An introduction to scaling limits in interacting particle systems

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Stochastic methods in finance and physics IV
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Outline

- Hydrodynamics of the ‘bulk’ mass
  - Model descriptions
  - Hydrodynamics: micro to macro scaling
- ‘Replacement’ averaging principle methods
- Fluctuations of the ‘bulk’ mass and ‘occupation times’

Goals

Our aim today is to set the stage for a set of ideas,
going back to

- Guo-Papanicolaou-Varadhan 1988, and
- Yau 1991,

which allow to capture the continuum limit of the space-time evolution of the ‘bulk’ mass, in a rigorous across a variety of interactions, following physical intuitions.

Models

We will focus mostly on 'mass-conservative' systems of continuous-time RW's moving on a lattice $S = \mathbb{Z}^d$ or an approximating torus $S = \mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$.

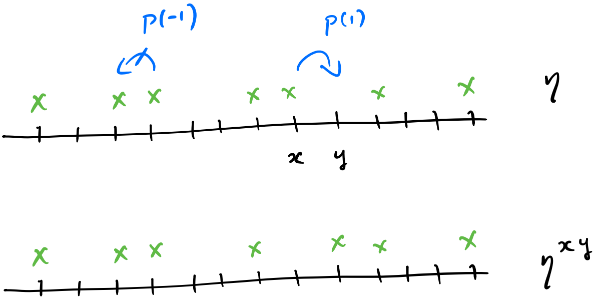
- Exclusion
- Zero-range

-These are well-studied systems, and good vehicles in which to study different physical phenomena.

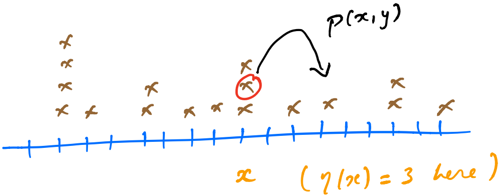
– By working on '$\mathbb{Z}^d$', we will be able to connect to limits (e.g. distributions, PDEs, SPDEs) on associated continuum spaces $\mathbb{R}^d$ or $\mathbb{T}^d$.

Exclusion interactions

Informally, the simple exclusion process on S consists of a collection of continuous time RW's, with jump probabilities $p(x, y)$ going from x to y, where jumps to occupied locations are suppressed.



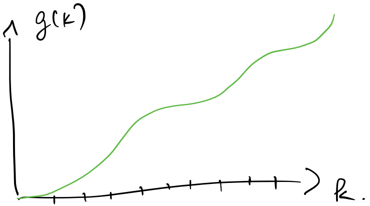
Zero-range interactions

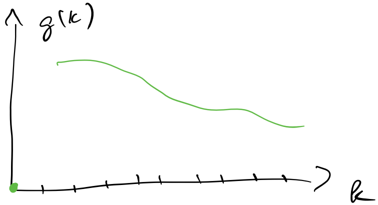


"At x, a clock rings at rate $g(\eta(x))$. Then, a particle at random is selected, which moves to y with chance $p(x, y)$"

–Here, g is a function, with $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$, which specifies the interaction.

–When $g(k) \equiv k$, the process is that of independent random walks.


g(k)
k.


g(k)
k

Nuts and bolts

1. We will be interested in the unlabeled evolution.

–Configuration

$$\eta_t = \{\eta_t(x) : x \in S\}$$

specifies numbers of particles at time t at sites in S.

–Configuration spaces

$$\Omega_{SE} = \{0, 1\}^S \text{ and } \Omega_{ZR} = \{0, 1, 2, \dots\}^S$$

2. We will focus on the ‘translation-invariant’ and ‘finite-range’ situation when $p(x, y) \equiv p(y - x)$, and $p(z) = 0$ for $|z| > R$ for some $R < \infty$.

3. In Zero-range, to be concrete, we specify that

$$a_1 k \leq g(k) \leq a_2 k$$

for all $k \geq 0$, although other rates can be considered.

3. The systems can be constructed as Markov processes, certainly on $S = \mathbb{T}_N^d$, and also on $S = \mathbb{Z}^d$.

Note: In the case $S = \mathbb{Z}^d$, Hille-Yosida theorems, and other finite system approximations are involved (Liggett book '85, Andjel '81).

$$L_{SE} f(\eta) = \sum_{x,y} \big(f(\eta^{xy}) - f(\eta)\big)\eta(x)(1 - \eta(y))p(y - x)$$

$$L_{ZR} f(\eta) = \sum_{x,y} \big(f(\eta^{xy}) - f(\eta)\big)g(\eta(x))p(y - x)$$

In the exclusion context, $\eta^{x,y}$ can be interpreted as 'exchange' of values at x and y.

–While, in zero-range, $\eta^{x,y}$ means we decrease and increase the particle numbers at x and y.

–Core: local functions $f : \Omega \to \mathbb{R}$ which depend only on a finite number of variables $\{\eta(x) : x \in S\}$.

Other interactions

A word about other models:

Exclusion and zero-range systems are members of a larger family of analyzable mass-conservative systems where

$\eta \to \eta^{x,y}$ with rate $b(\eta(x), \eta(y))p(x, y)$ (Coccoza '85).

–If interested in 'birth-death', Glauber dynamics may be considered where

$\eta \to \eta^{\pm,x}$ with rate $c(\eta, \pm, x)$.

–Combinations of 'exclusion' with 'Glauber', etc. have been good models to study 'reaction-diffusion' phenomena
(De Masi-Presutti book '91, Vares '91, Landim-Vares '96).

Invariant measures

Because of 'mass-conservation', there should be several invariant measures indexed to 'density':

$$\{\mu_\rho : \rho \in I\}.$$

–Exclusion. $\mu_\rho = \prod_{x\in S} \mathrm{Bern}(\rho)$

–Zero-range. $\bar{\mu}_\phi = \prod_{x\in S} \bar{m}_\phi$ where

$$\bar{m}_\phi(k) = \begin{cases} \frac{1}{Z}\frac{\phi^k}{g(1)\cdots g(k)} & k \geq 1 \\ \frac{1}{Z} & k = 0. \end{cases}$$

Here, $\bar{m}_\phi$ is well-defined as long as $\phi < \liminf_{k\uparrow\infty} g(k) := g_\infty$.

–Define $\mu_\rho = \bar{\mu}_\phi$ and $m_\rho = \bar{m}_\phi$
where ϕ is such that $E_{\bar{m}_\phi}[\eta(\cdot)] = \rho$.

This choice can be made as $\phi = \phi(\rho)$ increases in ρ,
so long as $\rho < \lim_{\beta\to g_\infty} \phi(\beta)$.

–When $g_\infty = \infty$ (our assumption), then $I = [0, \infty)$.

–When g is bounded, it may be that m_ϕ does not diverge at g_∞, in which case I is a finite interval.

Comments

In both exclusion and zero-range, μ_ρ is invariant,
no matter the structure of *p*,
as long as it is translation-invariant.

–That they are ‘product’ is of course helpful in calculations.

–In more general systems, one does not expect such a nice feature.

–Interacting systems with specified more general ‘Gibbs’ invariant measures may be constructed, e.g. ‘speed-change’ exclusion (Spohn book 1991)

–It is known that μ_ρ is an extreme point in the convex set of invariant measures. Depending on the form of p, there may be other extreme points, even in the translation-invariant case!

—For instance in $d = 1$ exclusion when $p(1) = 1$, there is no motion from the configuration $\eta(x) = 1$ for $x \geq 0$ and $\eta(x) = 0$ for $x < 0$, e.g. a 'blocking measure.'

Only in a few cases in low dimension $d = 1, 2$ have ALL the invariant measures of exclusion and zero-range been characterized.

–see Liggett book 1985, Andjel '81, SS 2001, Bramson-Mountford-Liggett 2002, Bramson-Mountford 2002, Bramson-Liggett 2005

Reversibility relation

Exclusion:

$$E_{\mu_\rho}[f(\eta^{xy})h(\eta)] = E_{\mu_\rho}[f(\eta)h(\eta^{xy})]$$

Zero-range:

$$E_{\mu_\rho}[g(\eta(x))f(\eta^{xy})h(\eta)] = E_{\mu_\rho}[g(\eta(y))f(\eta)h(\eta^{yx})]$$

–From these relations, one can deduce $E_{\mu_\rho}[h(Lf)] = E_{\mu_\rho}[(Lh)f]$ and therefore, as $L1 = 0$, that $E_{\mu_\rho}[Lf] = 0$.

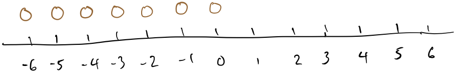
–The adjoint L^* with respect to μ_ρ may be computed as the 'reverse jump' processes, with jump probability $p^*(z) = p(-z)$.

Initial distributions

Let us focus on Exclusion to begin.

–Deterministic states: $\eta = \eta^N$, such as a 'step profile', or perhaps sequences of configurations.

Remark: 'Corner growth' motion corresponds to evolution from η such that $\eta(x) = 1$ for $x \leq 0$ and $\eta(x) = 0$ for $x \geq 1$.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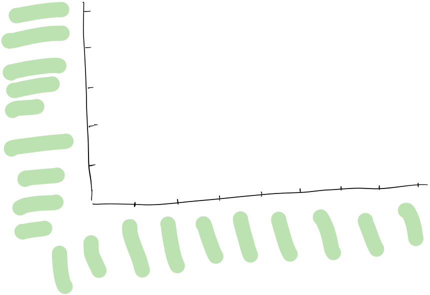

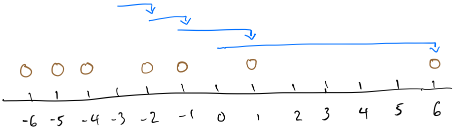
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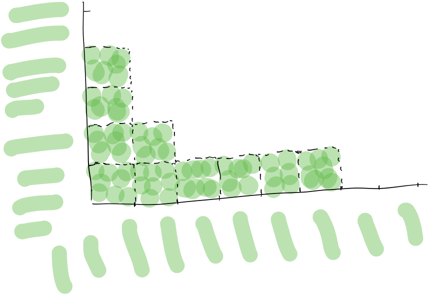


–Random 'local equilibrium' states:

Let $\rho_0 : S \to [0, 1]$ be a piecewise continuous function.
Define

$$\mu^N = \prod_{x \in S} \mathrm{Bern}(\rho_0(x/N))$$

–Of course, when $\rho_0 \equiv \rho$ is a constant function, $\mu^N = \mu_\rho$ is an invariant measure.

We choose initial configurations, whether deterministic or random, so that

$$\frac{1}{N}\sum_x J(x/N)\eta^N(x) \to \int J(u)\rho_0(u)du$$

in probability, for test functions J.

–In Zero-range, one can build deterministic or 'local equilibrium' states $\prod_{x\in S} m_{\rho_0(x/N)}$ similarly.

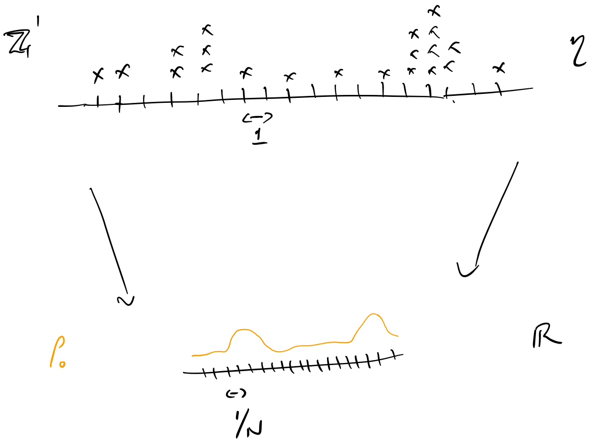
Hydrodynamics: Micro to Macro scaling

Question: How does the mass in the system evolve starting from one of these initial distributions?

–In the following, we will focus on $S = \mathbb{T}_N^d$.

–In infinite volume on $\mathbb{Z}^d$, results also hold , but there will be extra assumptions and more machinery.

Initial picture



Time scaling

In the bird's eye view, a small number of micro movements are not seen!

–Need to speed up time to correspond to the grid scaling of $1/N$.

–As suspected, the level of 'speed-up' depends on the form of the jump probability p.

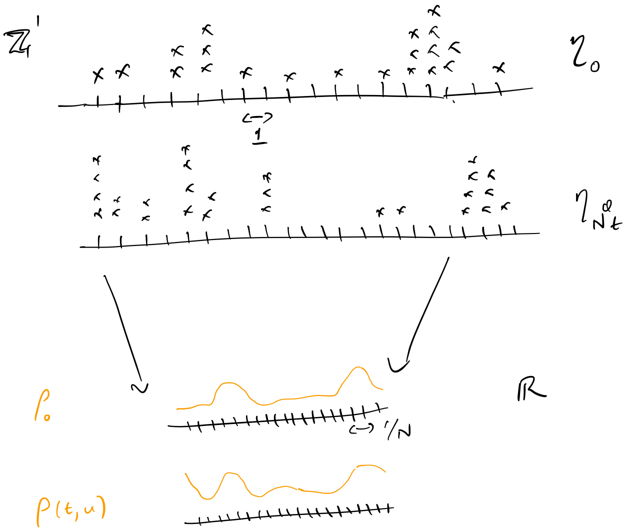
Suppose there is only one particle in the system.

Let X_t be its position at time t.

–If p has a drift, that is $v = \sum_x xp(x) \neq 0$, then in time Nt the position displaces by vNt which, in the macroscopic view, is near $vNt/N = vt$.

–If p is mean-zero (symmetric say), then $v = 0$, and in time N^2t the position displaces by $O(\sqrt{N^2t}) = O(N)$, which is macroscopically $O(1)$.

–For long-range p (not discussed) there may be other relevant scalings N^α, depending on the tails of p.



$\mathbb{Z}^1$
η_0
1
$\eta_{N^2 t}$
ρ_0
$\mathbb{R}$
1/N
$\rho(t,u)$

– In the following, we will consider the appropriately scaled evolution,

whether hyperbolic ($\theta = 1$)
or diffusive ($\theta = 2$), and

$$\eta_t^N = \eta_{N^\theta t}$$

generated by $N^\theta L$.

The mass evolution can be captured via the empirical measure.

$$\pi_t^N = \frac{1}{N^d}\sum_x \eta_t^N(x)\delta_{x/N}.$$

–Here,

$$\langle J, \pi_t^N \rangle = \frac{1}{N^d}\sum_x J(x/N)\eta_t^N(x)$$

–If $J = 1_A$, then $\langle J, \pi_t^N \rangle$ is the average number of particles in $NA \subset S$ at time $N^\theta t$.

Martingales

Let $T > 0$ be a finite time-horizon.

Recall for a Markov process Z_t
and $f : [0, T] \times S \to \mathbb{R}$ that

$$M_t^f = f(t, Z_t) - f(0, Z_0) - \int_0^t (\partial_t + L) f(s, Z_s) ds$$

and

$$(M_t^f)^2 - \int_0^t \left(Lf^2 + 2fLf \right) ds$$

are martingales.

First calculations in Exclusion

To reduce notation,
let us focus on $d = 1$ and nearest-neighbor p.

Let

$$f(\eta) = \frac{1}{N} \sum_{x \in S} J(x/N)\eta(x).$$

Discrete evolution equation

Write

$$\begin{aligned}\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle &= \int_0^t N^\theta L f(\eta_s^N) ds + M_t^f \\ &= \int_0^t N^\theta L f(\eta_s^N) ds + O(1/N).\end{aligned}$$

Some calculations in Exclusion

We have

$$
\begin{aligned}
&N^\theta Lf(\eta) \\
&\quad = \frac{N^\theta}{N} \sum_x \big(J((x+1)/N) - J(x/N) \big) \\
&\qquad\quad \times \big\{ \eta(x)(1-\eta(x+1))p(1) - \eta(x+1)(1-\eta(x))p(-1) \big\}.
\end{aligned}
$$

To analyze further

If $p(1) = p(-1)$, then

$$\begin{aligned}&\eta(x)(1-\eta(x+1))p(1) - \eta(x+1)(1-\eta(x))p(-1)\\ &\qquad = \big\{\eta(x) - \eta(x+1)\big\}p(1),\end{aligned}$$

permitting another summation-by-parts.

On the other hand, if $p(1) > p(-1)$,

with $\gamma = p(1) - p(-1)$,

can write

$$\begin{aligned} p(1) &= \big(p(1) - p(-1)\big) + p(-1) \\ &= \gamma + p(-1). \end{aligned}$$

Then,

$$\begin{aligned} &\eta(x)(1 - \eta(x+1))p(1) - \eta(x+1)(1 - \eta(x))p(-1) \\ &\quad = p(-1)\big\{\eta(x) - \eta(x+1)\big\} + \underbrace{\gamma\eta(x)(1 - \eta(x+1))}. \end{aligned}$$

In the symmetric case $p(1) = p(-1)$,
with $\theta = 2$.

Then,

$$\begin{aligned}
N^\theta Lf(\eta) &= \frac{N^\theta}{N} \sum_x \big(J((x+1)/N) - J(x/N)\big)\big\{\eta(x) - \eta(x+1)\big\}p(1) \\
&= \frac{N^\theta}{N} \sum_x \big(J((x+1)/N) - 2J(x/N) + J((x-1)/N)\big)\eta(x)p(1) \\
&\sim \frac{p(1)}{N}\frac{N^\theta}{N^2} \sum_x J''(x/N)\eta(x) \\
&= p(1)\langle J'', \pi_t^N\rangle.
\end{aligned}$$

Whereas, in the asymmetric case $p(1) > p(-1)$,
with $\theta = 1$.

Then, to dominant order,

$$N^\theta Lf(\eta) \sim \frac{\gamma}{N}\frac{N^\theta}{N}\sum_{x\in S} J'(x/N)\eta(x)(1-\eta(x+1)).$$

Quadratic variation calculation

With respect to the martingale M_t^f, we compute

$$
\begin{aligned}
&E\left[(M_t^f)^2\right] \\
&\quad = N^\theta E \int_0^t \left(Lf^2 + 2fLf\right) ds \\
&\quad = \frac{1}{N}\frac{N^\theta}{N^2}\int_0^t \sum_x \sum_\pm \left(J((x\pm 1)/N) - J(x/N)\right)^2 \\
&\qquad\qquad\qquad\qquad \times \eta_s^N(x)(1-\eta_s^N(x\pm 1)p(\pm 1)ds \\
&\quad = O(1/N),
\end{aligned}
$$

whether $\theta = 2$ or 1.

Putting together in Exclusion

–When p is symmetric,

$$\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle = p(1) \int_0^t \langle J'', \pi_t^N \rangle ds + O(1/N).$$

–When p is asymmetric,

$$\begin{aligned} &\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle \\ &= \int_0^t \frac{\gamma}{N} \sum_x J'(x/N) \eta_s^N(x)(1 - \eta_s^N(x+1)) ds + O(1/N). \end{aligned}$$

One can ‘see’ the equation that ‘$\rho(t, u)$’ should satisfy from these computations.

–Consider first the symmetric case. Here, the discrete equation is ‘closed’.

Ingredients to identify limit:

- Tightness/compactness of $\{\pi_t^N : t \in [0, T]\}$ in $D([0, T], \mathcal{M})$
- Absolute continuity of limit point $\pi_t^\infty(dx) = \rho(t, x)dx$
- Uniqueness of weak solution of the underlying PDE, and other properties of the solution.

With these in hand, via a subsequence of N, we have

$$\langle J, \pi_t^\infty \rangle - \langle J, \pi_0^\infty \rangle = p(1) \int_0^t \langle J'', \pi_s^\infty \rangle ds.$$

–This is a weak form of

$$\partial_t \rho = p(1) \Delta \rho.$$

–Since there is a unique bounded, weak solution, the limit trajectory

$$\pi^\infty = \left\{ \pi_t^\infty = \delta_{\rho(t,u)du} : t \in [0, T] \right\}$$

is identified.

–Moreover, from continuity of $\rho(t, u)$, at time $t > 0$, we have

$$\pi_t^N \Rightarrow \delta_{\rho(t,u)du}.$$

Replacement

In the asymmetric case, the discrete equation isn't closed.

–We have to deal with the term

$$`\eta(x)\big(1-\eta(x+1)\big)'$$

–Such a case is not restricted to the asymmetric situation.

In fact, making the calculation for Zero-range in $d = 1$
with nearest-neighbor symmetric p,
we get

$$\langle J, \pi_t^N \rangle - \langle J, \pi_0^N \rangle$$
$$= p(1) \int_0^t \frac{1}{N} \sum_x J''(x/N) g(\eta_s^N(x)) ds + O(1/N).$$

–Would like $g(\eta_t^N(x))$ to be approximated by a function of the empirical measure.

Main idea

Let τ_x be the shift of variables by x.

For example, $\tau_x \eta(y) = \eta(x+y)$.

–We may posit

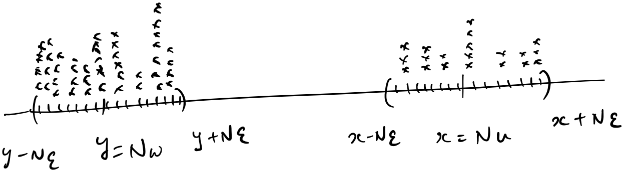
$$h\big(\tau_x \eta_s^N\big) \sim E\Big[h\big(\tau_x \eta_s^N\big) | \frac{1}{2N\epsilon + 1} \sum_{|z-x| \leq N\epsilon} \eta_s^N(z)\Big].$$

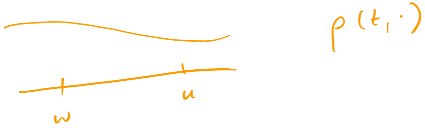
Rationale

The system has had time to mix up.

–At micro $x = Nu$, at micro time $N^\theta s$, in a micro window of width $O(N\epsilon)$, the system behaves, approximately, according to the (random) local density

$$\eta^{(N\epsilon)}_{N^\theta s}(x) := \frac{1}{2N\epsilon + 1} \sum_{|z-x|\le N\epsilon} \eta^N_s(z).$$


$y-N\varepsilon$
$y=Nw$
$y+N\varepsilon$
$x-N\varepsilon$
$x=Nu$
$x+N\varepsilon$


$\rho(t,\cdot)$
w
u

To first order, we might think the system is approximately in an invariant state with this density:

$$E\Big[h\big(\tau_x\eta^N_s\big)\big|\eta^{(N\epsilon)}_{N^\theta s}(x)\Big] \sim E_{\mu_{\eta^{(N\epsilon)}_{N^\theta s}(x)}}[h]$$
$$= H\Big(\eta^{(N\epsilon)}_{N^\theta s}(x)\Big).$$

Note that $\eta^{(N\epsilon)}_{N^\theta s}(x)$ is a function of the empirical measure:

$$\eta^{(N\epsilon)}_{N^\theta s}(x) = \frac{1}{2N\epsilon + 1} \sum_{|z-x| \le N\epsilon} \eta^N_s(z)$$
$$\sim \langle i_\epsilon, \pi^N_s \rangle,$$

where $i_\epsilon(u) = (2\epsilon)^{-1} 1(|u| \le \epsilon)$.

–This allows us to ‘replace’ and close the equation.

With such replacement, along with the other ingredients, applied to

$$h(\eta) = \eta(0)(1-\eta(1))$$
$$h(\eta) = g(\eta(0))$$

in the asymmetric Exclusion and
symmetric Zero-range models,

one arrives at

$$\partial_t \rho + \gamma\partial_x\big(\rho(1-\rho)\big) = 0, \text{ and}$$
$$\partial_t \rho = p(1)\Delta G(\rho).$$

–Here, by a calculation, $H(\rho) = \rho(1-\rho)$ and $G(\rho) = \phi(\rho)$.

Symmetric results

Theorem. Consider symmetric Exclusion or Zero-range on $\mathbb{T}_N^d$ with nearest-neighbor $p(e) = 1/(2d)$, starting from local equilibrium μ^N associated to $\rho_0(\cdot)$.

For each $t \geq 0$, we have $\pi_t^N \Rightarrow \rho(t,u)du$ where respectively

$$\partial_t \rho = p(e)\Delta\rho$$
$$\partial_t \rho = p(e)\Delta\phi(\rho)$$

such that $\rho(0,u) = \rho_0(u)$.

Comments

As mentioned, these results go back to
Guo-Papanicolaou-Varadhan 1988 and Yau 1991.

–There are generalizations:

- p may be finite, or infinite range with different time-scalings
- Initial conditions may be deterministic or other random measures associated to ρ_0.
- S may be $\mathbb{Z}^d$, with additional growth assumptions on $\rho_0(\cdot)$

See for instance Kipnis-Landim 1999, Seppäläinen 2008,
Fritz 1990, Yau 1994,
Lu 1995, Jara 2008, SS-Shahar 2018.

Asymmetric results

Theorem. Consider asymmetric Exclusion on $\mathbb{T}_d^N$ with nearest-neighbor p, starting from local equilibrium μ^N associated to $\rho_0(\cdot)$.

For each $t \geq 0$, we have $\pi_t^N \Rightarrow \rho(t,u)du$ where $\rho(t,u)$ is the unique 'entropy' solution of

$$\partial_t \rho + \gamma \cdot \nabla\big(\rho(1-\rho)\big) = 0$$

such that $\rho(0,u) = \rho_0(u)$.

–Here, $\gamma = \sum zp(z)$.

Comments

Here, ‘entropy’ solution is the ‘physical’
or ‘viscosity’ solution of the equation:

Add $\epsilon\Delta\rho$ to the equation. The limit of $\rho = \rho^{\epsilon}$, as $\epsilon \downarrow 0$, is the desired solution.

–There are other ways to define the ‘entropy’ solution, e.g. via Kruzkov, Hopf-Lax, etc. formulations (not discussed).

The proof, based on Young measures and Diperna-Lions theory,

works for Exclusion, and

Zero-range models where g is an increasing function,

e.g. particle systems

which allow ‘monotone particle couplings’.

See Rezakhanlou (1991), Bahadoran-Guiol-Ravishankar-Saada (2017) for instance, and also Loulakis-Stamatakis (2019).

–An open problem, however, is to show hydrodynamics for more general asymmetric interacting particle systems.

Summary

We have discussed ‘hydrodynamics’ on $\mathbb{Z}^d$ for systems with one conservation law. Although Euler equations have been derived in related models (see Olla-Varadhan-Yau 1993), less is known say about the rigorous derivation of Navier-Stokes equations.

Less is known also about phenomena on other graphs $\{G_n\}$ or when there are boundaries, and other inhomogeneities.

References

There are several good books on interacting particle systems:

- De Masi-Presutti: Mathematical methods for hydrodynamic limits 1991
- Kipnis-Landim: Scaling limits of interacting particle systems 1999
- Liggett: Interacting particle systems 1985
- Spohn: Large scale dynamics of interacting particles 1991

A word on mean-zero Exclusion

Finally, we comment on mean-zero Exclusion,
e.g. $p(2) = 1/3$ and $p(-1) = 2/3$ in $d = 1$.

In this case, the technical 'twice sum-by-parts' used for the symmetric model does not hold! Such a model is called 'non-gradient'.

However, diffusive scaling is still correct, and a hydrodynamic (quasilinear) heat equation limit can be proved. The diffusion coefficient will depend on ρ.

–See Varadhan (1994), Quastel (1992), Funaki-Uchiyama-Yau (1995). Bannai-Kametani-Sasada (2020)

Thank you!