PART 9: A TAGGED PARTICLE IN SYMMETRIC SIMPLE EXCLUSION ON Z^d

Understanding the motion of a tracer particle, as it interacts with others, is a basic applied concern. Einstein's famous 1905 paper established Brownian motion as the diffusive limit of a pollen particle. There have been many efforts to derive a rigorous physical basis of Brownian motion and other diffusions, however, the question is still largely open.

We consider here the problem in the finite range symmetric simple exclusion processes when started under an equilibrium ν_ρ. Except for the case dimension $d = 1$ when the jump law p is nearest-neighbor, the tagged particle motion is diffusive and converges to a Brownian motion. In the exception case, it can be shown the motion is subdiffusive and converges to a fractional Brownian motion of Hurst parameter 1/4.

First, we discuss LLNs. Then, the diffusive behavior is considered using the Kipnis-Varadhan theorem. Finally, we discuss the exceptional subdiffusive case.

1. Basic problem

Consider the simple exclusion process η_t with finite-range jump probability p on $\mathbb{Z}^d$. The system consists typically of an infinite number particles. Let us fix one initially at the origin. How to capture its evolution x_t? With respect to its own history, it is not Markovian because of influence of the other particles. However, if we consider x_t and η_t together, then the joint process (x_t, η_t) is Markovian with generator

$$\begin{aligned}\tilde{L}f(x,\eta) &= \sum_{u,v\neq x} p(v-u)\eta(u)(1-\eta(v))\big[f(x,\eta^{u,v}) - f(x,\eta)\big] \\ &\quad + \sum_v p(v)(1-\eta(x+v))\big[f(x+v,\eta^{x,x+v}) - f(x,\eta)\big].\end{aligned}$$

It will be convenient to consider "Lagrangian" coordinates, or those in the reference frame of the tagged motion. Define $\zeta_t = \tau_{x_t}\eta_t$, that is $\zeta_t(y) = \eta_t(y + x_t)$ for all $y \in \mathbb{Z}^d$ where τ_x is shift by x. Then, the joint process (x_t, ζ_t) is also Markovian and has generator

$$\begin{aligned}\hat{L}f(x,\zeta) &= \sum_{u,v\neq 0} p(v-u)\zeta(u)(1-\zeta(v))\big[f(x,\zeta^{u,v}) - f(x,\zeta)\big] \\ &\quad + \sum_v p(v)(1-\zeta(v))\big[f(x+v,\theta_v\zeta) - f(x,\zeta)\big]\end{aligned}$$

where $\theta_v\zeta$ is the configuration which displaces the particle at the origin, namely the tagged particle, by v, and then shifts the reference frame to this new origin.

Exercise 1.1. Explicitly describe $\theta_v\zeta$. In particular, show that

$$(\theta_v\zeta)(y) = \begin{cases} \zeta(y+v) & \text{for } y \neq -v, 0 \\ \zeta(v) & \text{for } y = -v \\ 1 & \text{for } y = 0. \end{cases}$$

Both joint processes can be constructed on bounded local functions as before with the zero-range process. Interestingly, the process ζ_t by itself is a Markov process. This can be seen as the generator L acting on functions of ζ alone,

$$\begin{aligned} Lf(\zeta) &= \sum_{u,v\neq 0} p(v-u)\zeta(u)(1-\zeta(v))\big[f(\zeta^{u,v})-f(\zeta)\big] \\ &\quad +\sum_v p(v)(1-\zeta(v))\big[f(\theta_v\zeta)-f(\zeta)\big], \end{aligned}$$

does not depend on x, and hence the associated semigroup also does not depend on x.

The process ζ_t also "drives" the process x_t in that x_t can be recovered in terms of reference frame shifts. Define $N_v(t)$ as the count of shifts of $\zeta_\cdot$ of displacement v up to time t. Then,

$$x_t = \sum_v vN_v(t). \tag{1.1}$$

For instance, in one dimension, when p is nearest-neighbor, x_t is the number of right shifts minus the number of left shifts.

Morevoer, it is not difficult to find invariant measures for ζ_t generated by L.

Lemma 1.2. *The Bernoulli product measures conditioned to have a particle at the origin, $\nu'_\rho = \nu_\rho(\cdot|\eta(0)=1)$, are all invariant for ζ_t.*

The Dirichlet form can be computed also on local functions:

$$\begin{aligned} D(f) &= E_{\nu'_\rho}[f(-Lf)] \\ &= \frac{1}{2}\sum_{u,v\neq 0} s(v-u)E_{\nu'_\rho}[(f(\zeta^{u,v})-f(\zeta))^2] \\ &\quad +\frac{1}{2}\sum_v s(v)E_{\nu'_\rho}[(1-\zeta(v))f(\theta_v\zeta)-f(\zeta))^2] \\ &= D_e(f)+D_t(f) \end{aligned}$$

where s is the symmetrized probability, $s(v) = (p(v)+p(-v))/2$.

Lemma 1.3. *ν'_ρ is extremal for the process ζ_t.*

Exercise 1.4. Since ν_ρ is invariant to exchanges of coordinates and also the operator θ_v, the proof of Lemma 1.2 is similar to the proof which shows $\nu_r ho$ is invariant for η_t given in an earlier Part. As before, for symmetric exclusion, the proof simplifies. Make this precise.

Exercise 1.5. Show the computations for the Dirichlet form, and deduce Lemma 1.3 from its form following the proof of extremality of ν_ρ.

Now, as before, with zero-range processes, we can extend the process ζ_t to an $L^2(\nu_\rho(\cdot|\eta(0)=1))$ process. The problem now is to understand the LLN and fluctuations for x_t in terms of the formula (1.1).

1.1. **Martingales for x_t.** Each count $N_v(t)$, as with a Poisson process, can be compensated by its intensity, $\int_0^t p(v)(1-\zeta_s(v))ds$, to form a martingle

$$M_v(t) = N_v(t) - \int_0^t p(v)(1-\zeta_s(v))ds$$

which quadratic variation

$$\langle M_v\rangle(t) \;=\; \int_0^t p(v)(1-\zeta_s(v))ds.$$

Then,

$$\mathcal{M}_v := M_v^2(t) - \int_0^t p(v)(1-\zeta_s(v))ds$$

are martingales.

How to verify the above statements? Technically, we should have formed the generator $\vec{L}$ for the process $(\{N_v(t)\}, \zeta_t)$. Then, with $f(\{N_v\}, \zeta) = N_v$, we may compute $\vec{L}f$ and $\vec{L}f^2 - 2f\vec{L}f$ to see that

$$M_v(t) \;=\; f(\{N_v(t)\}, \zeta_t) - f(\{N_v(0)\}, \zeta_0) - \int_0^t \vec{L}f ds$$

and

$$\langle M_v(t)\rangle \;=\; \int_0^t \vec{L}f^2 - 2f\vec{L}f ds.$$

Although f is not a bounded function, one may apply truncations to bring it in the domain of the generator $\vec{L}$.

Exercise 1.6. Make these computations.

Now, we may write

$$x_t \;=\; \sum_v vM_v(t) + \sum_0^t \sum vp(v)(1-\zeta_s(v))ds. \tag{1.2}$$

The first term, $\hat{M}(t) = \sum vM_v(t)$ is another (vector valued) martingale such that quadratic variation

$$\langle \hat{M}\cdot\ell\rangle(t) \;=\; \int_0^t \sum_v (v\cdot\ell)^2 p(v)(1-\eta_s(v))ds. \tag{1.3}$$

At this point, we remark, we will need only formulas (1.2) and (1.3) in the following, which we remark could have been derived from the generator action of $\hat{L}$. The purpose here was to explain more physically their interpretation in terms of counts $\{N_v\}$.

We also note that since ν'_ρ is extremal, we have a.s. and in L^1 that

$$\frac{\langle \hat{M}\cdot\ell\rangle(t)}{t} \;\to\; (1-\rho)\sum(v\cdot\ell)^2 p(v).$$

2. LLN for x_t

The laws of large numbers for x_t is already interesting. In some sense x_t should be a random walk, but it is impeded by the other particles. How much is the question.

Theorem 2.1. *Starting under ν'_ρ, we have a.s. and in L^1 that*

$$\frac{x_t}{t} \;\to\; (1-\rho)\sum vp(v).$$

Proof. Write

$$\frac{x_t}{t} \;=\; \frac{\hat{M}(t)}{t} + \frac{1}{t}\int_0^t \sum vp(v)(1-\zeta_s(v))ds.$$

Since ν'_ρ is extremal, we have by the ergodic theorem that the right side converges to its mean which is equal to $(1-\rho)\sum vp(v)$ a.s. and in L^1. □

3. CLT for x_t except in the exceptional case

We now show a central limit theorem for x_t when the jump probability p is symmetric. It can be extended to a functional CLT to a Brownian motion, but that is not done here [6].

Theorem 3.1. *Let p be symmetric. Starting under ν'_ρ, we have*

$$\frac{x_t}{\sqrt{t}} \;\Rightarrow\; \mathrm{N}(0,C)$$

where $C = C(\rho,p)$ is covariance matrix.

The limiting covariance C is not explicit, and there are physics conjectures, and some rigorous results for its behavior as a function of ρ [7]. However, it is only nondegenerate excluding a particular case.

Proposition 3.2. *Except in the case $d=1$ and p is nearest-neighbor ($p(1) = p(-1) = 1/2$), the covariance C is nondegenerate.*

We first prove Theorem 3.1 and then later Proposition 3.2.

Proof of Theorem 3.1. From the martingale decomposition (1.2), we have

$$\frac{x_t}{\sqrt{t}} \;=\; \frac{\hat{M}(t)}{\sqrt{t}} + \frac{1}{\sqrt{t}}\int_0^t \sum vp(v)(1-\eta_s(v))ds.$$

Since p is symmetric, the process ζ_t is reversible with respect to ν'_ρ. We want to apply the Kipnis-Varadhan theorem to the second term.

In this effort, we need to show that $h(\zeta) = \sum vp(v)(1-\zeta(v)) = -\sum vp(v)\zeta(v)$ belongs to H_{-1}. Since p is symmetric, h is a linear combination of $\zeta(w)-\zeta(-w)$ for w in the support of p. Consider, for local function (mean-zero) ϕ that

$$\begin{aligned}
\langle \zeta(w)-\zeta(-w), \phi\rangle_{\nu'_\rho} &= E_{\nu'_\rho}[\zeta(w)\phi(\zeta)] - E_{\nu'_\rho}[\theta_w(\zeta)(-w)\phi(\theta_w(\zeta))]\\
&= E_{\nu'_\rho}[\zeta(w)(\phi(\zeta)-\phi(\theta_w(\zeta))]\\
&\leq E_{\nu'_\rho}[(1-\zeta(w)^2(\phi(\zeta)-\phi(\theta_w(\zeta))^2]^{1/2}\\
&= E_{\nu'_\rho}[(1-\zeta(w)(\phi(\zeta)-\phi(\theta_w(\zeta))^2]^{1/2}.
\end{aligned}$$

Now, the last quantity is less than $D(\phi)^{1/2}$. Hence, h belongs to H_{-1}.

Therefore, there is a square integrable $M(t)$ and error $\chi(t)$ such that $|\chi(t)|/\sqrt{t}$ vanishes in L^2, and also

$$\int_0^t h(\zeta_s)ds \;=\; M(t)+\chi(t).$$

Hence,

$$\frac{x_t}{\sqrt{t}} \;=\; \frac{\hat{M}(t)+M(t)}{\sqrt{t}} + \frac{\chi(t)}{\sqrt{t}}.$$

At this point, one can take inner product with a vector ℓ and complete the argument by martingale central limit theorem since by the ergodic theorem

$$\frac{\langle \hat{M}\cdot\ell + M\cdot\ell\rangle(t)}{t} \ \to\ E\langle \hat{M}\cdot\ell + M\cdot\ell\rangle(1)$$

a.s. and in L^1. □

3.1. **Proof of Proposition 3.2.** In the exceptional case, the two martingales $\hat{M}$ and M cancel each other, leading to subdiffusive fluctuations for x_t. However, in all other situations, full cancellation does not occur.

The argument relies on the following bound.

Lemma 3.3. *We have*

$$|\langle f,\phi\rangle_{\nu'_\rho}| \ \le\ CD_e(\phi)^{1/2}.$$

Proof. From the form of h, we need only show the result for $\zeta(w)-\zeta(-w)$. The key point is that, aside from the exceptional case, one can either go around, in terms of exchanges of coordinates, the origin in $d\ge 2$ or hop over it in $d=1$, without disturbing the tagged particle sitting at the origin.

One can rewrite, by irreducibility of p, that

$$\zeta(w)-\zeta(-w) \ =\ \sum \zeta(u_k)-\zeta(u_{k+1})$$

in terms of a sequence so that $p(u_{k+1}-u_k)>0$. Then,

$$\begin{aligned}
|\langle\zeta(u_k)-\zeta(u_{k+1}),\phi\rangle| \ &=\ E_{\nu'_\rho}[\zeta(u_k)[\phi(\zeta)-\phi(\zeta^{u_k,u_{k+1}})] \\
&\le\ \rho^{1/2}E_{\nu'_\rho}[\zeta(u_k)(\phi(\zeta)-\phi(\zeta^{u_k,u_{k+1}}))^2]^{1/2} \\
&\le\ CD_e(\phi)^{1/2}.
\end{aligned}$$

□

Proof of Proposition 3.2. From the formula for the limiting variance, we need to show that

$$E\langle\hat{M}+M\rangle(1) \ =\ \lim_{\lambda\downarrow 0} E(\hat{M}(1)+M_\lambda(1))^2 \ >\ 0.$$

Step 1. Consider the function $g(x,\zeta)=x+u_\lambda(\zeta)$. Then,

$$\hat{M}(t)+M_\lambda(t) \ =\ g(x_t,\zeta_t)-g(x_0,\zeta_0)-\int_0^t \hat{L}g(x_s,\zeta_s)ds.$$

Then, by computing the quadratic variation,

$$E_{\nu'_\rho}(\hat{M}(1)+M_\lambda(1))^2 \ =\ 2E_{\nu'_\rho}[g\hat{L}g].$$

Now, an explicit computation gives that the right hand side equals

$$\begin{aligned}
&E_{\nu'_\rho}\sum_{u,v\neq 0} p(v-u)\big[g(x,\zeta^{u,v})-g(x,\zeta)\big]^2 \\
&\quad +E_{\nu'_\rho}\sum_v p(v)(1-\zeta(v))\big[g(x+v,\theta_v\zeta)-g(x,\zeta)\big]^2.
\end{aligned}$$

The first term is $2D_e(u_\lambda)$. Then, by dropping the second term, we have the lower bound

$$E(\hat{M}(1)+M_\lambda(1))^2 \ \ge\ 2D_e(u_\lambda).$$

Step 2. Consider the equation

$$\lambda u_\lambda - Lu_\lambda = h.$$

Multiplying by u_λ and integrating, we obtain the inequality

$$\|u_\lambda\|_1^2 \;\le\; \langle h, u_\lambda\rangle \;\le\; CD_e(u_\lambda)$$

as $u_\lambda \in H_1$.

Now, if $\|u_\lambda\|_1$ vanishes as $\lambda \downarrow 0$, we have that

$$EM_\lambda^2(1) \;=\; \|u_\lambda\|_1^2 \;\to\; 0$$

and so the limiting variance is just $E\hat{M}^2(1)$ which is explicit and positive.

On the other hand, if $\|u_\lambda\|_1 \not\to 0$, then $D_e(u_\lambda)$ also does not vanish. Hence, still the limiting variance is positive. □

4. Subdiffusive CLT for the exceptional case

Since particles are ordered in $d = 1$ when the transition probability p is nearest-neighbor, it is not difficult to suspect that the expected displacement in time t may be less than diffusive, that is on order $\sqrt{t}$. This is the case when p is additionally symmetric, $p(1) = p(-1)$.

Theorem 4.1. *Starting under ν'_ρ, we have*

$$\frac{x_t}{t^{1/4}} \;\Rightarrow\; \mathrm{N}(0, \sigma^2)$$

where $\sigma^2 = \sqrt{2/\pi}[(1-\rho)/\rho]$.

A functional CLT is also known, but here the limit will be a fractional BM with Hurst parameter 1/4 [10].

There are a couple of ways to prove this subdiffusive CLT which was first proved in [1]. Soon after, another proof was given in [11]. We present a method which makes a connection with the current through the bond $(-1, 0)$ given in [2].

Define

$$J_{-1,0}(t) \;=\; N_+(t) - N_-(t)$$

where $N_+(t)$ is the number of particles which start on the left of $x = 1/2$ and move across the bond $(-1, 0)$ up to time t, and $N_-(t)$ is the number of particles to the right of $x = 1/2$ crossing to the left.

Then, we have the following relations between x_t and $J_{-1,0}(t)$:

$$\begin{aligned}
\{x_t \ge A\} &\;=\; \{J_{-1,0}(t) \ge \sum_{x=1}^{A-1} \eta_t(x)\} \;\text{ when } A \ge 1\\
\{x_t = 0\} &\;=\; \{J_{-1,0}(t) = 0\}\\
\{x_t \le A\} &\;=\; \{J_{-1,0}(t) \le \sum_{x=A+1}^{0} \eta_t(x)\} \;\text{ when } A \le -1.
\end{aligned} \tag{4.1}$$

We will show the following result which will imply Theorem 4.1. The proof of Theorem 4.2 is deferred to a later subsection.

Theorem 4.2. *Starting under ν'_ρ, we have*

$$\frac{J_{-1,0}(t)}{t^{1/4}} \;\Rightarrow\; \mathrm{N}(0, \rho^2\sigma^2).$$

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1. Write, for $A > 0$, that

$$P(x_t \geq At^{1/4}) \;=\; P\big(J_{-1,0}(t) \geq \sum_{x=1}^{At^{1/4}-1} \eta_t(x)\big)$$
$$= \; P\Big(t^{-1/4}J_{-1,0}(t) \geq t^{-1/4}\sum_{x=1}^{At^{1/4}-1} \eta_t(x)\Big).$$

Since, under ν'_ρ, $\{\eta_t(x)\}$ are iid Bernoulli random variables,

$$t^{-1/4}\sum_{x=1}^{At^{1/4}-1} \eta_t(x) \;\to\; A\rho \quad \text{a.s.}$$

Then, from Theorem 4.2, we have that

$$P(x_t \geq At^{1/4}) \;=\; P\big(t^{-1/4}J_{-1,0}(t) \geq A\rho\big) \;\Rightarrow\; P\big(\mathrm{N}(0,\sigma^2\rho^2) \geq A\rho\big).$$

The right hand side is the same as $P(\mathrm{N}(0,\sigma^2) \geq A)$.

A similar argument works for $A < 0$. □

4.1. **Stirring representation.** One way to give a proof of Theorem 4.2 is to use an interesting representation of symmetric simple exclusion processes due to Ted Harris. We will limit ourselves to the nearest-neighbor situation in $d = 1$.

Recall that the generator for symmetric simple exclusion in this case is given by

$$\mathcal{A}\phi(\eta) \;=\; \sum_x \big[\phi(\eta^{x,x+1}) - \phi(\eta)\big].$$

In other words, the process evolves by "exchanging" values with neighboring coordinate variables.

Now, consider a collection of Poisson processes with intensity $1/2$ indexed by the bonds $\{(x, x+1) : x \in \mathbb{Z}\}$. Place a particle at each vertex in $\mathbb{Z}$. At the event times of the Poisson process with index $(x, x+1)$, the particles at x and $x+1$ exchange positions. Let ξ^x_t be the location of the particle intially at x at time $t \geq 0$. Marginally, each ξ^x_t has the statistics of a nearest-neighbor symmetric random walk, although jointly they are dependent.

We claim that

$$\eta_t(x) \;=\; 1\big\{x \in \{\xi^i_t : \eta_0(i) = 1\}\big\},$$

that is $\eta_t(x) = 1$ exactly when there is an $i \in \mathbb{Z}$ such that $\xi^i_t = x$ and $\eta_0(i) = 1$. Infinitesimally, the possible transitions are exchanges of nearest-neighbor values which is the same as given in the generator $\mathcal{A}$. Then,

$$J_{-1,0}(t) \;=\; \sum_{i<0} 1(\xi^i_t \geq 0)\eta_0(i) - \sum_{i\geq 0} 1(\xi^i_t < 0)\eta_0(i).$$

Define now

$$K_+(t) = \sum_{i<0} 1\{\xi^i_t \geq 0\}, \quad \text{and} \quad K_-(t) = \sum_{i\geq 0} 1\{\xi^i_t < 0\}$$

which represent the number of stirring particles starting from the left of the origin which end up on the right at time t, and vice versa.

Since, in the stirring process, a crossing of the bond $(-1, 0)$ in one direction corresponds to a crossing in the other direction, as all sites are occupied, we have that $K_+(t) - K_-(t)$ is constant in time. But, $K_+(0) = K_-(0) = 0$, and hence $K(t) := K_+(t) = K_-(t)$ for all $t \geq 0$. When $K(t) \geq 1$, let $i_1 < i_2 < \cdots < i_{K(t)} < 0$ be the random locations where $\xi^{i_k} \geq 0$, and $0 \leq j_1 < \cdots < j_{K(t)}$ be the random locations where $\xi^{j_k} < 0$. Then, we can write the current as

$$J_{-1,0}(t) \ = \ \sum_{k=1}^{K(t)} a_k$$

where $a_k(t) = \eta_0(i_k) - \eta_0(j_k)$; Note the time dependence on t comes from the random locations i_k, j_k which depend on t. Conditionally on $K(t)$, and the random locations, the variables $\{a_k\}$ are iid with mean 0 and variance $2\rho(1-\rho)$. Moreover, one can calculate the explicit probabilities a_k takes values $-1, 0, 1$ which is left as an exercise.

4.2. **Estimates on K.** We have the following properties of K.

Lemma 4.3. *We have that*

$$t^{-1/2}E[K(t)] \ \to \ (2\pi)^{-1/2}.$$

Proof. Write

$$\begin{aligned} E[K(t)] &= \sum_{i<0} P(\xi_t^i \geq 0) \\ &= \sum_{i<0} P(\xi_t^0 \geq -i) \\ &= \sum_{i>0} P(\xi_t^0 \geq i) \\ &= E[(z(t))^+] \end{aligned}$$

where $z(t)$ is a simple random walk starting at the origin and $(z(t))^+$ is the positive part.

Now, $\{t^{-1/2}z(t)\}$ is a uniformly integrable sequence of random variables which converge in distribution to $N(0,1)$. Hence,

$$t^{-1/2}E[K(t)] \ \to \ \int_0^\infty (2\pi)^{-1/2} x e^{-x^2/2} dx \ = \ 1.$$

□

Exercise 4.4. Show the uniform integrability mentioned in the above lemma.

We also have that the variables $1(\xi_t^x \geq 0)$ and $1(\xi_t^y \geq 0)$ are negatively correlated, which makes intuitive sense since the exchange mechanism is repulsive to an extent.

Lemma 4.5. *For $x, y < 0$, we have that*

$$P(\xi_t^x, \xi_t^y \geq 0) \ \leq \ P(\xi_t^x \geq 0)P(\xi_t^y \geq 0).$$

We refer the reader to Lemma 4.12 [8] or page 365 [1] for proofs of the lemma (in a more general context). *Challenge:* In this simple context, a challenge is to find another proof.

A consequence of the lemma is that $\mathrm{Var}(K(t)) \leq E[K(t)]$.

4.3. **Proof of Theorem 4.2.** To prove the theorem, we show that the difference

$$W(t) \ = \ t^{-1/4}\Big[\sum_{k=1}^{K(t)} a_k - \sum_{k=1}^{(2\pi)^{-1/2}t^{1/2}} a_k\Big] \ \to \ 0$$

as $t \uparrow \infty$. Now the variance

$$\begin{aligned}\mathrm{Var}(W(t)) \ &= \ E\big[E[W(t)^2|K(t),\{i_k,j_k\}]\big] \\ &= \ 2\rho(1-\rho)t^{-1/2}E[|K(t)-(2\pi)^{-1/2}t^{1/2}|] \\ &= \ 2\rho(1-\rho)t^{-1/2}E[|K(t)-E[K(t)]|]+\rho(2-\rho)|E[K(t)]-(2\pi)^{-1/2}|.\end{aligned}$$

Since $K(t)$ is the sum of negatively correlated random variables, we have that $\mathrm{Var}(K(t)) \leq E[K(t)] = O(t^{1/2})$. Hence, we can apply Chebychev's inequality to get

$$P(W(t) > b) \ \leq \ b^{-2}\mathrm{Var}(W(t)) \ \leq \ Cb^{-2}.$$

Now,

$$\begin{aligned}&E\Big[\exp\Big\{i\theta\frac{1}{t^{-1/4}}\sum_{k=1}^{(2\pi)^{-1/2}t^{1/2}} a_k\Big\}\Big] \\ &= \ E\Big[\Big[\exp\Big\{i\theta\frac{1}{t^{-1/4}}\sum_{k=1}^{(2\pi)^{-1/2}t^{1/2}} a_k\Big\}\Big]K(t),\{i_k,j_k\}\Big] \\ &= \ E\Big[\psi\big(\theta t^{-1/4}\big)^{(2\pi)^{-1/2}t^{1/2}}\Big] \\ &= \ \psi\big(\theta t^{-1/4}\big)^{(2\pi)^{-1/2}t^{1/2}}.\end{aligned}$$

where ψ is the characteristic function of a_k conditioned on $K(t)$ and the random locations, which has an explicit distribution on values $0,-1,1$. One now follows the strategy for the usual CLT to finish the proof.

One also sees from this calculation that the variance of the limiting Normal distribution is $\sqrt{2/\pi}\rho(1-\rho)$ as desired. □

5. Notes

The tagged particle problem is somewhat complete at the level of LLN and CLT for symmetric exclusion processes with finite range. See [8] and [9] for excellent treatments. See also [3] for longer range symmetric exclusion processes.

For asymmetric simple exclusion, however, there are some nice open problems. A CLT has been shown in diffusive scale when $d \geq 3$ [13], or when $d = 1$ and the jumps are nearest-neighbor [5]. Also, when the jumps are mean-zero but asymmetric, the CLT also holds [15].

Although the Laplace transform of the variance of the tagged particle at time t is shown to be diffusive, that is bounded below and above in terms of constants by the Laplace transform of t, in the open asymmetric cases [12], when $d = 2$ and when $d = 1$ and the jumps are not nearest-neighbor, the CLT is open.

We comment that the tagged particle problem is an old problem going back to Einstein's 1905 opus. Deriving rigorously Brownian motion from a Hamiltonian dynamics, where the initial positions of particles are random, is a largely open problem although there has been some progress. We refer to [14] for a review.

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