On fluctuations in interacting particle systems

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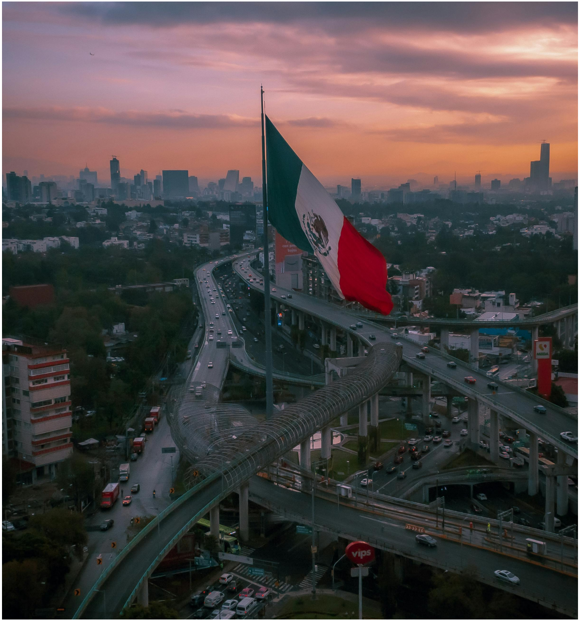
Nov 10 - 14, 2025

Symposium of probability and stochastic processes
Universidad de Sonora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Outline

- Local statistics: Occupation times
  - Model descriptions
  - Occupation time and other functionals
- Local statistics: Tagged particles
- Bulk limits: LLN of the ‘bulk’ mass and hydrodynamics
- Bulk limits: Fluctuations of the ‘bulk’ mass

Goals

Today, we will first set the stage for later discussions of problems, both ‘local’ and ‘bulk’, which connect ‘micro’ to ’macro’ scales, across a variety of interactions.

–Then, we discuss certain ‘local’ functionals, say the time spent at a point, and results/methods, as well as open problems.

Models

We will focus mostly on 'mass-conservative' systems of continuous-time RW's moving on a lattice $S = \mathbb{Z}^d$ or an approximating torus $S = \mathbb{T}_N^d = \mathbb{Z}^d/N\mathbb{Z}^d$.

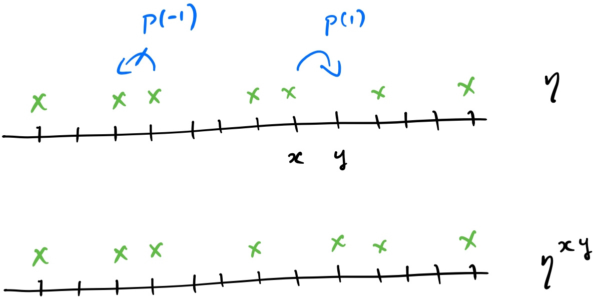
- Exclusion
- Zero-range

-These are well-studied systems, and good vehicles in which to study different physical phenomena.

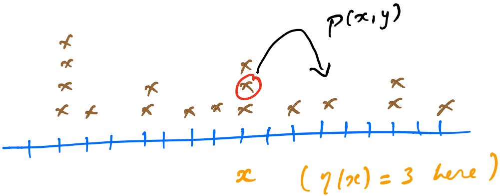
– By working on $\mathbb{T}_N^d$ or $\mathbb{Z}^d$, we will be able to connect to limits on associated continuum spaces $\mathbb{T}^d$ or $\mathbb{R}^d$.

Exclusion interactions

Informally, the simple exclusion process on S consists of a collection of continuous time RW's, with jump probabilities $p(x, y)$ going from x to y, where jumps to occupied locations are suppressed.



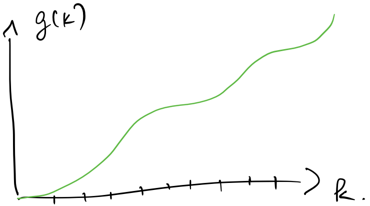
Zero-range interactions

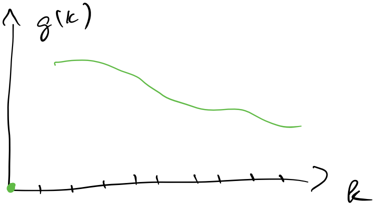


"At x, a clock rings at rate $g(\eta(x))$. Then, a particle at random is selected, which moves to y with chance $p(x,y)$"

–Here, g is a function, with $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$, which specifies the interaction.

–When $g(k) \equiv k$, the process is that of independent random walks.


g(k)
k


g(k)
k

Nuts and bolts

1. We will be interested in the unlabeled evolution.

–Configuration

$$\eta_t = \{\eta_t(x) : x \in S\}$$

specifies numbers of particles at time t at sites in S.

–Configuration spaces

$$\Omega_{SE} = \{0, 1\}^S \text{ and } \Omega_{ZR} = \{0, 1, 2, \ldots\}^S$$

2. We will focus on the 'translation-invariant' and 'finite-range' situation when $p(x, y) \equiv p(y - x)$, and $p(z) = 0$ for $|z| > R$ for some $R < \infty$.

3. In Zero-range, to be concrete, we specify that

$$a_1 k \leq g(k) \leq a_2 k$$

for all $k \geq 0$, although other rates can be considered.

3. The systems can be constructed as Markov processes, certainly on $S = \mathbb{T}_N^d$, and also on $S = \mathbb{Z}^d$.

Note: In the case $S = \mathbb{Z}^d$, Hille-Yosida theorems, and other finite system approximations are involved (Liggett book '85, Andjel '81, L6).

$$L_{SE} f(\eta) = \sum_{x,y} \big(f(\eta^{xy}) - f(\eta) \big) \eta(x)(1 - \eta(y)) p(y - x)$$

$$L_{ZR} f(\eta) = \sum_{x,y} \big(f(\eta^{xy}) - f(\eta) \big) g(\eta(x)) p(y - x)$$

In the exclusion context, $\eta^{x,y}$ can be interpreted as 'exchange' of values at x and y.

–While, in zero-range, $\eta^{x,y}$ means we decrease and increase the particle numbers at x and y.

–Core: local functions $f : \Omega \to \mathbb{R}$ which depend only on a finite number of variables $\{\eta(x) : x \in S\}$.

Other interactions

A word about other models:

Exclusion and zero-range systems are members of a larger family of analyzable mass-conservative systems where

$\eta \to \eta^{x,y}$ with rate $b(\eta(x), \eta(y))p(x, y)$ (Coccoza ’85).

–If interested in ‘birth-death’, Glauber dynamics may be considered where

$\eta \to \eta^{\pm,x}$ with rate $c(\eta, \pm, x)$.

–Combinations of ‘exclusion’ with ‘Glauber’, etc. have been good models to study ‘reaction-diffusion’ phenomena

De Masi-Presutti book ’91; see Funaki-Landim-SS 2024 for more recent references.

Invariant measures

Because of ‘mass-conservation’, there should be several invariant measures indexed to ‘density’:

$$\{\mu_\rho : \rho \in I\}.$$

–Exclusion. $\mu_\rho = \prod_{x\in S} \mathrm{Bern}(\rho)$

–Zero-range. $\bar{\mu}_\phi = \prod_{x\in S} \bar{m}_\phi$ where

$$\bar{m}_\phi(k) = \begin{cases} \frac{1}{Z}\frac{\phi^k}{g(1)\cdots g(k)} & k \geq 1 \\ \frac{1}{Z} & k = 0. \end{cases}$$

Here, $\bar{m}_\phi$ is well-defined as long as $\phi < \liminf_{k\uparrow\infty} g(k) := g_\infty$.

–Define $\mu_\rho = \bar{\mu}_\phi$ and $m_\rho = \bar{m}_\phi$
where ϕ is such that $E_{\bar{m}_\phi}[\eta(\cdot)] = \rho$.

This choice can be made as $\phi = \phi(\rho)$ increases in ρ,
so long as $\rho < \lim_{\beta\to g_\infty} \phi(\beta)$.

–When $g_\infty = \infty$ (for our assumption), then $I = [0, \infty)$.

–When g is bounded, it may be that m_ϕ does not diverge at g_∞, in which case I is a finite interval.

Comments

In both exclusion and zero-range, μ_ρ is invariant,
no matter the structure of *p*,
as long as it is translation-invariant.

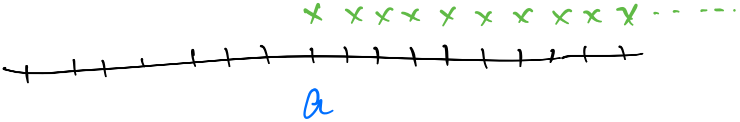
–That they are ‘product’ is of course helpful in calculations.

–In more general systems, one does not expect such a nice feature.

–Interacting systems with specified more general ‘Gibbs’ invariant measures may be constructed, e.g. ‘speed-change’ exclusion (Spohn book 1991)

–It is known that μ_ρ is an extreme point in the convex set of invariant measures. Depending on the form of *p*, there may be other extreme points, even in the translation-invariant case!

—For instance in $d = 1$ exclusion when $p(1) = 1$ (e.g. TASEP), there is no motion from the configuration $\eta(x) = 1$ for $x \geq 0$ and $\eta(x) = 0$ for $x < 0$, e.g. a 'blocking measure.'



Only in a few cases in low dimension $d = 1, 2$ have ALL the invariant measures of exclusion and zero-range been characterized.

–see Liggett book 1985, Andjel ’81, SS 2001,
Bramson-Mountford-Liggett 2002,
Bramson-Liggett 2005,
Amir-Bahadoran-Busani-Saada 2025

Adjoint/Reversibility relation

One may compute:

Exclusion.

$$E_{\mu_\rho}[f(\eta^{xy})h(\eta)] = E_{\mu_\rho}[f(\eta)h(\eta^{yx})]$$

Zero-range.

$$E_{\mu_\rho}[g(\eta(x))f(\eta^{xy})h(\eta)] = E_{\mu_\rho}[g(\eta(y))f(\eta)h(\eta^{yx})]$$

From these relations, one can deduce

$$E_{\mu_\rho}[h(Lf)] = E_{\mu_\rho}[(L^* h)f]$$

where the adjoint L^* with respect to μ_ρ
may be computed as the 'reverse jump' processes,
with jump probability $p^*(z) = p(-z)$.

Therefore, as $L^* 1 = 0$, we have $E_{\mu_\rho}[Lf] = 0$.

Moreover, when $p(\cdot)$ is symmetric, the process is reversible.

‘Local’ statistics

Let’s now discuss a problem of ‘local’ statistics.

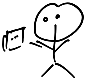
–Such statistics are of interest and include

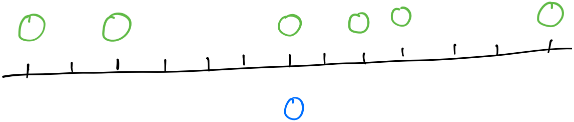
‘occupation time’ at a site,

‘current’ through a bond, or

the motion of a distinguished particle, a ‘tagged’ particle, etc.

– Let’s look at ‘occupation times’ with respect to Exclusion processes, say at the origin: $\int_0^t \eta_s(0)ds$.





General problem

In the Exclusion process on $\mathbb{Z}^d$,
let $f : \Omega \to \mathbb{R}$ be a local function.

What is the behavior of

$$A_f(t) = \int_0^t f(\eta_s)ds$$

as $t \uparrow \infty$?

–We will start the process under an invariant measure μ_ρ, where some calculations can be made and things are already interesting.

Since μ_ρ is extremal, the process is ergodic with respect to time-shifts.

So, the a.s. law of large numbers holds:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\eta_s) ds = E_{\mu_\rho}[f].$$

–One can ask about the fluctuations. Are they diffusive and Gaussian?

Fluctuations of the occupation time

We will now concentrate on the centered function

$$f(\eta) = \eta(0) - \rho$$

to develop some ideas.

–More general functions $f(\eta)$ can also be considered.

See Bernardin-Gonçalves-SS 2015,
Bernardin-Gonçalves-SS 2016 for background references.

It turns out the answers depend on

- the dimension d,
- the structure of jump probability (e.g. symmetric or asymmetric), and
- the density ρ.

–There are still a few open questions.

Variance

Let's try to compute the variance of $A_f(t)$:

$$Var(A_f(t)) = E_{\mu_\rho}\Big[\Big(\int_0^t f(\eta_s)ds\Big)^2\Big]$$
$$= 2\int_0^t \int_r^t E_{\mu_\rho}\big[f(\eta_s)f(\eta_r)\big]\,drds.$$

By stationarity, since

$$E_{\mu_\rho}\big[f(\eta_s)f(\eta_r)\big] = E_{\mu_\rho}\big[f(\eta_{s-r})f(\eta_0)\big],$$

we have further

$$Var(A_f(t)) = 2\int_0^t (t-s)E_{\mu_\rho}\big[f(\eta_s)f(\eta_0)\big]\,ds.$$

Two point function

With

$$f(\eta) = \eta(0) - \rho,$$

the centered occupation variable,

we may compute

$$E_{\mu_\rho}\big[(\eta_s(0) - \rho)(\eta_0(0) - \rho)\big]$$
$$= \rho(1-\rho)\Big\{E_{\mu_\rho}[\eta_s(0)|\eta_0(0) = 1] - E_{\mu_\rho}[\eta_s(0)|\eta_0(0) = 0]\Big\}.$$

This can be viewed in terms of ‘coupling’:

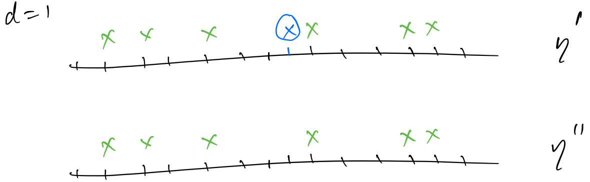
-On the RHS, in the first expectation, the origin is occupied initially.

–In the second expectation, it is conditioned to be empty.

Basic coupling

Couple two copies of the Exclusion process, starting from configurations $\eta' \geq \eta''$, such that

$$\eta'(x) = \eta''(x) \text{ for all } x \neq 0, \text{ and } \eta'(0) = 1, \text{ while } \eta''(0) = 0.$$



The coupled system (η'_t, η''_t) has generator

$$\bar{L}f(\eta', \eta'')$$
$$= \sum_{x,y} p(y-x) 1(\eta'(x) = \eta''(x) = 1)\big[f\big((\eta')^{xy}, (\eta'')^{xy}\big) - f\big(\eta', \eta''\big)\big]$$
$$+ \sum_{x,y} p(y-x) 1(\eta'(x) = 1, \eta''(x) = 0)\big[f\big((\eta')^{xy}, \eta''\big) - f\big(\eta', \eta''\big)\big].$$

Second-class particle

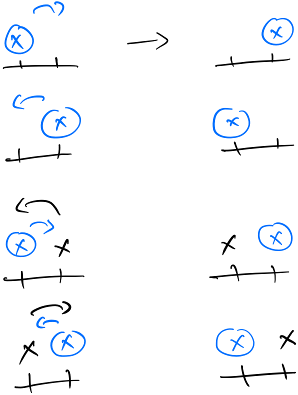
Here, the process η'_t, for times $t \geq 0$, majorizes η''_t.

–There is exactly one discrepancy, which we label R_t.

The dynamics of R_t is as follows:

It displaces $R_t \to R_t + z$
with rate

$$p(z)(1 - \eta''(R_t + z)) + p(-z)\eta''(R_t + z).$$



rate $p(1-\eta'(+))$

rate $q(1-\eta'(-))$

rate $q\,\eta'(+)$

rate $p\,\eta'(-)$

Then, the two-point function

$$
\begin{aligned}
&E_{\mu_\rho}\big[(\eta_s(0)-\rho)(\eta_0(0)-\rho)\big] \\
&\quad = \rho(1-\rho)\Big\{E_{\mu_\rho}[\eta_s(0)|\eta_0(0)=1] - E_{\mu_\rho}[\eta_s(0)|\eta_0(0)=0]\Big\} \\
&\quad = \rho(1-\rho)\bar{P}(R_s=0).
\end{aligned}
$$

In the symmetric model, the displacement rate

$$p(z)(1-\eta'(R_t+z))+p(-z)\eta'(R_t+z)$$
$$=p(z)$$

does not depend on the underlying configuration!

–So, the statistics of R_t,
in this case,
is that of a fair random walk.

In the asymmetric model, however,
formally the drift of the second-class particle is

$$(1 - 2\rho) \sum zp(z).$$

–Its statistics are more involved.
Let $\gamma = \sum zp(z)$.

–In $d = 1$, for nearest-neighbor systems, the a.s. law of large numbers holds:

$$\frac{1}{t} R_t \to \gamma(1 - 2\rho)$$

Ferrari '92, Rezakhanlou '95, Balazs-Nagy 2017.

Variances

Fluctuations of the second-class particle are also of interest.

–In $d = 1$, for nearest-neighbor asymmetric systems, these connect to fluctuations of the current and KPZ class scalings
Ferrari-Spohn 2007, Quastel-Valko 2007.

In particular,

$$\mathrm{Var}(R_t) \sim t^{4/3}$$

(Balazs-Seppalainen 2010).

–In the integrable probability literature, there is much more detailed information for TASEP, etc.
e.g. Ferrrari-Ghoshal-Nejjar 2019.

–In $d = 2$, for $(\uparrow, \rightarrow)$ systems,

$$E\left|R_t - \gamma(1-2\rho)t\right|^2 \sim t(\log t)^{2/3}$$

(Yau 2002, Cannizzaro-Moulard-Toninelli 2025).

–In $d \geq 3$, for finite-range systems,

$$E\left|R_t - \gamma(1-2\rho)t\right|^2 \sim t$$

(Landim-Olla-Varadhan 2004).

Back to occupation time

Recall $f(\eta) = \eta(0) - \rho$.

In the finite-range symmetric model, since

$$\mathrm{Var}(A_f(t)) = C(\rho)t\int_0^t (t-s)\bar{P}(R_s = 0)ds,$$

we have

$$\mathrm{Var}(A_f(t)) \sim \begin{cases} t^{3/2} & d = 1 \\ t\log(t) & d = 2 \\ t & d \geq 3. \end{cases}$$

Let us now consider the asymmetric model.
When $\rho \neq 1/2$,
the 'velocity' $\gamma(1 - 2\rho)$ of R_t does not vanish.

In this case, it turns out

$$\mathrm{Var}(A_f(t)) \sim t$$

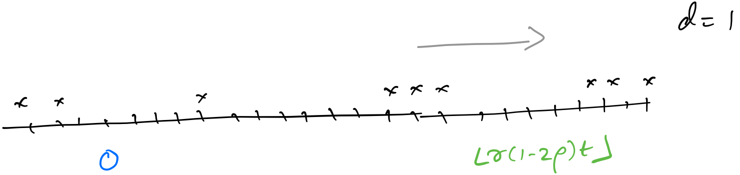
–The velocity $\gamma(1 - 2\rho)$, corresponds to Burgers equation

$$\partial_t \rho = m \nabla \rho(1 - \rho) = \gamma(1 - 2\rho) \nabla \rho,$$

which is the 'bulk' hydrodynamic limit (discussed in a later talk).

However, when $\rho = 1/2$, based on the orders of $E|R_t|^2$, one conjectures that the transition probability

$$\bar{P}(R_t = 0) \sim \frac{1}{\sqrt{E|R_t|^2}} \sim \begin{cases} t^{-2/3} & d = 1 \\ t^{-1/2}(\log t)^{-1/3} & d = 2 \\ t^{-1/2} & d \geq 3. \end{cases}$$



Accordingly, one conjectures that

$$\mathrm{Var}(A_f(t)) \sim t \int_0^t \bar{P}(R_s = 0) ds$$

should match the orders of $E|R_t|^2$ as ρ and d vary.

–This has been proved for all (ρ, d), except in $d \leq 2$ for asymmetric Exclusion when $\rho = 1/2$.

Here, only superdiffusive bounds have been shown:

$$Var(A_f(t)) \geq \begin{cases} t \log\log(t) & d = 2 \\ t^{5/4} & d = 1. \end{cases}$$

–There is more 'fluctuation' because of rigidity in $d \leq 2$ in the $\rho = 1/2$ asymmetric model.

–See background reference
Bernardin-Gonçalves-SS 2015,
which also considers general $f(\eta)$ and $A_f(t)$.

Limit distributions

Given the variance orders, what are the distributional limits of the scaled, centered occupation time?

Consider symmetric finite-range Exclusion.

–When $d = 1$, it can be shown

$$\frac{1}{N^{3/4}} \int_0^{Nt} (\eta_s(0) - \rho) ds \Rightarrow \sigma(\rho) fBM_{3/4}(t)$$

Here, $\sigma(\rho)$ is the (non-explicit) limiting standard deviation.

–When $d = 2$,

$$\frac{1}{\sqrt{N \log N}} \int_0^{Nt} \big(\eta_s(0) - \rho\big) ds \Rightarrow \sigma(\rho) BM(t).$$

–When $d \geq 3$,

$$\frac{1}{\sqrt{N}} \int_0^{Nt} \big(\eta_s(0) - \rho\big) ds \Rightarrow \sigma(\rho) BM(t).$$

Kipnis-Varadhan CLT

The limit in $d \geq 3$ can be understood in a larger setting:

Recall

$$A_f(t) = \int_0^t f(\eta_s)ds$$

and for occupation times, $f(\eta) = \eta(0) - \rho$.

If one could solve $f = -Lu$, write

$$A_f(t) = M_t^u + \xi_t$$

where

$$M_t^u = u(\eta_t) - u(\eta_0) - \int_0^t Lu(\eta_s)ds$$

and ξ_t is an error.

Then, martingale CLT, etc. might apply.

–Gordin-Lifsic 1978, Bhattacharya 1982

However, generally one cannot solve the Poisson equation.

–But, one can solve the resolvent equation

$$\lambda u - Lu = f$$

whose (abstract) solution is

$$\int_0^\infty e^{-\lambda t} P_t f dt.$$

–Then,

$$A_f(t) = M_t^{u,\lambda} + \lambda \int_0^t u(\eta_s) ds + u(\eta_0) - u(\eta_t).$$

KV statement

Let η_t be a reversible Markov processes, starting in an ergodic invariant measure μ.

Theorem (Kipnis-Varadhan '87). Suppose

$$\lim_{t\uparrow\infty} \frac{1}{t}\mathrm{Var}(A_f(t)) = \sigma_f^2 < \infty.$$

Then,

$$\frac{1}{\sqrt{N}}A_f(Nt) = \frac{1}{\sqrt{N}}\int_0^{Nt} f(\eta_s)ds \Rightarrow \sigma_f BM(t).$$

–See also Merlevede-Peligrad-Utev 2006, Landim-Komorowski-Olla 2012, Cuny-Lin 2025 for work on the general 'CLT' problem.

Application to occupation times

Hence, in $d \geq 3$,
for symmetric (reversible) Exclusion,
starting in μ_ρ,
since the variance converges $\frac{1}{t}\mathrm{Var}(A_f(t)) \to \sigma(\rho)$,
we verify

$$\frac{1}{\sqrt{N}} \int_0^{Nt} \big(\eta_s(0) - \rho\big) ds \Rightarrow \sigma(\rho) BM(t).$$

Dimensions $d = 1, 2$

There are different proofs of the fractional BM limit in $d = 1$.

–see SS 2000, Gonçalves-Jara 2013, Erhad-Franco-Xu 2024.

–We will also discuss it in a later talk, as a consequence of a ‘bulk’ fluctuations limit.

–For the limit in $d = 2$, it turns out one can compute things explicitly via the resolvent decomposition (see Kipnis ’87).

Comment on limits in the asymmetric model

In the asymmetric model,
when $\rho \neq 1/2$ or $d \geq 3$, the Brownian motion limit holds:

$$\frac{1}{\sqrt{N}} \int_0^{Nt} \big(\eta_s(0) - \rho\big) ds \Rightarrow \sigma(\rho) BM(t).$$

–Here, for $d \leq 2$, other 'CLT's (Newman-Wright) are used, and in $d \geq 3$ a modified KV theorem is applied.

–see Bernardin-Gonçalves-SS 2015 for references.

–When $\rho = 1/2$ and $d \leq 2$, one can formulate conjectures of the limit, based on the KPZ fixed point, but these 'local' fluctuations are open.

A word on mean-zero nonreversible Exclusion

We comment now on mean-zero Exclusion,
e.g. $p(2) = 1/3$ and $p(-1) = 2/3$ in $d = 1$.

In this case, μ_ρ is still invariant but nonreversible.

With respect to occupation times, the scalings still follow that of symmetric Exclusion.

–In $d \geq 3$ a modified KV theorem applies (Varadhan 1990).

–In $d = 1$, one can use the method in Gonçalves-Jara 2013.

–In $d = 2$, however, finding the variance order with precise prefactor constant and the associated *BM* limit is open.

In Zero-range models

Finally, we comment that variances and scaled limits of $A_f(t) = \int_0^t f(\eta_s)ds$ have also been considered in Zero-range.

–Results in the symmetric setting are similar to that in Exclusion, but less calculation is possible.

–See Quastel-Jankowski-Sheriff 2002 (which makes use of $L^2(\mu_\rho)$ decay estimates of $P_t f$, of their own interest).

–In the asymmetric case, there are also a few results, but things are more open.

–See Bernardin-Gonçalves-SS 2016.

References

Books on interacting particle systems include

- De Masi-Presutti: Mathematical methods for hydrodynamic limits 1991
- Kipnis-Landim: Scaling limits of interacting particle systems 1999
- Liggett: Interacting particle systems 1985; see also book in 1999.
- Spohn: Large scale dynamics of interacting particles 1991

–There are also some notes (in construction) at
$http://www.math.arizona.edu/\sim sethuram/hermosillo/$
There, also these slides will be available.

Calculation in $d = 2$

Let

$$g_\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(0,x)\,dt$$

where p_t is the transition probability of RW on $\mathbb{Z}^2$.
Note $p_t(0,0) \sim t^{-1}$ and

$$(\lambda - \Delta_{RW}) g_\lambda = \delta(0)$$

where Δ_{RW} is the random walk generator.

Form

$$G_\lambda(\eta) = \sum_x g_\lambda(x)(\eta(x) - \rho).$$

A calculation shows $\lambda G_\lambda - L G_\lambda = \eta(0) - \rho$.

The error between $A_f(t)$ and the martingale $G_\lambda(\eta_0) - G_\lambda(\eta_t) - \int_0^t LG_\lambda(\eta_s)ds$ is

$$\zeta(\lambda, t) = \lambda \int_0^t G_\lambda(\eta_s)ds + G_\lambda(\eta_0) - G_\lambda(\eta_t).$$

One can calculate explicitly that

$$\frac{1}{\sqrt{N \log N}} \|\zeta(\lambda = 1/N, Nt)\|_{\mu_\rho}^2 \sim C/\log N$$

so that martingale CLT can be applied.