PART 3: RIGOROUS PROOF OF HYDRODYNAMICS FOR SYMMETRIC SIMPLE EXCLUSION PROCESSES

After stating the main theorem, we provide an outline of the proof and provide details on the associated steps. Such an outline can be used to prove 'hydrodynamic limits' in other models.

1. Statement of the main theorem.

Recall the notation from the last section, with respect to symmetric simple exclusion processes with finite-range symmetric jump rate p, in particular the definition of the operator

$$\triangle_C \;=\; \sum_{1\leq i,j\leq d} C_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}$$

where covariances $C_{i,j} = \sum_{z\in\mathbb{T}^d_N} z_i z_j p(z)$.

Theorem 1.1. *Consider the symmetric simple exclusion process with finite-range jump probabilities. Then, starting from the local equilibrium measure μ^N, associated with profile $\rho_0(\cdot)$, the empirical measure $\pi^N_{N^2t}$ converges in probability to the measure $\rho(t,u)du$ where $\rho(t,u)$ satisfies the hydrodynamic equation*

$$\partial_t\rho(t,u) \;=\; \frac{1}{2}\triangle_C\rho(t,u), \quad \text{and} \quad \rho(0,u) = \rho_0(u). \tag{1.1}$$

The proof is given through the following rough steps, which are then explained in more detail in later sections.

Step 1. Consider the trajectory of empirical measures indexed in an interval of time, $\pi^N = \langle \pi^N_{N^2t} : t\in[0,T]\rangle$. Here, $T>0$ is time length, fixed throughout. Let Q^N be the law of the trajectory. The first step is to show that the laws $\{Q^N : N\geq 1\}$ are tight in the space of measure-valued right-continuous trajectories with left limits, $\mathcal{D}([0,T];\mathcal{M}_+)$. Moreover, this tightness will be shown in the uniform topology.

Step 2. Given tightness, we will show that any subsequential limit Q of Q^N must be supported on trajectories $\langle \pi_t : t\in[0,T]\rangle$ satisfying

$$\langle G(t,\cdot),\pi_t\rangle - \langle G(0,\cdot),\rho_0\rangle \;=\; \int_0^t \langle(\partial_s + \frac{1}{2}\triangle_C)G(s,\cdot),\pi_s\rangle ds. \tag{1.2}$$

This equation is similar to what was derived in the last section when G did not depend on time. However, to input into PDE uniqueness results, we will need to derive the equation with respect to this wider class of functions G. Moreover, after also showing in Step 3 below that trajectories under a limit point Q are absolutely continuous, we will be able to conclude the associated densities satisfy a weak formulation of (1.1).

Step 3. We will show that Q supports trajectories such that for each $t\in[0,T]$, π_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}^d$, and hence can be written as $\pi_t = m(t,u)du$. Therefore, from Step 2, $m(t,u)$ is a weak solution of the hydrodynamic equation (1.1). By uniqueness of weak solutions to the heat equation in the class of bounded solutions, we see that $m(t,u)$ is not random, but deterministic. In particular, all subsequential limits of Q^N converge to the point mass supported on the trajectory $\langle \rho(t,u)du : t \in [0,T]\rangle$. Since tightness was proved in the uniform topology, this trajectory is continuous in time. [This can also be inferred from regularity results in PDE.] Hence, it can be concluded, at a fixed time $t \in [0,T]$, that $\pi^N_{N^2t}$ converges weakly to the constant measure $\rho(t,u)du$, and hence in fact converges in probability.

The strategy now taken is to recall that we have already almost shown Step 2 in the last section. To this end, let $G : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}$ be a smooth function. By the same methods as in the last section, we obtain

$$\begin{aligned}
\langle G(t,\cdot), \pi^N_{N^2t}\rangle &= \langle G(0,\cdot), \pi^N_0\rangle + \int_0^t (\partial_s + N^2 L)\langle G(s,\cdot), \pi^N_s\rangle ds + M^G_{N^2t} \\
&= \langle G(0,\cdot), \pi^N_0\rangle + \int_0^t \langle (\partial_s + \frac{N^2N^{-2}}{2}\triangle_C) G(s,\cdot), \pi^N_s\rangle ds + o(1) \\
&= \langle G(0,\cdot), \pi^N_0\rangle + \int_0^t \langle (\partial_s + \frac{1}{2}\triangle_C) G(s,\cdot), \pi^N_s\rangle ds + o(1).
\end{aligned}$$

Therefore, if $\langle \pi_t : t \in [0,T]\rangle$ is a limit point of $\langle \pi^N_t : t \in [0,T]\rangle$, we obtain (1.2).

We now discuss more carefully some topological considerations and argue Step 1, the most difficult part. Afterwards, we concentrate on Step 3 in a subsequent section.

2. Topology and Compactness

Before making calculations with Step 1, we first recall some important definitions and results on weak convergence, and the spaces $C([0,T];\mathcal{M}_+)$ and Skorohod space $\mathcal{D}([0,T];\mathcal{M}_+)$ with a view toward characterizing when a sequence of probability measures Q on these spaces is tight. More details can be found in [7][Section 4.1] and [1].

First, we recall that a family of probability measures Q^N on a metric space is relatively compact if any subsequence of the family has a weakly convergent subsequence. We say that the family is tight if for each $\epsilon > 0$ there is a compact set K_ϵ such that all measures give at least weight $1-\epsilon$ to K_ϵ. Recall that E is a complete, separable metric space if all Cauchy sequences converge in E, and E contains a countable dense set of points.

Proposition 2.1 (Prokhorov's theorem)**.** *Let E be a complete, separable metric space. Then, a family Q^N of probability measures on E is relatively compact exactly when the family is tight.*

We will consider in the following a generic metric space E with metric δ. Often, E will be $\mathcal{M}_+$ the space of probability measures on $\mathbb{T}^d$ equipped with the metric

$$\delta(\mu,\nu) \;=\; \sum_{k\geq 1} \frac{1}{2^k} \frac{|\langle f_k,\mu\rangle - \langle f_k,\nu\rangle|}{1 + |\langle f_k,\mu\rangle - \langle f_k,\nu\rangle|}$$

where $\{f_k\}$ is a countable dense set of function in $C(\mathbb{T}^d)$, the space of continuous functions on the torus $\mathbb{T}^d$. At other times, E may be $\mathbb{R}$ with usual Euclidean metric.

Now, we consider the space $C([0,T];E)$ with the 'uniform' distance,

$$d(\pi,\chi) \;=\; \sup_{t\in[0,T]} \delta(\pi_t,\chi_t).$$

It is known that this space is a complete separable space, and therefore amenable to Prokhorov's theorem.

Since our basic building blocks in our study of hydrodynamics involve jump processes, the space of continuous trajectories is not sufficient for our purposes. We will often focus on the space $\mathcal{D}([0,T],E)$ of right-continuous trajectories with left limits. Unfortunately, the uniform distance will not make this space a complete, separable metric space. Define Λ to be the set of strictly increasing continuous functions λ of $[0,T]$ into itself, and

$$\|\lambda\| \;=\; \sup_{s\neq t}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right|.$$

Then, define the Skorohod distance between elements in $\mathcal{D}([0,T],\mathcal{M}_+)$ as

$$d(\pi,\chi) \;=\; \inf_{\lambda\in\Lambda}\max\left\{\|\lambda\|,\ \sup_{t\in[0,T]}\delta(\pi_t,\chi_{\lambda(t)})\right\}.$$

In some sense, the Skorohod distance compares two trajectories allowing small variation both in space and in time. To contrast, the uniform distance only compares variation in space. With the Skorohod distance, $\mathcal{D}([0,T],E)$ is a complete, separable metric space.

How to characterize compact sets in these spaces? Consider the following moduli of continuity:

$$\begin{aligned} w_\pi(\gamma) \;&=\; \sup_{\substack{|t-s|\leq\gamma\\ s,t\in[0,T]}} \delta(\pi_t,\pi_s)\\ w'_\pi(\gamma) \;&=\; \inf_{\{t_i\}_{i=0}^r}\ \max_{0\leq i\leq r-1}\ \sup_{t_i\leq s<t<t_{i+1}} \delta(\pi_s,\pi_t)\end{aligned}$$

where $\{t_i\}_{i=0}^r$ refers to a partition $0=t_0<\cdots<t_r=T$ such that $t_{i+1}-t_i>\gamma$ for $0\leq i\leq r-1$.

Exercise 2.2. Relate the two moduli by showing that

$$w'_\pi(\gamma) \;\leq\; w_\pi(2\gamma).$$

Then, $\pi\in C([0,T],E)$ exactly when $\lim_{\gamma\downarrow 0} w_\pi(\gamma)=0$, and π belongs to $\mathcal{D}([0,T],E)$ exactly when $\lim_{\gamma\downarrow 0} w'_\pi(\gamma)=0$.

Exercise 2.3. Show, when $E=\mathbb{R}$ and $\pi = 1\big([0,T/2)\big)$ that $\lim_{\gamma\downarrow 0} w'_\pi(\gamma)=0$ but $\liminf_{\gamma\downarrow 0} w_\pi(\gamma)>0$.

Compact sets in these spaces, since they are complete, are characterized as follows.

Proposition 2.4. *A set A belonging to $C([0,T],E)$ or $\mathcal{D}([0,T],E)$ is compact exactly when*

- $\{\pi_t:\pi\in A, t\in[0,T]\}$ *is tight/compact in E.*
- $\lim_{\gamma\downarrow 0}\sup_{\pi\in A}\bar{w}_\pi(\gamma)=0$ *where $\bar{w}_\pi=w_\pi$ on $C([0,T],E)$ and $\bar{w}_\pi=w'_\pi$ on $\mathcal{D}([0,T],E)$.*

We remark, when $A \subset C([0,T],\mathbb{R})$, the above characterization reduces to the Ascoli-Arzela condition with respect to equicontinuous families of trajectories.

Recall now Exercise 2.2.

Proposition 2.5. *A family Q^N of probability measures on $\mathcal{D}([0,T],E)$ is tight/relatively compact exactly when*

(1) *For each $t \in [0,T]$, the distributions of π_t under Q^N are tight/relatively compact.*
(2) *For every $\epsilon > 0$, $\lim_{\gamma\downarrow 0}\lim_{N\uparrow\infty} Q^N(\pi : w'_\pi(\gamma) > \epsilon) = 0$.*

Moreover, a sufficient condition for (2) is

(2') *In condition (1), replace w'_π with w_π.*

Exercise 2.6. Observe that any limit point Q of $\{Q^n\}$ satisfying $(2')$ is supported on continuous paths.

It is not so easy to work with $(2')$ directly. However, when $E = \mathcal{M}_+$, one can understand a family Q^N of probability measures on $\mathcal{D}([0,T],\mathcal{M}_+)$ by their actions on smooth functions in $C(\mathbb{T}^d)$.

Proposition 2.7. *Let $G \in C^2(\mathbb{T}^d)$. Then, a family $\{Q^N\}$ of probability measures on $\mathcal{D}([0,T],\mathcal{M}_+)$ is tight/relatively compact if the distributions of $\{\langle G, \pi^n_t\rangle : t \in [0,T]\}$ under Q^N for $N \geq 1$ are tight/compact in $\mathcal{D}([0,T],\mathbb{R})$.*

3. Step 1

We are now back to considering the tightness of $\pi^N = \langle \pi^N_{N^2t} : t \in [0,T]\rangle$ for $N \geq 1$ which are elements of $\mathcal{D}([0,T],\mathcal{M}_+)$. From Proposition 2.7, we need only show for a smooth function $G : \mathbb{T}^d \to \mathbb{R}$, tightness of the distributions of $\{\langle G, \pi^N_{N^2t}\rangle : t \in [0,T]\}$ for $N \geq 1$ which are elements of $\mathcal{D}([0,T],\mathbb{R})$.

From Proposition 2.5, we need to show conditions (1) and $(2')$. Condition (1) is the simplest, and follows straightforwardly as for each $t \in [0,T]$,

$$|\langle G, \pi^N_{N^2t}\rangle| \;\leq\; \|G\|_{L^\infty} \cdot \frac{1}{N^d} \sum_{x \in \mathbb{T}^d_N} \eta_{N^2t}(x) \;\leq\; \|G\|_{L^\infty}.$$

Then, $\langle G, \pi^N_{N^2t}\rangle$ is a tight sequence in $\mathbb{R}$ as the sequence is uniformly bounded in N with full probability.

Condition $(2')$ is a little more involved. Since

$$\langle G, \pi^N_{N^2t}\rangle \;=\; \langle G, \pi^N_0\rangle + \frac{1}{2}\int_0^t \frac{1}{N^d} \sum_{x,y \in \mathbb{T}^d_N} \eta_{N^2s}(x) p(y) \triangle^N_{x,y} G ds + M^G_t,$$

we need only show condition (2') for each term separately. The initial term $\langle G, \pi^N_0\rangle$ does not contribute in this respect.

We now bound the second term. Condition (2') follows as

$$\sup_{|t-s|\leq\gamma} \left| \int_0^t \frac{1}{N^d} \sum_{x,y \in \mathbb{T}^d_N} \eta_{N^2s}(x) p(y) \triangle^N_{x,y} G ds \right| \;\leq\; \gamma R \|\triangle G\|_{L^\infty}$$

where R is the range of the probability p.

For the third term, to treat condition (2'), we first recall Doob's inequality (cf. [3]): For a martingale $(M_t, \mathcal{F}_t)$ and given $t_0 > 0$, we have

$$\begin{aligned} P\Big(\sup_{t\in[a,T]} |M_t - M_a| > \lambda\Big) &\le \frac{1}{\lambda^2} E\Big[\sup_{t\in[a,T]} |M_t - M_a|^2\Big] \\ &\le \frac{1}{\lambda^2} E\Big[|M_T - M_a|^2\Big]. \end{aligned} \tag{3.1}$$

Then, after partitioning the interval $[0,T]$ into divisions of sublength γ, and noting when $|t-s| \le \gamma$ that either both t, s lie in the same subinterval or lie in adjacent intervals, we have

$$\begin{aligned} &\mathbb{P}_{\mu^N}\Big(\sup_{|t-s|\le\gamma} |M_t^G - M_s^G| > \lambda\Big) \\ &\le \mathbb{P}_{\mu^N}\Big(\sup_{\substack{l_k<t\le l_{k+1}\\ 1\le k\le \lfloor T/\gamma\rfloor+1}} |M_t^G - M_{l_k}^G| > \lambda/3\Big) \\ &\le \sum_{k=1}^{\lfloor T/\gamma\rfloor+1} \mathbb{P}_{\mu^N}\Big(\sup_{l_k<t\le l_{k+1}} |M_t^G - M_{l_k}^G| > \lambda/3\Big) \\ &\le \frac{9}{\lambda^2}\sum_{k=1}^{\lfloor T/\gamma\rfloor+1} \mathbb{E}_{\mu^N}\Big[\sup_{l_k<t\le l_{k+1}} |M_t^G - M_{l_k}^G|^2\Big]. \end{aligned}$$

We now use Doob's inequality and the quadratic variation bound for $M_t^G - M_{l_k}^G$,

$$\begin{aligned} \mathbb{E}_{\mu^N}\Big[(M_t^G - M_{l_k}^G)^2\Big] &\le \mathbb{E}_{\mu^N}\Big[N_t^G - N_{l_k}^G\Big] \\ &= \mathbb{E}_{\mu^N} \frac{N^2}{2N^{2d+2}} \int_s^t \sum_{x,y\in\mathbb{T}_N^d} p(y)(\nabla_{x,y}^N G)^2 du \\ &= O(N^{-d}|t-s|), \end{aligned}$$

to bound (3.1) on order $O(\gamma^{-1}\gamma N^{-d})$ which vanishes as $N \uparrow \infty$.

4. Step 3

To show π_t is absolutely continuous, we note for any continuous function $G : \mathbb{T}^d \to \mathbb{R}$ and any realization, that

$$\begin{aligned} \sup_{t\in[0,T]} |\langle G, \pi_{N^2t}^N\rangle| &\le \frac{1}{N^d}\sum_{x\in\mathbb{T}_N^d} |G(x/N)|\eta_{N^2t}(x) \\ &\le \|G\|_{L^1} \end{aligned}$$

since at most one particle is allowed per site. Hence, since the function

$$\langle \pi_t : t\in[0,T]\rangle \mapsto \sup_{t\in[0,T]} |\langle G, \pi_t\rangle|$$

is continuous with respect to the Skorohod topology, any limit point satisfies, with full probability (see Portmanteau theorem [1][Chapter 1]),

$$\sup_{t\in[0,T]} |\langle G, \pi_t\rangle| \le \|G\|_{L^1}.$$

Hence, any limit point Q of $\{Q^N\}$ is supported on trajectories with the property that π_t is absolutely continuous for all $t \in [0,T]$.

Exercise 4.1. Show more carefully the claims in the previous paragraph.

At this point, for each $t \in [0,T]$, π_t can be written as $\pi_t = m(t,u)du$ where m may be random! However, by Step 2, m is a weak solution to the hydrodynamic equation, which has a unique bounded solution. Therefore, m is deterministic, and $\pi_t = \rho(t,u)du$ where ρ is the hydrodynamic density. We remark when $\rho_0 \in C^2(\mathbb{T}^d)$ then $m(t,u)$ is actually a 'classical' solution given in terms of a Gaussian kernel convolution with ρ_0.

What we have shown is that the law of $\langle \pi^N_{N^2t} : t \in [0,T] \rangle$, where initial configurations are distributed according to μ^N, converges to the point mass at $\langle \rho(t,u)du : t \in [0,T] \rangle$. To conclude convergence at a fixed time $t \in [0,T]$, we note that in Step 1, tightness of Q^N was obtained in the uniform topology. Hence, the trajectory is supported by the limit Q is continuous for all $t \in [0,T]$. For $t \in [0,T]$, let h_t be the projection function,

$$\langle \pi_t : t \in [0,T] \rangle \ \mapsto \ \pi_t.$$

Now, h_t is not continuous on $\mathcal{D}([0,T];\mathcal{M}_+)$. However, since the limit Q is supported on a continuous trajectory, h_t is continuous on the support of Q. Now it is known that if $Q^N \Rightarrow Q$ and $h = h_t$ is a continuous function almost surely on the support of Q, then $Q^N \circ h_t^{-1} \Rightarrow Q \circ h_t^{-1}$ (see [1][Chapter 1]). In other words, the projection $\pi^N_{N^2t}$ converges in law to the point mass at $\rho(t,u)du$, and therefore also in probability.

5. Notes

There are other proofs of hydrodynamics for symmetric simple exclusion processes, for instance using correlation functions as in [2] or superexponential estimates as in [8]. We have followed mostly the treatment with some caveats in [7], which follows the strategy, easy in the symmetric simple exclusion context, expounded in [5], a classic in the field.

Besides [1], another good reference for 'weak convergence' in general spaces is [6].

It is worth noting that we have shown existence of weak solutions to the PDE (1.1) by our method.

We note that a subject of recent interest has been hydrodynamics of exclusion processes where the jump parameters are chosen from random environments (cf. [4]).

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