PART 7: INVARIANT MEASURES: ERGODICITY AND EXTREMALITY

We discuss the invariant measures in zero-range processes. To determine "all" of them, since the set of invariant measures is a convex set, one tries to prescribe the extreme points of this set. If an invariant measure is known to be an extreme point, the process run under it has interesting ergodic properties. In this subject, many things are known, but there are several open questions.

1. Invariant measures for zero-range processes

We say that a probability measure μ on Ω is an *invariant measure* if

$$\int P_t f d\mu \;=\; \int f d\mu$$

for all bounded Lipschitz functions $f \in \mathcal{L}$. Let $\mathcal{I}$ denote the set of invariant measures.

To simplify the discussion, we will assume that p is translation-invariant. However, a rich theory exists when p is not translation-invariant which are remarked upon in the Notes section.

Recall the discussion of invariant measures for the zero-range process on a torus $\mathbb{T}_N^d$. We extend these notions and define marginal

$$\bar{\mu}_\alpha(k) \;=\; \begin{cases} \frac{1}{Z(\alpha)}\frac{\alpha^k}{g(k)!} & \text{for } k \geq 1 \\ \frac{1}{Z(\alpha)} & \text{for } k = 0 \end{cases}$$

for $0 \leq \alpha < \liminf g(k)$. Correspondingly, define

$$\bar{\nu}_\alpha \;=\; \prod_{x\in\mathbb{Z}^d} \bar{\mu}_\alpha.$$

As before $\rho = \rho(\alpha) = E_{\bar{\mu}_\alpha}[\eta(0)]$ is a strictly increasing function of α, and hence the inverse exists. We then define

$$\nu_\rho \;=\; \bar{\nu}_{\alpha(\rho)}$$

for $\rho < \rho^* := \lim_{\alpha\uparrow\liminf g(k)} \rho(\alpha)$.

Our first result is that these measures are invariant.

Theorem 1.1. *For $\rho < \rho^*$, we have $\nu_\rho \in \mathcal{I}$.*

Moreover, these measures fully charge Ω', that is

$$\nu_\rho(\Omega') \;=\; 1. \tag{1.1}$$

This is seen by integrating $\|\eta\|$,

$$E_{\nu_\rho}\big[\|\eta\|\big] \;=\; \sum_{x\in\mathbb{Z}^d} E_{\nu_\rho}[\eta(x)]\beta(x) \;=\; \rho \sum_{x\in\mathbb{Z}^d}\beta(x)$$

and noting that

$$\sum_{x\in\mathbb{Z}^d}\beta(x) \;=\; \sum_{x\in\mathbb{Z}^d}\sum_{n\geq 0}\frac{p^{(n)}(x,0)}{2^n} \;=\; 2 \tag{1.2}$$

given $p^{(n)}(x,0) = p^{(n)}(0,-x)$ by translation-invariance of p. Hence, $\|\eta\| < \infty$ a.s. under ν_ρ.

2. Proof of Theorem 1.1

We first give a generator characterization of invariance of a measure.

Proposition 2.1. *Let μ be a probability measure on Ω such that $E_\mu[\|\eta\|] < \infty$. Then, the following are equivalent:*

(1) $\int Lf d\mu = 0$ *for all* $f \in \mathcal{L}$
(2) $\mu \in \mathcal{I}$.

In what follows, we will only use that "(1) implies (2)", and so we will prove this part of the result.

Proof. For $f \in \mathcal{L}$ and $\eta \in \Omega'$, we have

$$\begin{aligned} |LP_sf(\eta)| &\leq a_0 \sum_{x,y\in\mathbb{Z}^d} \eta(x)p(x,y)\big|P_sf(\eta^{x,y}) - P_sf(\eta)\big| \\ &\leq 3a_0c(f)e^{4a_0s}\|\eta\|. \end{aligned}$$

The inequalities use that the Lipschitz bound on g, and that $c(P_sf) \leq 3e^{4a_0s}c(f)$ from Theorem 2.4 in Part 6.

When (1) holds, for $f \in \mathcal{L}$ which is bounded,

$$\int \Big[P_tf - f\Big]d\mu \;=\; \int \Big[\int_0^t LP_sfds\Big]d\mu.$$

Since $P_sf \in \mathcal{L}$, and the bound given above, we may interchange the integrals by Fubini's theorem. Also, by (1), we have therefore

$$\int \Big[\int_0^t LP_sfds\Big]d\mu \;=\; \int_0^t \Big[\int LP_sfd\mu\Big]ds \;=0$$

and so may conclude (2). □

Exercise 2.2. To prove the converse, "(2) implies (1)", we refer the reader to [1][Lemma 2.9].

Lemma 2.3 (Finite set invariance)**.** *Suppose that p is a transition probability on $A_n = \{x : |x_i| \leq n, 1 \leq i \leq d\}$ which is doubly stochastic: $\sum_{x\in A_n} p(x,y) = 1$ for all $y \in A_n$. Then, ν_ρ is invariant for the zero-range process on A_n.*

Proof. We need only show that $E_{\nu_\rho}[Lf(\eta)] = 0$ for all functions f. Recall

$$Lf(\eta) \;=\; \sum_{x,y\in A_n} g(\eta(x))p(x,y)\big[f(\eta^{x,y}) - f(\eta)\big].$$

Since

$$E_{\nu_\rho}\big[g(\eta(x))f(\eta^{x,y})\big] \;=\; \alpha(\rho)E_{\nu_\rho}\big[f(\eta+\delta_y)\big] \;=\; E_{\nu_\rho}\big[g(\eta(y))f(\eta)\big],$$

we have that

$$\begin{aligned} E_{\nu_\rho}\sum_{x,y\in A_n} g(\eta(x))p(x,y)f(\eta^{x,y}) &= \sum_{x,y\in A_n} p(x,y)E_{\nu_\rho}\big[g(\eta(y))f(\eta)\big] \\ &= \sum_{y\in A_n} E_{\nu_\rho}\big[g(\eta(y))f(\eta)\big]. \end{aligned}$$

On the other hand, clearly,

$$\sum_{x,y\in A_n} p(x,y)E_{\nu_\rho}\big[g(\eta(x))f(\eta)\big] = \sum_{x\in A_n} E_{\nu_\rho}\big[g(\eta(x))f(\eta)\big].$$

Hence, combining these estimates, $E_{\nu_\rho}[Lf] = 0$. $\square$

Proof of Theorem 1.1. For a transition probability p on $\mathbb{Z}^d$, define the 'truncation' (different than in the previous Part 6),

$$p_n(x,y) \;=\; \begin{cases} 1 & \text{if } x = y \notin A_n \\ p(x,y) + Q_n^{-1}\big[\sum_{z\notin A_n} p(x,z)\big]\big[\sum_{z\notin A_n} p(z,y)\big] & \text{if } x,y\in A_n \\ 0 & \text{otherwise} \end{cases}$$

where

$$Q_n = \sum_{z\in A_n, y\notin A_n} p(z,y) = \sum_{z\in A_n} 1 - \sum_{z,y\in A_n} p(z,y) = \sum_{y\in A_n, z\notin A_n} p(z,y).$$

We have that (A) p_n is a transition probability: $\sum_y p_n(x,y) = 1$ for all $x\in\mathbb{Z}^d$. If $x\notin A_n$, it is trivial. If $x\in A_n$, noting that $Q_n = \sum_{z\notin A_n, y\in A_n} p(z,y)$, we have $\sum_y p_n(x,y) = \sum_{y\in A_n} p_n(x,y) = 1$. Moreover, p_n has no transitions from A_n to its complement.

In addition, (B) $\sum_x p_n(x,y) = 1$ for all $y\in\mathbb{Z}^d$: If $y\notin A_n$, the claim is trivial. If $y\in A_n$, write

$$\sum_x p_n(x,y) \;=\; \sum_{x\in A_n} p(x,y) + Q_n^{-1}\big[\sum_{x\in A_n, z\notin A_n} p(x,z)\big]\big[\sum_{z\notin A_n} p(z,y)\big] \;=\; \sum_x p(x,y).$$

The last quantity equals 1 if p is doubly-stochastic which is the case if p is translation-invariant.

Also, (C) $p_n \to p$ for all $x,y\in Z^d$. Eventually, $x,y\in A_n$. The claim follows as $Q_n^{-1}\sum_{z\notin A_n} p(x,y) \le 1$.

From Lemma 2.3, as nothing moves off A_n, we have that $E_\nu[L_n f] = 0$ for $f\in\mathcal{L}$ where L_n is the generator for the zero-range process on A_n according to transition probability p_n. Therefore, to show $E_{\nu_\rho}[Lf] = 0$, it is enough to show

$$E_{\nu_\rho}\big[|L_n f - Lf|\big] \;\to\; 0.$$

It is straightforward to write

$$\begin{aligned} E_{\nu_\rho}\big[|L_n f - Lf|\big] \;&\le\; a_0 c(f) E_{\nu_\rho}\big[\sum_{x\in\mathbb{Z}^d}\eta(x)\sum_{y\neq x}|p(x,y) - p_n(x,y)|\big(\beta(x)+\beta(y)\big) \\ &=\; I_1 + I_2 \end{aligned}$$

corresponding to the two factors $\beta(x)$ and $\beta(y)$.

We bound I_1 by dominated convergence. We evaluate I_1 as

$$\rho\sum_x \beta(x)\sum_{y\neq x}|p(x,y) - p_n(x,y)|.$$

For fixed x, eventually $x\in A_n$ for large n, and so

$$\sum_y |p(x,y) - p_n(x,y)| \;\le\; \sum_{y\in A_n}|p(x,y) - p_n(x,y)| + \sum_{y\notin A_n} p(x,y) \;=\; 2\sum_{y\notin A_n} p(x,y)$$

using the form of Q_n. Hence, since $\sum_x \beta(x) < \infty$ and the last display vanishes as $n\uparrow\infty$ for each x, by dominated convergence I_1 vanishes as $n\uparrow\infty$.

We now bound I_2 in a similar manner. Write I_2 as

$$\rho \sum_x \sum_{y \neq x} \beta(y)|p(x,y) - p_n(x,y)| \ = \ \rho \sum_y \beta(y) \sum_{x \neq y} |p(x,y) - p_n(x,y)|.$$

For each y eventually $y \in A_n$. Then, using a form of Q_n, we have that

$$\sum_{x \neq y} |p(x,y) - p_n(x,y)| \ \leq \ 2 \sum_{x \notin A_n} p(x,y)$$

which vanishes as p is doubly-stochastic. Hence, as $\sum_y \beta(y) < \infty$, I_2 vanishes by dominated convergence. □

3. Extremality, Harmonicity and Ergodicity

First, what does it mean to be extremal? Let Σ be a space with Borel sets $\mathcal{B}$. Let Q be an invariant proability measure on Σ for the Markov process $\eta(t)$. Let $\mathbb{P}_Q$, as before, be the probability on the path space with initial distribution Q. Let T_t be the process semigroup.

Definition 3.1. *We say Q is an extremal invariant measure if the the following property holds. When for $0 < \epsilon < 1$ and invariant probability measures Q_1 and Q_2 we have $Q = \epsilon Q_1 + (1-\epsilon)Q_2$, then $Q = Q_1 = Q_2$.*

We note if the process has only one invariant measure Q, then of course it is extremal. The import of the definition comes when the process is reducible in some way. For instance, in finite state Markov chains, with two irreducible components C_1 and C_2, the extreme invariant measures are exactly the unique invariant measures supported on C_1 and C_2 respectively.

One might ask why is it useful to know when an invariant measure is extremal. It turns out there is an interesting connection with harmonic functions and shift-ergodicity.

Lemma 3.2. *The space $L^2(Q)$ can be decomposed into the direct sum of*

$$H_0 \ = \ \{g \in L^2(Q) : T_t g = g, t \geq 0\}, \quad \text{and} \quad H_0^c \ = \ \{T_t h - h : h \in L^2(Q), t \geq 0\}.$$

Proof. Let f be perpendicular to all functions in H_0^c. Then, $\langle f, T_t h - h\rangle = 0$ for all h and $t \geq 0$. This means $T_t^* f = f$ or $T_t f = f$ for all $t \geq 0$. Hence, $f \in H_0$. □

Since T_t is a contraction on $L^2(Q)$, we have the following ergodic theorem due to Von Neumann.

Proposition 3.3. *We have for $f \in L^2(Q)$ that*

$$\frac{1}{t}\int_0^t T_s f \, ds \ \to \ \hat{f}$$

converges in $L^2(Q)$ to $\hat{f} \in L^2(Q)$ which is a projection of f onto the subspace H_0. That is, $\hat{f}$ is the conditional expectation of f with respect to the time shift invariant sets.

Proof. Decompose $f = \hat{f} + g$ where $g \in H_0^c$. Clearly, $T_t \hat{f} = \hat{f}$. We need to understand the contribution of the part g which is in form $g = T_u h - h$ for some $h \in L^2(Q)$ and $u \geq 0$.

Then,

$$\frac{1}{t}\int_0^t T_s g\,ds \;=\; \frac{1}{t}\Big[\int_t^{t+u} T_s h\,ds - \int_0^u T_s h\,ds\Big] \;=\; O(t^{-1})$$

as T_t is a contraction. $\square$

Proposition 3.4 (Equivalences)**.** *All are equivalent:*

(a) *For sets $A \in \mathcal{B}$, $T_t I(A) = I(A)$ Q a.s. $\Rightarrow Q(A) = 0$ or 1.*

(b) *$\mathbb{P}_Q$ is ergodic: For each $f \in L^2(Q)$, $\hat{f} = E_Q[f]$ Q a.s.*

(c) *Q is extremal.*

Note that part (b) is the same as 'time shift ergodicity' where the shift invariant σ-field is trivial: Shift invariant sets Λ satisfy $\mathbb{P}_Q(\Lambda) = 0$ or 1.

Proof. "b⇒c" Let Q be an invariant measure whose path measure is ergodic. Write $Q = \epsilon Q_1 + (1-\epsilon)Q_2$ for $0 < \epsilon < 1$ and Q_1 and Q_2 invariant measures. Let now f be a bounded function. Then, as $t \to \infty$, $\frac{1}{t}\int_0^t (T_s f) ds$ converges to $E_Q[f]$ in both $L^2(Q)$ and $L^2(Q_1)$. Moreover, $\frac{1}{t}\int_0^t (T_s f) ds$ converges to $\hat{f}$ in $L^2(Q_1)$. Hence, $\hat{f} = E_Q[f]$ Q_1 a.s. and taking expectation, $E_{Q_1}[f] = E_Q[f]$. This gives $Q_1(B) = Q(B)$ for $B \in \mathcal{B}$ and therefore $Q_1 = Q$.

"a⇒b" Let Q be an invariant measure and suppose that P^Q is not ergodic. Then there exists an $f \in L^2(Q)$ such that $\hat{f}$ is not constant Q-a.s. Let c be such that $Q(A) = \epsilon$, $0 < \epsilon < 1$ where $A = \{\hat{f} > c\}$. Now, as $T_t\hat{f} = \hat{f}$ Q-a.s. and T_t is a positive contraction taking 1 into 1, we have that $T_t I(A) = I(A)$ Q-a.s.: First, as T_t is a positive operator, $|\hat{f}| = |T_t\hat{f}| \leq T_t|\hat{f}|$, so that, as T_t is an L^2 contraction, we have that Q-a.s. $T_t|\hat{f}| = |\hat{f}|$. Therefore, $\max\{0, \hat{f}\} = (\hat{f} + |\hat{f}|)/2$ is harmonic. Further, if $f, g \in L^2$ are harmonic, then $\max\{f, g\} = \max\{f - g, 0\} + g$ is harmonic. Correspondingly, $\min\{f, g\} = -\max\{-f, -g\}$ is harmonic. Of course, 1 is harmonic. All of this gives that $\min\{n\max(0, \hat{f} - c), 1\}$ for $n \geq 1$ is a sequence of bounded harmonic functions. The limit, as $n \to \infty$, is $I(A)$ which is therefore harmonic by dominated convergence. Hence, $I(A)$ is constant which is a contradiction.

"c⇒a" Let A be such that $T_t I(A) = I(A)$ Q-a.s. and $Q(A) = \epsilon$ for $0 < \epsilon < 1$. As the process begun on A stays in A with Q-probability 1, we have that $Q_1(B) = \epsilon^{-1}Q(B \cap A)$ and $Q_2(B) = (1-\epsilon)^{-1}Q(B \cap A^c)$ are distinct invariant probability measures such that $Q = \epsilon Q_1 + (1-\epsilon)Q_2$. Therefore Q is not extremal. $\square$

4. Extremality of ν_ρ

We now address the 'extremality' of ν_ρ in the convex set of invariant measures $\mathcal{I}$. To do this rigorously, we will need to extend the process so that the extended semigroup T_t^ρ and generator L^ρ act on $L^2(\nu_\rho)$ functions, not just Lipschitz functions $\mathcal{L}$. However, to present the main ideas we assume this extension has been done, and that adjoints can be taken. Later, we detail the extension to a $L^2(\nu_\rho)$ process.

Let

$$s(x, y) \;=\; \frac{1}{2}\big(p(x, y) + p(y, x)\big)$$

be the symmetrized transition probability.

Theorem 4.1. *When s is irreducible on $\mathbb{Z}^d$, the invariant measure ν_ρ is extremal.*

Proof. The idea is simple. We will need to understand the Dirichlet form of the process. In the next section, we show for $f \in \mathrm{Dom}(\rho)$, the domain of the extended generator L^ρ, that the associated Dirichlet form satisfies

$$D_\rho(f) := \langle f, -L^\rho f\rangle \;=\; \frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} s(x,y)E_{\nu_\rho}\big[g(\eta(x))\big(f(\eta^{x,y})-f(\eta)\big)^2\big].$$

Now, let f be a bounded harmonic function, that is $P_t^\rho f = f$ ν_ρ a.s. Hence, as the limit

$$\lim_{t\downarrow 0}\frac{1}{t}\big[P_t^\rho f - f\big] \;=\; 0$$

exists ν_ρ a.s., $f \in \mathrm{Dom}(\rho)$, and $L^\rho f = 0$ ν_ρ a.s.

Therefore, $D_\rho(f) = 0$, and so all summands where $s(x,y) > 0$ vanish. In particular, as $g(k) > 0$ exactly when $k \geq 1$,

$$f(\eta^{x,y}) = f(\eta)\;\;\text{for all } x,y \text{ such that } s(x,y)>0 \text{ when } \eta_x \geq 1.$$

Consequently, f is invariant to the motion of particles! So, by irreducibility of s, one has

$$f(\eta_{x,y}) = f(\eta)\;\;\text{for all } x,y \text{ a.s.}$$

where $\eta_{x,y}$ is the configuration which exchanges values $\eta(x)$ and $\eta(y)$.

Therefore, f is finite permutation invariant. Since ν_ρ is a product measure with iid marginals, by Hewitt-Savage $0-1$ law, f is constant ν_ρ a.s. Inputting into Proposition 3.4 shows extremality. □

5. Extension to a $L^2(\nu_\alpha)$ process

We first extend the semigroup defined on $\mathcal{L}$ to $L^2(\nu_\rho)$. We define the concept of a Markov semigroup on $X = L^2(\nu_\rho)$.

Definition 5.1. *A family of linear operators P_t on X is a Markov semigroup if (a) $P_0 = I$; (b) If $f \in X$, $P_t f$ is right-continuous in t on X; (c) P_t satisfies the semigroup property $P_{t+s} = P_t P_s$; (d) $P_t 1 = 1$ for $t \geq 0$; (e) $P_t f \geq 0$ if $f \in X$ is nonnegative.*

Lemma 5.2. *P_t on $\mathcal{L}$ extends by continuity to a Markov semi-group P_t^α on $L^2(Z_{\alpha(\cdot)})$.*

Proof. For $f \in \mathcal{L}$, as $P_t f(\eta) = E_\eta[f(\eta(t))]$, we have that P_t is a contraction: By Schwarz inequality,

$$[P_t f(\eta)]^2 \leq P_t f^2(\eta).$$

Then

$$\begin{aligned}\|P_t f\|^2_{L^2(\nu_\rho)} &= \int [P_t f]^2 dnu_\rho\\ &\leq \int P_t f^2 d\nu_\rho\\ &= \int f^2 d\nu_\rho = \|f\|_{L^2(\nu_\rho)}.\end{aligned}$$

As simple functions are contained in $\mathcal{L}$, $\mathcal{L}$ is dense in $L^2(\nu_\rho)$. Therefore P_t extends to a Markov semigroup P_t^ρ on $L^2(\nu_\rho)$. □

There is a one-to-one correspondence between Markov semigroups and associated generators by the Hille-Yosida theorem.

Proposition 5.3 (Hille-Yosida Theorem)**.** *For a Markov semigroup T_t on $L^2(Q)$, define*

$$\begin{aligned} \mathrm{Dom} &= \big\{f \in L^2(Q) : \lim_{t\downarrow 0} \frac{T_t f - f}{t} \text{ exists}\big\} \\ Lf &= \lim_{t\downarrow 0} \frac{T_t f - f}{t} \text{ for } f \in \mathrm{Dom} \end{aligned}$$

Then, if $f \in \mathrm{Dom}$, $T_t f \in \mathrm{Dom}$ and $(d/dt)T_t f = LT_t f = T_t L f$.

Let now L^ρ be the generator associated with semigroup P^ρ_t with domain $\mathrm{Dom}(\rho)$.

Lemma 5.4. *We have $\mathcal{L} \in \mathrm{Dom}(\rho)$, L^ρ on $\mathrm{Dom}(\rho)$ extends L from $\mathcal{L}$, and L^ρ is the closure of L from $\mathcal{L}$ whose graph is the closure in $L^2(\nu_\rho) \times L^2(\nu_\rho)$ of the graph of L.*

Proof. For $f \in \mathcal{L}$ and $\eta \in \Omega'$, we have

$$\begin{aligned} \frac{1}{t}\big[P_t f - f\big] &= \frac{1}{t}\int_0^t LP_s f ds \\ &\leq C(a_0)c(f)\|\eta\|t^{-1}\int_0^t e^{4a_0 s} ds \ \leq\ C'(a_0,t)\|eta\|. \end{aligned}$$

But, by Schwarz inequality,

$$E_{\nu_\rho}\big[\|\eta\|^2\big] \ \leq\ \big[\sum_x \beta(x)\big]\sum_x E_{\nu_\rho}[\eta(x)^2]\beta(x) \ \leq C(\rho)\big[\sum_x \beta(x)\big]^2 \ <\ \infty.$$

Hence, as P^ρ_t extends P_t on $\mathcal{L}$, we have by dominated convergence for $f \in \mathcal{L}$ that

$$\frac{1}{t}\big[P^\rho_t f - f\big] \ \to\ Lf$$

in $L^2(\nu_\rho)$ as $t \downarrow 0$. Hence, $\mathcal{L} \in \mathrm{Dom}(\rho)$, $Lf = L^\alpha f$, and L^α is an extension of L from $\mathcal{L}$.

Finally, as $P_t : \mathcal{L} \to \mathcal{L}$, and $\mathcal{L}$ is dense in $\mathrm{Dom}(\rho)$, one may conclude that $\mathcal{L}$ is a 'core' for L^ρ, that is the closure of L on $\mathcal{L}$ is equal to L^ρ on $\mathrm{Dom}(\rho)$. [See [2, Chapter 1]]. □

We now give the formula for the Dirichlet form. Recall the symmetrized transition probability s.

Lemma 5.5. *For $f \in \mathrm{Dom}(\rho)$, we have that*

$$D_\rho(f) \ =\ \frac{1}{2}\sum_{x,y} s(x,y)E_{\nu_\rho}\big(f(\eta^{x,y}) - f(\eta)\big)^2\big].$$

Proof. The formula is easily obtained for $f \in \mathcal{L}$, from our previous experience, and is left as an exercise.

We now extend the representation to $\mathrm{Dom}(\rho)$. Let $R(f)$ be the right-hand side of the display in the lemma. For $f \in \mathrm{Dom}(\rho)$, take $f_n \in \mathcal{L}$ so that $f_n \to f$ and $L^\rho f_n \to L^\rho f$ in $L^2(\nu_\rho)$. Then

$$\lim_{n\to\infty} D_\rho(f_n) = D_\rho(f), \quad \text{and}$$

$$\liminf_{n\to\infty} R(f_n) \geq R(f)$$

by Fatou's lemma. Therefore, $R(f) \leq D_\rho(f)$ and in particular, $R(f) < \infty$ for $f \in \mathrm{Dom}(\rho)$. However also,

$$0 \leq D_\rho(f - f_n) \leq \|f - f_n\|_{L^2} \cdot \|L^\rho f - L^\rho f_n\|_{L^2}$$

which vanishes as $n \to \infty$. Hence, $\lim_{n\to\infty} R(f - f_n) = 0$, and so $\lim_{n\to\infty} R(f_n) = R(f)$, to finish the proof. □

6. Notes

In these notes, we have followed [1] for the invariance of ν_ρ and [4] for the extension to L^2 and extremality of ν_ρ. A natural question is what are all the extremals of the process. Even in the translation-invariant situation, much is not known. In [1], it is shown in $d = 1, 2$ when g is an increasing function that $\{\nu_\rho : \alpha < \liminf g(k)\}$ are all the extremals! When p is not translation-invariant, also in $d = 1, 2$ all the extremals are found–in this case, the invariant measures are not necessarily translation-invariant [1]. When p is positive-recurrent, the extremals concentrate on configurations with a finite number of particles [5], [1].

In simple exclusion, also much is known, but there are interesting open problems. See [3].

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