LECTURE 2: EMPIRICAL MEASURES, AND A FIRST LOOK AT HYDRODYNAMICS OF THE SIMPLE EXCLUSION PROCESSES

We introduce the notion of empirical measures, and after a preliminary discussion of certain martingales in the Markov chain context, we discuss the hydrodynamics of simple exclusion processes.

1. Empirical measures and view of 'hydrodynamics' as a LLN

We now give a similar, but perhaps simpler derivation of 'hydrodynamics' in systems of independent random walks in terms of the empirical distribution

$$\pi^N_{v(N)t} \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}\delta_{x/N}\eta_{v(N)t}(x),$$

which is a probability measure on $\mathbb{T}^d$. Before, we considered what happens nearby a macroscopic point $u\in\mathbb{T}^d$. However, it will be easier in what follows to consider the weaker notion of the asymptotic behavior of the empirical distribution. In this sense, what is meant by 'hydrodynamics' is a law of large numbers which characterizes 'first-order' behavior.

Let G be a smooth, bounded function, and write

$$\langle G,\pi^N_{v(N)t}\rangle \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\eta_{v(N)t}(x). \tag{1.1}$$

Recall that the distribution of $\eta_{v(N)t}$ starting from μ^N is a product of Poisson measures with intensity $\psi_{N,v(N)t}(\cdot)$. Then, in mean-value, we have

$$\begin{aligned} E_{\mu^N}\Big[\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\eta_{v(N)t}(x)\Big] \;&=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)E_{\mu^N}[\eta_{v(N)t}(x)]\\ &=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)E\big[\rho_0\big(N^{-1}(x-Z_{v(N)t})\big)\big]. \end{aligned}$$

Depending on whether $m\neq 0$ or $m=0$, the last quantity as before tends respectively to

$$\int\rho_0(u-mt)G(u)du \;\;\text{or}\;\; \int G(u)\int\bar{\rho}_0(w)G_t(u-w)dwdu. \tag{1.2}$$

However, the variance of (1.1), since under μ^N the occupation numbers $\eta_{v(N)t}$ are independent Poisson variables with intensity $\psi_{N,v(N)t}(x)$, equals

$$\begin{aligned} \mathrm{Var}_{\mu^N}\Big[\frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N}G(x/N)\eta_{v(N)t}(x)\Big] \;&=\; \frac{1}{N^{2d}}\sum_{x\in\mathbb{T}^d_N}G^2(x/N)\psi_{N,v(N)t}(x)\\ &=\; O(N^{-d}) \;\to\; 0. \end{aligned}$$

Hence, $\langle G,\pi^N_{v(N)t}\rangle$ converges to (1.2) in probability depending on the drift m.

In particular, we have shown, with respect to the initial distribution μ^N for the process η_0, the empirical measure $\pi^N_{v(N)t}$, which is a random element of $\mathcal{M}_+(\mathbb{T}^d)$, converges in probability to the measure $\rho(t,u)du$ corresponding to the macroscopic space-time mass evolution. Here, the topology on $\mathcal{M}_+(\mathbb{T}^d)$ used is as follows: Let $\{f_k : k \geq 1\}$ be a dense, countable family of continuous functions on $\mathbb{T}^d$. Then, define distance $\delta(\mu,\nu)$ on $\mathcal{M}_+(\mathbb{T}^d)$ by

$$\delta(\mu,\nu) \ = \ \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}{1 + |\langle \mu, f_k\rangle - \langle \nu, f_k\rangle|}.$$

Hence, capturing the limit behavior $\langle G, \pi^N_{v(N)t}\rangle$ is enough for each continuous function G.

2. Simple exclusion processes

We now consider particles on $\mathbb{T}^d_N$ with the simple interaction that no particle can jump onto another one. Accordingly, a configuration of occupation numbers η_t belongs to state space $\Omega = \{0,1\}^{\mathbb{T}^d_N}$ where $\eta_t(x) = 0$ or 1 depending on whether $x \in \mathbb{T}^d_N$ is empty or occupied at time $t \geq 0$. Informally, η_t updates in that each particle is a continuous time random walk carrying an exponential 1 clock. When a clock rings, the particle tries to displace with skeleton jump probability $p(\cdot)$. However, if the site chosen is already occupied, the jump is suppressed, and the clock resets.

More formally, infinitessimally η_t can change to $\eta_t^{x,x+y}$ when a particle jumps from x displaces by y where

$$\eta^{a,b}(z) \ = \ \begin{cases} \eta(b) & \text{when } z = a \\ \eta(a) & \text{when } z = b \\ \eta(z) & \text{when } z \neq x, y \end{cases}$$

with rate $\eta(x)(1-\eta(x+y))p(y)$. The factor '$\eta(x)(1-\eta(x+y))$' is 1 exactly when x is occupied and the destination $x+y$ is unoccupied.

The generator of the process η_t is given by

$$(Lf)(\eta) \ = \ \sum_{x\in\mathbb{T}^d_N} \sum_{y\in\mathbb{T}^d_N} \eta(x)(1-\eta(x+y))p(y)\big[f(\eta^{x,x+y}) - f(\eta)\big]$$

for bounded functions $f : \Omega \to \mathbb{R}$.

In the following, we will assume that p is finite-range, that is $p(z) = 0$ for $|z| > R$ for some R. To check calculations, it may be helpful to assume that p is nearest-neighbor, that is when the range $R = 1$.

When p is symmetric, the process is called the 'symmetric simple exclusion process'. When p is asymmetric, η_t is termed the 'asymmetric exclusion process'.

There is a simplification of the form of L when p is symmetric. Namely, since $\eta^{x,x+y} = \eta^{x+y,x}$,

$$\begin{aligned} (Lf)(\eta) \ &= \ \frac{1}{2}\sum_{x\in\mathbb{T}^d_N} \sum_{y\in\mathbb{T}^d_N} \big\{\eta(x)(1-\eta(x+y)) + \eta(x+y)(1-\eta(x))\big\}p(y)\big[f(\eta^{x,x+y}) - f(\eta)\big] \\ &= \ \frac{1}{2}\sum_{x\in\mathbb{T}^d_N} \sum_{y\in\mathbb{T}^d_N} p(y)\big[f(\eta^{x,x+y}) - f(\eta)\big]. \end{aligned}$$

The last line follows as the term in curly braces equals $|\eta(x)-\eta(x+y)|$ exactly when the difference in square brackets vanishes.

Let ν_α^N be the product measure on $\mathbb{T}_N^d$ with Bernoulli marginals with success probability α.

Proposition 2.1. *The measures $\{\nu_\alpha^N : 0\le\alpha\le 1\}$ are invariant measures for η_t.*

Proof. Under the change of measure $\zeta=\eta^{x,y}$, which exchanges values $\eta(x)$ and $\eta(y)$, the measure ν_α^N remains the same. Hence, for bounded functions f,g, the term

$$E_{\nu_\alpha^N}\big[g(\eta)\eta(x)(1-\eta(x+y))p(y)f(\eta^{x,x+y})\big] \;=\; E_{\nu_\alpha^N}\big[g(\eta^{x,x+y})\eta(x+y)(1-\eta(x))p(y)f(\eta)\big].$$

Hence, by collecting terms,

$$E_{\nu_\alpha^N}\big[gLf\big] \;=\; \sum_{x,y\in\mathbb{T}_N^d} E_{\nu_\alpha^N}\big[(L^*g)f\big]$$

where L^* is the exclusion generator corresponding to jump rate $q(z)=p(-z)$.

Now, since $L^*\mathbf{1}=0$, by inspection (here $\mathbf{1}$ is the constant function 1), we have that $E_{\nu_\alpha^N}[Lf]=0$ for all bounded f. This shows ν_α^N is invariant. $\square$

We remark this proof also shows that ν_α^N is reversible when p is symmetric. Also, when there are exactly K particles in the system, $\nu_\alpha^N(\cdot|\sum_{x\in\mathbb{T}_N^d}\eta(x)=K)$ is the unique 'canonical' invariant measure.

Let $\rho_0:\mathbb{T}^d\to\mathbb{R}_+$ be a continuous function. We will denote by $\nu^N_{\rho_0(\cdot)}$ as the product measure with Bernoulli marginal at site x with success probability $\rho_0(x/N)$.

Exercise 2.2. When p is symmetric, show that the space of linear combinations of occupation variables $\eta(x)$ for $x\in\mathbb{T}_d^N$ remains invariant under action by generator L. Hint: First compute the action on the variable $\eta(x)$.

3. Martingales and Markov chains

Recall that a martingale M_t corresponding to sigma-fields $\mathcal{F}_t$ is a random process which satisfies

$$E[M_t|\mathcal{F}_s] \;=\; M_s \;\text{ and }\; E|M_t|<\infty$$

for all $t\ge s\ge 0$.

Exercise 3.1. Let $N(t)$ be a Poisson process with rate λ. Then, $M_t=N(t)-\lambda t$ is a martingale with respect to the 'natural' sigma-fields $\mathcal{F}_t=\sigma\{N_u : u\le t\}$. Also, $Q_t=M_t^2-\lambda t$ is also a martingale with respect to $\mathcal{F}_t$.

Let X_t be a Markov process on a countable state space Ω. Let $F:\mathbb{R}_+\times\Omega\to\mathbb{R}$ be a twice continuously differentiable function whose first and second partial time derivatives are uniformly bounded. Define

$$\begin{aligned} M_t^F &= F(t,X_t)-F(0,X_0)-\int_0^t\Big(\frac{\partial}{\partial_s}+L\Big)F(s,X_s)ds \\ N_t^F &= (M_t^F)^2-\int_0^t(LF^2)(s,X_s)-2F(s,X_s)(LF)(s,X_s)ds. \end{aligned}$$

These two processes, which we show below are martingales, are ubiquitous in stochastic analysis of Markov systems. Often, the corrector term,

$$\langle M_t^F\rangle \;=\; \int_0^t (LF^2)(s,X_s) - 2F(s,X_s)(LF)(s,X_s)ds$$

is referred to as the 'quadratic variation' of the martingale M_t^F.

Proposition 3.2. *With respect to natural sigma-fields $\mathcal{F}_t = \sigma\{X_u : u \leq t\}$, both M_t^F and N_t^F are martingales.*

Proof. We will show that M_t^F is a martingale when F does not depend on time. Generalizations and verification of N_t^F as a martingale are left to the reader. We need only show that

$$E\big[F(X_t)|\mathcal{F}_s\big] - F(X_s) - \int_s^t E\big[(LF)(X_u)|\mathcal{F}_s\big]du \;=\; 0.$$

Now, $E[F(X_t)|\mathcal{F}_s] = P_{t-s}F(X_s)$ and $E[(LF)(X_u)|\mathcal{F}_s] = P_{u-s}(LF)(X_s)$ from the Markov property. From the forward equation, the derivative of the left-side of the above display in t equals

$$P_{t-s}(LF)(X_s) - P_{t-s}(LF)(X_s) \;=\; 0.$$

At time $t = s$, the left-side also vanishes. This concludes the proof. $\square$

4. Sketch of the hydrodynamics for simple exclusion processes

Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ be a continuous function, and let μ^N be a local equilibrium sequence with respect to ρ_0. For instance $\nu^N_{\rho_0(\cdot)}$ is such a sequence.

Our goal now is to analyze the asymptotic behavior of the empirical measure with respect simple exclusion process η_t,

$$\pi^N_{v(N)t} \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} \eta_t(x)\delta_{x/N}$$

in time scale $v(N)$ to be chosen later. Instead of computing the mean and variance, as with independent particles, we will use the martingale formulation. The variance with respect to the exclusion interaction is not so easy to handle as before.

Let $G : \mathbb{T}^d \to \mathbb{R}$ be a smooth function. Treating

$$\langle G, \pi^N_{v(N)t}\rangle \;=\; \frac{1}{N^d}\sum_{x\in\mathbb{T}^d_N} G(x/N)\eta_{v(N)t}(x)$$

as a function on the state space Ω, we have that

$$\langle G, \pi^N_{v(N)t}\rangle \;=\; \langle G, \pi^N_0\rangle + \int_0^{v(N)t} L\langle G, \pi^N_s\rangle ds + M_{v(N)t}$$

where M_t is a martingale (cf. Proposition 3.2) with quadratic variation

$$\langle M_t\rangle \;=\; \int_0^{v(N)t} L\big(\langle G,\pi^N_s\rangle\big)^2 - 2\langle G,\pi^N_s\rangle\big(L\langle G,\pi^N_s\rangle\big)ds.$$

Now, the martingale is negligible in the $N \uparrow \infty$ limit: We compute that

$$\begin{aligned} E_{\mu^N}\left[(M^G_{v(N)t})^2\right] &= \int_0^{v(N)t} \frac{1}{N^{2(d+1)}} \sum_{x,y\in\mathbb{T}^d_N} \left(\nabla^N_{x,x+y}G\right)^2 p(y)\eta_s(x)(1-\eta_s(x+y)) \\ &\leq \frac{v(N)t}{N^{2(d+1)}} \sum_{x,y\in\mathbb{T}^d_N} \left(\nabla^N_{x,x+y}G\right)^2 p(y) \qquad (4.1) \\ &= O(v(N)N^{-d-2}). \end{aligned}$$

In the last line, we have used that the occupation variables are bounded by 1 and that the jump probabilities are finite-range.

A calculation shows that

$$\begin{aligned} L\langle G, \pi^N_s\rangle &= \frac{1}{N^{d+1}} \sum_{x,y\in\mathbb{T}^d_N} \eta_s(x)(1-\eta_s(x+y))p(y)\left[\nabla^N_{x+y,x}G\cdot(\eta_s(x)-\eta_s(x+y))\right] \\ &= \frac{1}{N^{d+1}} \sum_{x,y\in\mathbb{T}^d_N} \eta_s(x)(1-\eta_s(x+y))p(y)\left[\nabla^N_{x+y,x}G\right] \qquad (4.2) \end{aligned}$$

where $\nabla^N_{u,v}G = N[G(u/N)-G(v/N)] \sim (u-v)\cdot\nabla G(v/N)$.

Exercise 4.1. Verify the form of the quadratic variation given in (4.1).

4.1. **Symmetric case.** When p is symmetric, a further summation by parts is possible and we obtain in this case that

$$\begin{aligned} L\langle G, \pi^N_s\rangle &= \frac{1}{2N^{d+1}} \sum_{x,y\in\mathbb{T}^d_N} \big(\eta_s(x)-\eta_s(x+y)\big)p(y)\nabla^N_{x+y,x}G \\ &= \frac{1}{2N^{d+2}} \sum_{x,y\in\mathbb{T}^d_N} p(y)\eta_s(x)\triangle^N_{x,y}G \end{aligned}$$

where $\triangle^N_{x,y}G = N^2[G(x+y/N) - 2G(x/N) + G(x-y/N)]$.

Now, $\triangle^N_{x,y}G = \triangle_C G(x/N) + o(1)$ where

$$\triangle_C = \sum_{1\leq i,j\leq d} C_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}, \quad \text{and covariances } C_{i,j} = \sum_{z\in\mathbb{T}^d_N} z_i z_j p(z).$$

Hence, if $v(N) = N^2$, we have that

$$\langle G, \pi^N_{v(N)t}\rangle = \langle G, \pi^N_0\rangle + N^{-2}\int_0^{v(N)t} \langle \triangle_C G, \pi^N_s\rangle ds + M^G_t$$

Putting these estimates together, for symmetric p, we have 'closed' the equation:

$$\langle G, \pi^N_{v(N)t}\rangle = \langle G, \pi^N_0\rangle + \int_0^t \langle \triangle_C G, \pi^N_{v(N)s}\rangle ds + o(1).$$

This suggests in the $N \uparrow \infty$ limit that the empirical measure π^N_t converges in a sense to a solution of the Heat equation $\partial_t\rho = \triangle_C\rho$.

The goal of the next lecture is to make precise this statement for symmetric simple exclusion.

4.2. **Drift case.** When p is asymmetric, say $m = \sum_z zp(z) \neq 0$, we choose $v(N) = N$. But, one cannot 'close' the equation. One has to deal with the term

$$\frac{1}{N^{d+1}} \sum_{x,y \in \mathbb{T}_N^d} \eta_s(x)(1-\eta_s(x+y))p(y)\big[\nabla^N_{x+y,x} G\big]$$

composed of 'two-point' functions $\eta(x)\eta(x+y)$. In the limit, such a term due to 'local averaging' should be replaced by a quadratic function of the empirical density. Formally, one would obtain the Burger's equation

$$\partial_t + m \cdot \nabla \rho(1-\rho) \;=\; 0.$$

4.3. **Asymmetric, mean-zero case.** In the final case when p is asymmetric, but mean-zero, that is $p(z) \neq p(-z)$ for some z, but $\sum_x xp(x) = 0$, things are more complicated. Although, we should speed up time by $v(N) = N^2$, a second summation-by-parts as in the symmetric situation cannot be done. Namely, multiplying (4.2) by N^2, we have

$$\frac{N^2}{2N^{d+1}} \sum_{x,y \in \mathbb{T}_N^d} \nabla G(x/N) \cdot \big\{\eta_s(x)(1-\eta_s(x+y))yp(y) - \eta_s(x+y)(1-\eta_s(x))yp(-y)\big\}$$

$$\frac{N^2}{2N^{d+1}} \sum_{x,y \in \mathbb{T}_N^d} \nabla G(x/N) \cdot \big\{\eta_s(x-y)yp(y) - \eta(x)\big(\eta(x+y)+\eta(x-y)\big)yp(y)\big\}.$$

The second term in braces would vanish if p were symmetric, closing the equation. However, another factor of N has to be squeezed from it in some way. In some sense, it can be shown that the second term can be written as the difference of a function and its translate, namely a 'gradient'. The hydrodynamic equation, after local averaging of this function, could be written down as a certain nonlinear Heat equation. Even to write down the equation is beyond the scope of the course, and we refer to [10] and [3].

5. Notes

The material on Martingales can be found in [1] for instance, and other places. Similar treatments of the hydrodynamics of simple exclusion can be found in [3] and [10].

The simple exclusion process, introduced in [9] (see [2] for a retrospective), has many properties which make it amenable to calculation. It has proved to be a versatile model, which can be defined on very general graphs, in applications and theoretical studies as a web search reveals. See [5], [6], [7], [8] for detailed studies.

As one can see in the sketch of the hydrodynamics for symmetric simple exclusion processes, one can weaken requirements on the initial measure. In fact, what is needed is a guarantee of a law of large numbers at time 0. The concept of 'very weak local equilibrium' given below is sufficient and somewhat general.

Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ be a function. We say that a sequence of probability measures μ^N on $\mathbb{T}_N^d$ is a 'very weak local equilibrium' according to profile ρ_0 if

$$\lim_{N\uparrow\infty} \Big[\Big|\frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G(x/N)\eta(x) - \int_{\mathbb{T}^d} G(u)\rho_0(u)du\Big|\Big] \;=\; 0.$$

for all bounded, continuous $G : \mathbb{T}^d \to \mathbb{R}$.

We remark that μ^N may be degenerate, that is supported on a single configuration, and that the sequence $\{\mu^N\}$ may consist of deterministic configurations which satisfy the law of large numbers in the definition above.

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