PART 11: BOLTZMANN-GIBBS PRINCIPLE FOR SYMMETRIC ZERO-RANGE PROCESSES

We discuss the equilibrium fluctuation field of symmetric zero-range processes. Unlike for symmetric exclusion models, a "replacement" at the fluctuation level must be made in order to identify the limit field as a generalized OU process. The proof we give of this replacement, called the "Boltzmann-Gibbs" principle, uses a "spectral gap" estimate which is of its own interest.

1. Introduction

Recall that the rate function $g : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$ and the jump probability $p(\cdot)$ define the zero-range process. We will, as before, assume that p is symmetric and finite-range. Also, we will impose that g satisfies

(Lip) $|g(k+1) - g(k)| \leq a_0$ for all $k \geq 0$

(M) There exists k_0 and $\epsilon_0 > 0$ such that $g(k + k + 0) - g(k) \geq \epsilon_0$ for all $k \geq 0$.

The first assumption is something we have already seen in the construction of the process on $\mathbb{Z}^d$. The second assures a uniform "spectral gap" which will be discussed later.

We will also fix an invariant measure ν_ρ which is again a product of "Poisson" like marginals, and the process will be assumed to begin under this extremal equilibrium. The reader may assume the process is an $L^2(\nu_\rho)$ process on $\mathbb{Z}^d$, although strictly speaking we assumed a bounded g before. There is no difficulty in assuming if preferred that the process is on the torus $\mathbb{T}_N^d$ where there are no construction issues.

Recall that $W_t^N(G)$, for fixed G smooth with compact support, stands for the fluctuation field,

$$W_t^N(G) \;=\; \frac{1}{N^{d/2}} \sum_x G(x/N)\big(\eta_{N^2 t}(x) - \rho\big).$$

To derive the limit field, write as before

$$W_t^N(G) \;=\; W_0^N(G) + N^2 \int_0^t LW_s^N(G)ds + \mathcal{M}_t^N(G)$$

where, after the usual summation-by-parts,

$$LW_t^N(G) \;=\; \frac{1}{N^{d/2}} \sum \triangle_N G(x/N)\big(g\big(\eta_{N^2 s}(x)\big) - \varphi(\rho)\big),$$

where $\varphi(a) = E_{\nu_a}[g(\eta(0))]$, and $\mathcal{M}_t^N(G)$ is a martingale such that

$$\mathcal{N}_t^N(G) \;=\; (\mathcal{M}_t^N(G))^2 - N^2 \int_0^t L(Y_s^N(G))^2 - 2Y_s^N(G)LY_s^N(G)ds$$

is a martingale. The last integral in the display can be evaluated as

$$N^2 \int_0^t L(Y_s^N(G))^2 - 2Y_s^N(G)LY_s^N(G)ds$$

$$= \frac{1}{2N^d} \int_0^t \sum [\nabla_N G(x/N)]^2 g\big(\eta_{N^2 s}(x)\big)ds.$$

Following the method described for exclusion processes, we have two steps:

Step 1: Show tightness of $\{W_t^N : t \in [0, T]\}$ in an appropriate space, and continuity of limit trajectories under limit points.

Step 2: Identify the limit points in terms of a unique "infinite dimensional Ornstein-Uhlenbeck" process

As before, the space of trajectories is $D([0, T], \mathcal{H}_{-k})$, for large enough k. [Recall the notion of Hermite spaces $\mathcal{H}_k$ and $\mathcal{H}_{-k}$.] Step 1, tightness, is accomplished as before, and it is left to the reader to verify the proof in the zero-range setting.

The more interesting part, for zero-range processes, is Step 2 where instead of the occupation variable $\eta(x)$ we have a function of it, namely $g(\eta(x))$ in both martingales above. If the normalization were N^{-d} instead of $N^{-d/2}$, replacing the nonlinear function with a linear combination of linear functions is the "standard" hydrodynamic replacement. Here, we have to work a little harder–however, we start in equilibrium ν_ρ which helps.

The following "Boltzmann-Gibbs" estimate allows the replacement in $\mathcal{M}_t^N(G)$.

Theorem 1.1. *For smooth, compactly supported G, we have*

$$\lim_{N\uparrow\infty} E_{\nu_\rho}\Big[\Big|\int_0^t \frac{1}{N^{d/2}} \sum \triangle_N G(x/N)\Big(g\big(\eta_{N^2 s}(x)\big)-\varphi(\rho)-\varphi'(\rho)\big(\eta_{N^2 s}(x)-\rho\big)\Big]ds\Big|^2\Big] = 0.$$

The replacement, however, in the square martingale $\mathcal{N}_t^N(G)$ will be a consequence of the L^2 ergodic theorem since ν_ρ is extremal. Namely, we know

$$\lim_{N\uparrow\infty} E_{\nu_\rho}\Big[\Big|\int_0^t \sum [\nabla_N G(x/N)]^2 \Big[g\big(\eta_{N^2 s}(x)\big) - \varphi(\rho)\Big]ds\Big|^2\Big] = 0. \qquad (1.1)$$

Hence, we arrive at the following equation: W_t^N converges to W_t where

$$dW_t = \frac{\varphi'(\rho)}{2}\triangle W_t dt + \sqrt{\varphi(\rho)}\nabla d\mathcal{B}_t. \qquad (1.2)$$

The characterization of W_t in terms of the generalized OU martingale problem of Holley and Stroock is as before with simple exclusion. Here, the operators $U = (\varphi'(\rho)/2)\triangle$ and $B = \varphi(\rho)\nabla$ only change with some prefactor constants.

Exercise 1.2. Show that the initial field W_0 is again a generalized Brownian motion with covariance

$$E_{\nu_\rho}[W_0(G)W_0(H)] = \sigma^2\langle G, H\rangle$$

where $\sigma^2(\rho) = E_{\nu_\rho}[(\eta(0)-\rho)^2]$. Also, compute the covariance of $W_t(G)$ and $W_s(H)$.

One way to look at (1.2) is to relate it in terms of the hydrodynamic equation:

$$\frac{\partial}{\partial_t}\rho = \frac{1}{2}\triangle\varphi(\rho)$$

with initial condition $\rho(0, x) = \rho_0(x)$. The drift of the limit field can be seen as a "linearization" of the hydrodynamic equation about the equilibrium density ρ

where $\varphi(\rho(t,x)) \sim \varphi(\rho) + \varphi'(\rho)\rho(t,x)$. See [10] for more physical intuition behind this intepretation.

2. Proof of Theorem 1.1

The following lemma is of independent interest.

Lemma 2.1. *For all local $L^2(\nu_\rho)$ functions, we have*

$$E_{\nu_\rho}\Big[\Big(\int_0^t f(\eta_s)ds\Big)^2\Big] \ \leq \ 12t\|f\|_{-1}^2.$$

Proof. Write the resolvent equation, for $\lambda > 0$,

$$\lambda u_\lambda - L u_\lambda \ = \ f.$$

Multiplying by u_λ and integrating, we have

$$\lambda\|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 \ = \ \langle f, u_\lambda\rangle_{\nu_\rho}.$$

Now, consider the martingale

$$M(t) \ = \ u_\lambda(\eta_t) - u_\lambda(\eta_0) - \int_0^t L u_\lambda(\eta_s)ds$$

and write

$$\int_0^t f(\eta_s)ds \ = \ M(t) - \lambda\int_0^t u_\lambda(\eta_s)ds + u_\lambda(\eta_0) - u_\lambda(\eta_t).$$

Since $E_{\nu_\rho}[M^2(t)] = t\|u_\lambda\|_1^2$, from squaring the left hand side of the above display, by stationarity, we have

$$E_{\nu_\rho}\Big[\Big(\int_0^t f(\eta_s)ds\Big)^2\Big] \ = \ 4\big[t\|u_\lambda\|_1^2 + \lambda^2 t^2\|u_\lambda\|_0^2 + 2\|u_\lambda\|_0^2\big].$$

Choosing $\lambda = t^{-1}$, we see that the left hand side is bounded by

$$12t\langle f, u_\lambda\rangle_{\nu_\rho} \ = \ 12t\int_0^\infty e^{-\lambda s}\langle P_s f, f\rangle_{\nu_\rho}ds \ \leq \ 12t\|f\|_{-1}^2.$$

$\square$

Now, to show that the variance of

$$\int_0^t \frac{1}{N^{d/2}}\sum G(x/N)\Big(g\big(\eta_{N^2 s}(x)\big) - \varphi(\rho) - \varphi'(\rho)\big(\eta_{N^2 s}(x) - \rho\big)\Big]ds$$

vanishes in the $N \uparrow \infty$ limit, we write it as the sum of two terms:

$$\int_0^t \sum_x N^{-d/2}G(x/N)\Big[g(\eta_{N^2 s}(x)) - E[g(\eta_{N^s}(x))|\sum_{y\in B_{\ell,x}}\eta_{N^2 s}(y)]\Big]ds$$

$$+ \int_0^t \sum_x N^{-d/2}G(x/N)\Big[E[g(\eta_{N^s}(x))|\sum_{y\in B_{\ell,x}}\eta_{N^2 s}(y)]\Big]$$

$$-\varphi(\rho) - \varphi'(\rho))\eta_{N^2 s}(x) - \rho\Big]ds \ = \ A_1 + A_2.$$

Here, $B_{\ell,x}$ is a block of width $\ell \geq 2$ around each x where $\ell << N$ is another scaling parameter. The strategy will be to bound the H_{-1} norm of A_1, and to use Schwarz inequality and Taylor expansions with A_2, a sort of "equivalence of ensembles" estimate.

2.1. **Bound on A_1.** To bound A_1, let ϕ be a local function and write

$$\langle N^{-d/2}G(x/N)\tau_x V(\eta), \phi\rangle \;=\; \sum_{k\geq 0} P(k,\ell)\langle N^{-d/2}G(x/N)\tau_x V(\eta), \phi\rangle_{\nu_{k,\ell}}$$

where τ_x is the shift operator, $V(\eta) = g(\eta(0) - E[g(\eta(0))]|\sum_{y\in B_{\ell,0}} \eta(y)]$, and we have conditioned on there being k particles in the block, $P(k,\ell) = P_{\nu_\rho}[\sum_{y\in B_{\ell,x}} \eta(y) = k]$, and $\nu_{k,\ell}$ is the canonical measure, $E_{\nu_{k,\ell}}[f] = E_{\nu_\rho}[f|\sum_{y\in B_{\ell,x}} \eta(y) = k]$.

The reason for this localization is that we can now solve a Poisson equation. Since $\nu_{k,\ell}$ is the unique invariant measure for zero-range dynamics localized to $B_{\ell,x}$ and V is mean-zero with respect to the canonical measure, a decomposition of $\mathbb{R}^{|B_{\ell,x}|(k+1)}$ in terms of the Null space and Range of $N^2 L_{\ell,k}$ is possible, and we can write

$$\tau_x V \;=\; -N^2 L_{\ell,k} u$$

for some function u.

Then,

$$\begin{aligned}
\langle N^{-d/2}G(x/N)\tau_x V(\eta), \phi\rangle_{\nu_{k,\ell}} \;&=\; \langle N^{-d/2}G(x/N)N^2 L_{\ell,k} u, \phi\rangle_{\nu_{k,\ell}} \\
&=\; \langle N^{-d/2}G(x/N)(-N^{-1}L_{\ell,k}^{-1/2})\tau_x V, -N L_{\ell,k}^{1/2}\phi\rangle_{\nu_{k,\ell}}.
\end{aligned}$$

At this point, we state a spectral gap inequality which is discussed later. Namely, for mean-zero functions, we have that

$$\|L_{\ell,k}^{-1/2}V\|_{L^2}^2 \;\leq\; W^{1/2}(\ell,k)\|V\|_{L^2(\nu_{\ell,k})}^2$$

where $W(\ell,k) \leq C\ell^2$ independent of k when (Lip) and (M) hold for the rate g [6].

Hence, by a form of Schwarz inequality, we have

$$\langle N^{-d/2}G(x/N)V(\eta), \phi\rangle_{\nu_{k,\ell}} \;\leq\; \frac{\epsilon}{2}N^{-d}G^2(x/N)N^{-2}C^2\ell^2\|V\|_{L^2}^2 + \frac{\epsilon^{-1}N^2}{2}D_{k,\ell}(\phi).$$

Here, the Dirichlet form can be evaluated:

$$\begin{aligned}
D_{k,\ell}(\phi) = \langle \phi, -L_{\ell,k}\phi\rangle_{\nu_{\ell,k}} \\
= \sum_{y,z\in B_{\ell,x}} E_{\nu_{\ell,k}}\big[g(\eta(y))p(z-y)\big(\phi(\eta^{y,z}) - \phi(\eta)\big)^2\big].
\end{aligned}$$

Note that

$$\sum_{k\geq 0} P(k,\ell)D_{k,\ell}(\phi) \;=\; \sum_{y,z\in B_{\ell,x}} E_{\nu_\rho}\big[g(\eta(y))p(z-y)\big(\phi(\eta^{y,z}) - \phi(\eta)\big)^2\big] \;:=\; D_{\ell,x}(\phi).$$

Also, by estimating the overcount, we have

$$\sum_x D_{\ell,x}(\phi) \;\leq\; C\ell^d D(\phi)$$

in terms of the full Dirichlet form.

Similarly,

$$\sum_{k\geq 0} P(k,\ell)\|V\|_{L^2(\nu_{k,\ell})}^2 \;=\; \|V\|_{L^2(\nu_\rho)}^2.$$

Then, A_1 is bounded as

$$E_{\nu_\rho}[A_1^2] \quad \leq \quad \frac{\epsilon C^2 \ell^2 \|V\|_{L^2}^2}{N^2 N^d} \sum_x G^2(x/N) + \frac{C\epsilon^{-1} N^2 \ell^d}{2} D(\phi)$$

$$\leq \quad C\|V\|_{L^2(\nu_\rho)} \|G\|_{L^2(\mathbb{R}^d)} \frac{\ell^{d/2+1}}{N} \quad \to 0$$

as $N \uparrow \infty$ for each ℓ fixed.

2.2. Bound on A_2.

Since G is smooth, we may replace $\eta(x)$ with

$$\eta^\ell(x) \quad = \quad \frac{1}{(2\ell+1)^d} \sum_{y \in B_{\ell,x}} \eta(y) :$$

$$\lim_{N \uparrow \infty} E_{\nu_\rho} \Big| \int_0^t \frac{1}{N^{d/2}} \sum_x G(x/N) \Big[\eta_{N^2 s}(x) - \eta^\ell_{N^2 s}(x) \Big] ds \Big|^2$$

$$= \quad \lim_{N \uparrow \infty} E_{\nu_\rho} \Big| \int_0^t \frac{1}{N^{d/2}} \sum_x \frac{1}{\ell^d} \sum_{y \in B_{\ell,x}} \Big[G(x/N) - G(y - x/N) \Big] (\eta_{N^2 s}(x) - \rho) ds \Big|^2$$

$$\leq \quad C(\rho) \ell^2 N^{-2} \quad \to \quad 0.$$

Now, since we are in equilibrium ν_ρ, we may simply bound $\|A_2\|_{L^2}^2$ by Schwarz inequality as follows:

$$E_{\nu_\rho} \Big| \int_0^t \sum_x N^{-d/2} G(x/N) \Big[E[g(\eta_{N^s}(x)) | \sum_{y \in B_{\ell,x}} \eta_{N^2 s}(y)] - \varphi(\rho) - \varphi'(\rho) \big(\eta^\ell_{N^2 s}(x) - \rho \big) \Big] ds \Big|^2$$

$$\leq \quad \frac{C t^2 \ell^d}{N^d} \Big[\sum_x G^2(x/N) \Big] E_{\nu_\rho} \Big[E[g(\eta(0)) | \sum_{y \in B_{\ell,0}} \eta(y)] - \varphi(\rho) - \varphi'(\rho) \big(\eta^\ell(0) - \rho \big) \Big]^2. \quad (2.1)$$

The factor ℓ^d arises since the summands are not independent when their indices are within $2\ell + 1$ distance of each other.

We now estimate the expectation. We first truncate the number of particles in $B_{\ell,0}$. Note that

$$|E[g(\eta(0)) 1(\eta^\ell(0) \geq K) | \sum_{y \in B_{\ell,0}} \eta(y)]| \quad \leq \quad |\eta^\ell(0)|^2 K^{-1}.$$

Then, by Schwarz inequality,

$$E_{\nu_\rho} \Big[E[g(\eta(0)) | \sum_{y \in B_{\ell,0}} \eta(y)] - \varphi(\rho) - \varphi'(\rho) \big(\eta^\ell(0) - \rho \big) \Big] 1(\eta^\ell \geq K) \Big]^2$$

$$\leq \quad C(\rho, a_0) \ell^{-d} d K^{-2}$$

which is enough given (2.1).

We now use a Taylor expansion. Let $M = \sum_{y \in B_{\ell,0}} \eta(y)$, and write

$$E[g(\eta(0)) | \sum_{y \in B_{\ell,0}} \eta(y)] \quad = \quad \varphi(M/(2\ell+1)^d) \frac{P_{\nu_\rho}[(2\ell+1)^d \eta^\ell(0) = M - 1]}{P_{\nu_\rho}[(2\ell+1)^d \eta^\ell(0) = M]}.$$

By a local limit theorem, uniformly over M, we have that

$$\Big| (2\ell+1)^{d/2} P_{\nu_\rho}[(2\ell+1)^d \eta^\ell(0) = L] - (2\pi\sigma^2)^{-d/2} e^{-(L/(2\ell+1)^{d/2})^2/2\sigma^2} \Big| \quad \leq \quad C\ell^{-d/2}$$

for all large ℓ.

Hence, since $\sup_{0 \le a \le K} \varphi''(\rho) < \infty$, on the set $\eta^\ell \le K$, we have

$$
\begin{aligned}
E[g(\eta(0))| \sum_{y \in B_{\ell,0}} \eta(y)] \quad &\sim \quad \varphi(L/(2\ell+1)^d) \\
&= \quad \varphi(\rho) + \varphi'(\rho)[\eta^\ell(0) - \rho) + O(|\eta^\ell(0) - \rho|^3).
\end{aligned}
$$

Therefore,

$$
E_{\nu_\rho}\Big[E[g(\eta(0))| \sum_{y \in B_{\ell,0}} \eta(y)] - \varphi(\rho) - \varphi'(\rho)(\eta^\ell(0) - \rho)\Big]1(\eta^\ell \le K)\Big]^2
$$

$$
\le \quad CE_{\nu_\rho}|\eta^\ell(0) - \rho|^3 \quad = \quad O(\ell^{-3d/2}).
$$

Noting, (2.1), we see that $\lim_{\ell \uparrow \infty} \lim_{N \uparrow \infty} E_{\nu_\rho}[A_2^2] = 0.$ □

3. Spectral gap

One can define the spectral gap for a reversible process in terms of a Poincare inequality:

$$
\mathrm{Var}_{\nu_{k,\ell}}[f] \quad \le \quad W(k,\ell)D_{k,\ell}(f).
$$

Then, the gap would be the reciprocal of the smallest $W(k,\ell)$ for which the inequality is true for all f.

Equivalently, the gap is the difference between the two largest eigenvalues of $-L_{k,\ell}$. Since 0 is the largest eigenvalue, the gap would be the negative of the second largest eigenvalue, $-\lambda_2$. Since $L_{k,\ell}$ is a matrix, one can in principle compute it exactly, but this is usually hard to do when the state space gets large.

The spectral gap has connections with the mixing time of the process. Namely, one can show that

$$
\mathrm{Var}_{\nu_{k,\ell}}(P_t f) \quad \le \quad Ce^{\lambda_2 t}.
$$

The larger $-\lambda_2$, the faster the approach of $P_t f$ to its mean $E_{\nu_{k,\ell}}[f]$.

In zero-range processes, since the jump times are controlled by the rate g, one expects the mixing time to depend on g. This is in fact the case. When $g(k) = k$, the independent case, the mixing is rapid, and corresponds to the mixing time of a single random walk on $B_{\ell,0}$ which is $O(\ell^{-2})$. For instance, consider $d = 1$, then a nearest-neighbor walk, in moving from one end of the interval to the other, has to take ℓ steps which usually takes ℓ^2 units of time.

If g is not too different from the independent case, one expects similar behavior, and this is the result quoted and used above, valid under our assumptions (Lip) and (M) [6].

If g is sublinear, that is $g(k) = k^\gamma$ for $0 < \gamma < 1$, then it has been shown the spectral gap depends on the number of particles k, namely the gap is $O((1 + \rho)^{-\gamma}\ell^{-2})$ where $\rho = k/(2\ell + 1)^d$ [9]. For instance, if $\rho \uparrow \infty$, movement becomes interminable, and the gap vanishes!

If $g(k) = 1(k \ge 1)$, then it can be seen that the gap is $O((1 + \rho)^{-2}\ell^{-2})$ [8]. In $d = 1$, this is the case where there is a connection between the zero-range process and simple exclusion: The number of spaces between particles in simple exclusion correspond to the number of particles in the zero-range process.

It is an open problem to characterize the gap when g is bounded in general. One presumably expects the behavior as in the last case.

4. NOTES

The Boltzmann-Gibbs principle was first given by Brox and Rost [3] which did not use a spectral gap assumption. See also [7] for a presentation along this line. We have concentrated on symmetric systems, although the equilibrium fluctuations for asymmetric systems in $d \geq 3$ are mostly well understood [4].

There are other scalings and limit fields which can be considered. Among these are the "Navier-Stokes" corrections [5], and KPZ limits [2], [1].

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