(\tfrac{1-\rho}{\rho}\right)^{1/2} fBM_{1/4}(t)$$

Arratia '83, Jara-Landim '06 (also in nonstationary setting), Peligrad-SS '08, Erhad-Franco-Xu 2024.

When $\gamma = 0$, in $d \geq 2$
or in $1D$ for non-nearest neighbor finite-range processes,

$$\frac{1}{\sqrt{N}}\big(X_{Nt} - \gamma(1-\rho)t\big) \Rightarrow \sigma(\rho, p)B_t$$

Kipnis-Varadhan '86, Varadhan '90;
see also Jara '07 for long-range jumps.

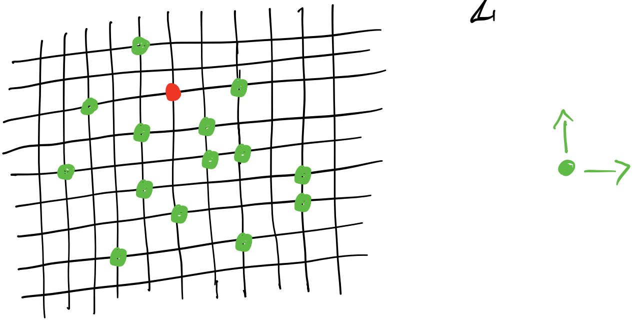
–When $\gamma \neq 0$ and $d \geq 3$,

$$\frac{X_{Nt} - \gamma(1-\rho)Nt}{\sqrt{N}} \Rightarrow \sigma(\rho, p)B_t$$

SS-Varadhan-Yau '00

Open question

The CLT, starting from $\nu_\rho(\cdot|\eta(0) = 1)$ for $d = 2$ and $d = 1$ with non nearest-neighbor jumps, when $\gamma \neq 0$, is open.



SS '07 shows that $Var(X_t) = O(t)$.

Recall of KV theorem

We will use the *proof* of the KV theorem to derive tagged particle CLTs in the reversible setting.

To this end, we discuss notions of H_1 and H_{-1} norms, as well as a useful 'helping' Proposition valid for nonreversible chains, of interest perhaps on its own.

KV Thm. Let Z_t be a stationary, reversible, ergodic Markov process on Σ with invariant measure π.

Let $f : \Sigma \to \mathbb{R}$ be such that $E_\pi[f] = 0$ and

$$\sigma_f^2 = \lim_{t\to\infty} \frac{1}{t} Var\left(\int_0^t f(Z_s)ds\right).$$

Then,

$$\frac{1}{\sqrt{t}} \int_0^t f(Z_s)ds \Rightarrow N(0, \sigma_f^2).$$

H_1 and H_{-1} norms

Let L and P_t be the $L^2(\pi)$ generator and semigroup for a reversible Markov process.

Define, for $\lambda > 0$,

$$\|\phi\|_{1,\lambda}^2 = \langle \phi, (\lambda - L)\phi \rangle_\pi = D(\phi) + \lambda \|\phi\|_{L^2(\pi)}^2.$$

Since $-L$ is nonnegative, self-adjoint,

$$\begin{aligned} \langle \phi, (\lambda - L)\psi \rangle_\pi &= \langle (\lambda - L)^{1/2}\phi, (\lambda - L)^{1/2}\psi \rangle_\pi \\ &\leq \|\phi\|_{1,\lambda} \|\psi\|_{1,\lambda}. \end{aligned}$$

Define also

$$\|\phi\|_{-1,\lambda} = \sup\left\{\frac{\langle \phi, \psi\rangle_\pi}{\|\psi\|_{1,\lambda}}\right\}.$$

Note that $(\lambda - L)^{-1}$ is a nonnegative, bounded operator for $\lambda > 0$. Then,

$$\|\phi\|^2_{-1,\lambda} = \langle \phi(\lambda - L)^{-1}\phi\rangle_\pi.$$

These norms are dual to each other:

$$\langle \phi, \psi\rangle_\pi \leq \|\phi\|_{1,\lambda}\|\psi\|_{-1,\lambda}.$$

Note that $\langle \phi, P_t \phi \rangle_\pi$ is nonnegative as P_t is self-adjoint.

The (monotone) limits $\lambda \downarrow 0$ of

$$\|\phi\|_{1,\lambda} = \lambda \|\phi\|^2_{L^2(\pi)} + D(\phi)$$
$$\|\phi\|_{-1,\lambda} = \int_0^\infty e^{-\lambda t} \langle \phi, P_t \phi \rangle_\pi$$

are the so-called H_1 and H_{-1} norms of ϕ.

In particular,

$$\|\phi\|_1 = \langle \phi, -L\phi \rangle_\pi = D(\phi)$$

and

$$\begin{aligned} 2\|\phi\|_{-1}^2 &= 2\lim_{\lambda\downarrow 0}\|\phi\|_{-1,\lambda}^2 \\ &= 2\lim_{\lambda\downarrow 0}\langle\phi,(\lambda-L)^{-1}\phi\rangle_\mu \\ &= 2\lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda t}E_\mu[\phi(P_t\phi)]dt \\ &= 2\int_0^\infty E_\mu[\phi(\eta_t)\phi(\eta_0)]dt \;=\; \sigma_\phi^2. \end{aligned}$$

–We have the KV condition

$$\sigma_f^2 < \infty \;\Leftrightarrow\; \|f\|_{-1} < \infty.$$

Back to KV theorem

Recall the resolvent equation

$$f = \lambda u_\lambda - L u_\lambda$$

where

$$u_\lambda(\eta) = \int_0^\infty e^{-\lambda t} P_t f(\eta) dt.$$

Then,

$$\int_0^t f(\eta_s) ds = M_\lambda(t) + \xi_\lambda(t)$$

with

$$\begin{aligned} M_\lambda(t) &= u_\lambda(\eta_t) - u_\lambda(\eta_0) - \int_0^t Lu_\lambda(\eta_s)ds \\ \xi_\lambda(t) &= u_\lambda(\eta_0) - u_\lambda(\eta_t) + \lambda \int_0^t u_\lambda(\eta_s)ds \end{aligned}$$

Note the mean quadratic variation

$$E_\pi \langle M_\lambda \rangle(t) = 2t \| u_\lambda \|_{1,\lambda}^2.$$

Proposition. Suppose the following Condition holds:

$$\lambda\|u_\lambda\|^2_{L^2(\pi)} \to 0 \text{ and } \{u_\lambda\} \text{ is Cauchy in } H_1.$$

Then, the statement of KV theorem holds.

–Moreover, for a reversible process, if $\|f\|_{-1} < \infty$, then the Condition holds.

–The Proposition gives a possible way to extend the Kipnis-Varadhan theorem to nonreversible models.

Question: Is there a good assumption, in the nonreversible setting, which would allow verification of the Condition?

Back to the reversible model.

–From the Condition, $M_\lambda(t) \to M_0(t)$ and we can write

$$\int_0^t f(\eta_s)ds = M(t) + (M_\lambda(t) - M_0(t)) + \xi_\lambda(t).$$

It is a calculation, choosing $\lambda = 1/t$, to see that the Condition implies

$$\frac{1}{\sqrt{t}}\Big\{(M_\lambda(t) - M_0(t)) + \xi_\lambda(t)\Big\} \to 0 \quad \text{in } L^2(\pi).$$

A way to show the Condition in the Proposition is the ‘sector’ inequality:

$$\|L\phi\|_1 \leq C\|\phi\|_1$$

always valid for reversible L with $C = 1$.

Multiply both sides of the resolvent equation

$$\lambda u_\lambda - L u_\lambda = f$$

by u_λ and take E_π expectation:

$$\lambda \|u_\lambda\|^2_{L^2(\pi)} + \|u_\lambda\|_1^2 = \langle f, u_\lambda \rangle_\pi .$$

Since $\langle f, u_\lambda \rangle_\pi \leq \|f\|_{-1} \|\phi\|_1$, we have for all $\lambda > 0$ that both

$$\lambda \|u_\lambda\|^2_{L^2(\pi)},\ \|u_\lambda\|_1^2 \ \leq\ \|f\|^2_{-1} .$$

Hence, there is a subsequence $u_{\lambda_n} \to w$ weakly in H_1, and by the sector inequality, $Lu_{\lambda_n} \to Lw$ weakly in H_{-1}.

–A few more functional analysis calculations will yield the Condition (omitted here).

For mean-zero asymmetric Exclusion,
the sector inequality is still valid with $C = C\big(p(\cdot)\big)$
(Varadhan 1990).

–For $d \geq 3$ asymmetric Exclusion, a 'graded sector' inequality is shown, suitable to show the Proposition.

SS-Varadhan-Yau 2000, Landim-Komorowski-Olla 2012

Back to tagged particle

Recall

$$X_t \; = \; \hat{M}(t) + \int_0^t \sum vp(v)(1 - \zeta_s(v))ds.$$

We claim when $p(v) = p(-v)$, that the function $f(\zeta) = \sum vp(v)(1 - \zeta(v))$ is in H_{-1},
for v in the support of $p(\cdot)$.

–Indeed, it is enough to show that $\zeta(v) - \zeta(-v) \in H_{-1}$.

The Dirichlet form can be computed also on local functions:

$$\begin{aligned}
D(\phi) &= E_{\nu_\rho}[\phi(-L\phi)] \\
&= \frac{1}{2}\sum_{u,w\neq 0} p(w-u)E_{\nu_\rho}[(\phi(\zeta^{u,w})-\phi(\zeta))^2] \\
&\quad +\frac{1}{2}\sum_{w} p(w)E_{\nu_\rho}[(1-\zeta(v))\phi(\theta_v\zeta)-\phi(\zeta))^2] \\
&= D_e(f)+D_t(f)
\end{aligned}$$

Write

$$
\begin{aligned}
&\langle \zeta(v) - \zeta(-v), \phi \rangle_{\nu_\rho} \\
&\quad = \langle (1 - \zeta(v)) - (1 - \zeta(-v)), \phi \rangle_{\nu_\rho} \\
&\quad = E_{\nu_\rho}[(1 - \zeta(v))\phi(\zeta)] - E_{\nu_\rho}[(1 - \theta_v(\zeta)(-v))\phi(\theta_v(\zeta))] \\
&\quad = E_{\nu_\rho}[(1 - \zeta(v))(\phi(\zeta) - \phi(\theta_v(\zeta))] \\
&\quad \leq E_{\nu_\rho}[(1 - \zeta(v))^2(\phi(\zeta) - \phi(\theta_v(\zeta))^2]^{1/2} \\
&\quad = E_{\nu_\rho}[(1 - \zeta(v))(\phi(\zeta) - \phi(\theta_v(\zeta))^2]^{1/2}.
\end{aligned}
$$

Now, the last quantity is less than $\frac{2}{p(v)} \times D(\phi)^{1/2}$.

–Hence, f belongs to H_{-1}.

By the KV theorem proof we have

$$\int_0^t \sum vp(v)(1-\zeta_s(v))ds = M_0(t) + \xi_\lambda(t).$$

–Then,

$$\frac{1}{\sqrt{t}} X_t \to N(0, \Lambda)$$

in all $d \geq 1$ where Λ is the limiting covariance of

$$\frac{1}{\sqrt{t}} \Big\{ \hat{M}(t) + M_0(t) \Big\}.$$

In the exceptional case,
that is the $d = 1$ nearest-neighbor model,

it turns out that the two martingales
$\hat{M}$ and M_0 cancel each other,

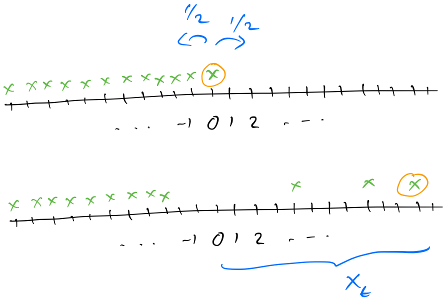
leading to subdiffusive fluctuations for X_t.

–However, in all other situations, full cancellation does not occur, and diffusive limits result.

Extreme value limits

We now turn to the second part of the talk.

–One may ask about the behavior of the ‘lead’ particle in 1D symmetric nearest-neighbor exclusion, starting from a step profile.



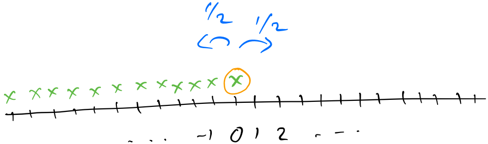
–We discuss that X_t has an anomalous displacement $O\left(\sqrt{t\log(t)}\right)$ in time t (Arratia 1983).

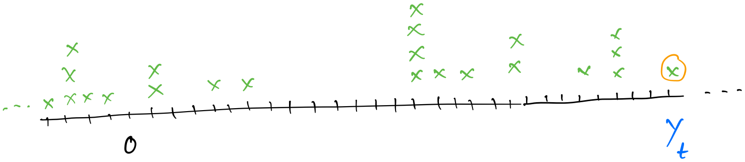
Question: What is the scaled fluctuation behavior of X_t?

To get intuition, consider the setting of *independent* particles, starting from a step profile.

–In this case, particles can pile up: Let

$$Y_t = \max\left\{\xi_t^\ell : \ell \leq 0\right\}.$$





It is known that

$$\frac{Y_t}{b_t} - a_t \Rightarrow e^{-e^{-x}}$$

where

$$b_t = \sqrt{\frac{t}{\log(t)}} \text{ and } a_t = \log\left(\frac{t}{\sqrt{2\pi}\log(t)}\right).$$

–Arratia 1983

A suggestion is the same limit holds for the lead particle position X_t in SSEP.

–Because of the interactions, there is a ‘misanthropic’ effect. When particles are apart, they move with their own rates.

So, comparing with independent particles may be reasonable, if one can deal with the correlations.

Gumbel limit

Recently, it was shown

Result.

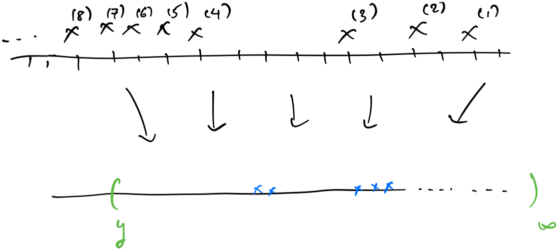
$$\frac{X_t}{b_t} - a_t \Rightarrow e^{-e^{-x}}$$

Conroy-SS 2023

Moreover, a Poisson point process for the order statistics can be shown:

$$\sum_k \delta\left(\frac{X_t^{(k)}}{b_t} - a_t\right) \Rightarrow PPP(e^{-x}dx).$$

–Conroy, Gonzalez-Casanova, SS 2025

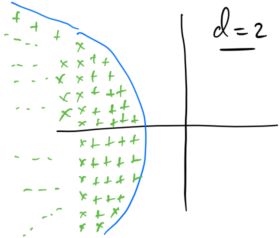


The initial condition can be generalized to that where

$$\lim_n \frac{1}{n} \sum_{i=-n}^{0} \eta(i) = v > 0.$$

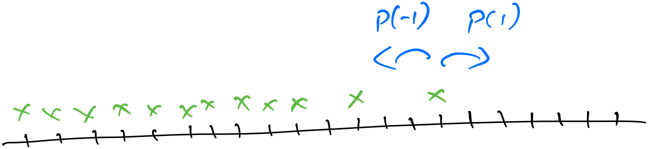
–Also, the symmetric dynamics can be finite-range or types of infinite-range.

–Also, *d*-dimensional symmetric Exclusion versions starting from ‘half space’ initial configurations have been considered. Conroy-SS 2025.



Comment in ASEP

Before a sketch of the argument, we remark in ASEP that the limits are different:



1. When $p(1) > p(-1)$, Tracy-Widom 2009 showed that

$$\frac{X_t - \gamma t}{\sqrt{t}} \Rightarrow F_\gamma,$$

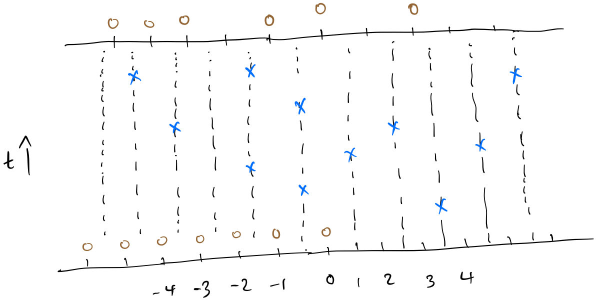
a family of determinantal laws indexed by $\gamma = p(1) - p(-1)$.
–When $p(1) = 1$, the law is Gaussian.

2. When $p(1) < p(-1)$, there is a limit, without scaling: $X_t \Rightarrow \zeta$.
The form of ζ can be computed.

Stirring representation

One can view the symmetric process in terms of 'stirrings': $\eta_t(x) = \sum_{\ell \le 0} 1(\xi_t^\ell = x)$.

–For each edge in $\mathbb{Z}$, at rate 1, neighbor values switch.
So, in the joint evolution $\{\xi_t^\ell : \ell \le 0\}$, there will be correlations, though, marginally, each ξ_t^ℓ is a random walk.



Finding the displacement

For a ‘scale’ $z(t)$, let

$$N_t = \sum_{\ell \leq 0} 1\left(\xi_t^\ell > z(t)\right)$$

be the number of particles, $\sum_{x>z(t)} \eta_t(x)$, beyond $z(t)$ at time t, starting from the step profile.

–Since

$$\{X_t > z(t)\} = \{N_t \geq 1\},$$

if $N_t \sim 1$,

then this would identify the displacement $z(t)$.

One calculates

$$\begin{aligned} E[N_t] &= \sum_{\ell \leq 0} P(\xi_t^\ell > z(t)) \\ &= E\left[\left(\xi_t^0 - z(t)\right)_+\right] \sim 1 \end{aligned}$$

when $z(t) \sim \sqrt{t \log(t)}$ from RW estimates.

–More refined, for $x \in \mathbb{R}$, if

$$z(t) = b_t(x + a_t) \ \ (\sim \sqrt{t \log(t)}),$$

then $E[N_t] \to e^{-x}$.

–This is enough for tightness in SSEP of the lead particle.

Moreover, if

$$N_t \Rightarrow \text{Poisson}(e^{-x}),$$

then,

$$\begin{aligned} P\left(\frac{X_t}{b_t} - a_t < x\right) &= P(N_t = 0) \\ &\to e^{-e^{-x}}, \end{aligned}$$

and the Gumbel limit would follow.

Strong Rayleigh property

Recently, Borcea-Brändén-Liggett 2009 showed that the symmetric exclusion system, when started from a product measure, is 'Strong Rayleigh',

a useful form of 'negative association',

which helps to deal with the correlations in $\{\xi_t^\ell : \ell \leq 0\}$.

–A consequence is a distributional property:

For each $t \geq 0$, and $B \subset \mathbb{Z}$,

there exists independent r.v.'s $\{\zeta_j\}_{j \in B}$

where

$$\sum_{j \in B} \eta_t(j) \overset{d}{=} \sum_{j \in B} \zeta(j).$$

Definition. The system is SR if, for all finite $A \subset \mathbb{Z}$, the generating function

$$Q(x) = E\left[\prod_{i\in A} x_i^{\eta_t(i)}\right]$$

satisfies

$$Q(x)\partial^2_{x_i x_j} Q(x) \le \partial_{x_i} Q(x) \partial_{x_j} Q(x)$$

for $i \neq j$ in A and $x \in \mathbb{R}^A$.

Example. If $A = \{i, j\}$, with $x = (1, 1)$, get

$$\begin{aligned} &P(\eta_t(i) = 1, \eta_t(j) = 1) \\ &\qquad \le P(\eta_t(i) = 1)P(\eta_t(j) = 1), \end{aligned}$$

a statement of ‘negative association’.

Hence, with SR in hand, one tries to verify the ‘Binomial to Poisson’ conditions in the iid framework.

–One needs to show that

$$\sum_{j>z(t)} E[\zeta(j)]^2 = E\Big[\sum_{i>z(t)} \zeta(i)\Big] - Var\Big(\sum_{i>z(t)} \zeta(i)\Big)$$

vanishes, or in other words

$$\begin{aligned}&E[N_t] - Var(N_t)\\&= \sum_{i>z(t)} \left(E[\eta_t(i)]\right)^2 + \sum_{i\neq j>z(t)} Cov\left(\eta_t(i), \eta_t(j)\right) \to 0\end{aligned}$$

as $t \to \infty$.

A main part is to show the covariances vanish. From the stirring representation, and other duality relations, it turns out

$$\begin{aligned}
&\sum_{i\neq j>z(t)} Cov\left(\eta_t(i),\eta_t(j)\right) \\
&\qquad\leq \; 2\sum_{k\in\mathbb{Z}}\int_0^t P(\xi_s^0 = k)^2 P(\xi_{t-s}^0 > z(t)-k-1)^2\, ds.
\end{aligned}$$

–This is further shown to vanish with involved combinations of local clt's, large deviation estimates, etc.

–See Conroy-SS 2023, Conroy, Gonzalez-Casanova, SS 2025

Other systems

We remark that in Zero-range and other ‘mass conservative’ particle systems, there are few tagged particle fluctuation limits shown.

–There are some $d = 1$ limits in Zero-range, starting from ν_ρ, SS 2007.

Another direction is to connect more with hydrodynamics.
For instance, starting from nonstationary initial conditions, the Exclusion LLN may be generalized to

$$X_{Nt}/N \to v_t$$

where

$$\dot{v}_t = \gamma\big(1 - \rho(t, v_t)\big)$$

and $\rho(t,u)$ is the hydrodynamic density.

But, for CLT's would need to capture 'non stationary' limits of

$$\frac{1}{\sqrt{N}} \int_0^{Nt} f(\eta_s) ds.$$

–At the moment, such limits are largely open. These would involve the hydrodynamic density $\rho(t, u)$, the subject of the next talk.