PART 10: EQUILIBRIUM FLUCTUATIONS OF SYMMETRIC SIMPLE EXCLUSION

We motivate the study of equlibrium fluctuations of symmetric simple exclusion through another proof of the $t^{1/4}$ scaling limit of the current in the one dimensional process. The equilibrium fluctuations capture the CLT behavior around the solution of the hydrodynamic equation, which in equilibrium is constant. The limiting equation is an infinite dimensional Ornstein-Uhlenbeck process.

1. Another proof of current fluctuations

As in Part 9, consider the current through the bond $(x, x+1)$, $J_{x,x+1}(t)$, which is the number of particles crossing the bond from left to right minus the number crossing from right to left up to time t.

A moment's thought gives that

$$J_{x-1,x}(t) - J_{x,x+1}(t) \;=\; \eta_t(x) - \eta_0(x),$$

the difference being equal to 0, 1 or -1. Then, formally,

$$J_{-1,0}(t) \;=\; \sum_{x\geq 0} J_{x-1,x}(t) - J_{x,x+1}(t) \;=\; \sum_{x\geq 0} \eta_t(x) - \eta_0(x).$$

However, the display does not make sense as typically there are an infinite number of particles in the system.

We may truncate however in the following way: Let

$$G_n(x) \;=\; \big(1 - \frac{x}{n}\big)1(0 \leq x \leq n).$$

Write, in terms of a scaling parameter N, that

$$\begin{aligned}
&\sum_{x\geq 0} G_n(x/N)\big[J_{x-1,x}(t) - J_{x,x+1}(t)\big] \\
&\quad = J_{-1,0}(t) + \sum_{x\geq 1} \big[G_n(x/N) - G_n(x-1/N)\big] J_{x-1,x}(t) \\
&\quad J_{-1,0}(t) + \frac{1}{nN}\sum_{x=1}^{N+1} J_{x-1,x}(t).
\end{aligned}$$

At the same time, we have

$$\sum_{x\geq 0} G_n(x/N)\big[J_{x-1,x}(t) - J_{x,x+1}(t)\big] \quad = \quad \sum_{x\geq 0} G_n(x/N)\big[\eta_t(x) - \eta_0(x)\big].$$

Following our previous custom, since space is scaled by N, we will speed up time by N^2, and define the "fluctuation field":

$$W_t^N(G) \;=\; \frac{1}{N^{d/2}}\sum_x G(x/N)\eta_{N^2 t}(x).$$

Then, scaling the current by the fourth root of the time scaling, we have for all $n \geq 1$ that

$$\frac{1}{\sqrt{N}} J_{-1,0}(N^2 t) \quad = \quad W_t^N(G_n) - W_0^N(G_n) - \frac{1}{nN^{3/2}} \sum_{x=1}^{N+1} J_{x-1,x}(N^2 t).$$

The idea now is that the first two terms on the right should be given in terms of a limit fluctuation field, which we must define. However, the last term on the right hand side should vanish as $N \uparrow \infty$ and $n \uparrow \infty$. One can adjust the construction of the process, to include counts $N^+_{x,x+1}(t)$ and $N^-_{x,x+1}(t)$ which keep track of the numbers of particles crossing $(x, x+1)$ from left to right and vice versa, so that

$$\begin{aligned} M_{x,x+1}(t) &= J_{x,x+1}(t) - \frac{1}{2}\int_0^t \eta_s(x)(1-\eta_s(x+1)) - \eta_s(x+1)(1-\eta_s(x))ds \\ &= J_{x,x+1}(t) - \frac{1}{t}\int_0^t \eta_s(x) - \eta_s(x+1)ds \end{aligned}$$

and

$$M_{x,x+1}(t)^2 - \frac{1}{2}\int_0^t \eta_s(x)(1-\eta_s(x+1)) + \eta_s(x+1)(1-\eta_s(x))ds$$

are martingales. Since jumps are not simultaneous in the process, $\{M_{x,x+1}(t)\}$ are orthogonal martingales.

Exercise 1.1. Show $E[M_{x,x+1}(t)M_{y,y+1}(t)] = 0$ by decomposing on the possible crossing times, which are stopping times, of bonds $(x, x+1)$ and $(y, y+1)$ which occur a.s. at distinct times, and the martingale property. That is, write $M_{x,x+1}(t) = \sum (M_{x,x+1}(\tau_{k+1}) - M_{x,x+1}(\tau_k))$ and a similar formula for $M_{y,y+1}(t)$ where $0 = \tau_0 < \cdots < \tau_{N(t)} < \tau_{N(t)+1} = t$ are the $N(t)$ jumps on these bonds. Then, we have

$$E[(M_{x,x+1}(\tau_{k+1}) - M_{x,x+1}(\tau_k))(M_{y,y+1}(\tau_{j+1}) - M_{y,y+1}(\tau_j))$$

equals zero if $k = j$ since jumps are not simultaneous. But, if $k < j$, then since $M_{y,y+1}(\tau_{j+1} \wedge t)$ is a martingale, the display also vanishes.

Lemma 1.2. *Starting in equilibrium, ν_ρ, we have*

$$\lim_{n\uparrow\infty} \sup_{N\geq 1} E_{\nu_\rho}\Big[\Big(\frac{1}{nN}\sum_{x=1}^{N+1} J_{x-1,x}(N^2 t)\Big)^2\Big] \;=\; 0.$$

Proof. Write

$$\begin{aligned} \frac{1}{nN}\sum_{x=1}^{N+1} J_{x-1,x}(N^2 t) &= \frac{1}{nN}\sum_{x=1}^{N+1}\frac{1}{2}\int_0^{N^2 t} \eta(x-1)(1-\eta(x)) - \eta(x)(1-\eta(x-1))ds \\ &\quad + \frac{1}{nN}\sum_{x=1}^{N+1} M_{x-1,x}(N^2 t) \qquad (1.1) \\ &= \frac{1}{nN^{3/2}}\frac{1}{2}\int_0^{N^2 t} \eta_s(0) - \eta_s(N)ds + \frac{1}{nN}\sum_{x=1}^{N+1} M_{x-1,x}(N^2 t). \end{aligned}$$

Here, we used that $\eta(x-1)(1-\eta(x)) - \eta(x)(1-\eta(x-1)) = \eta(x-1) - \eta(x)$.

Now, from an H_{-1} norm estimate, we have for all y that

$$E_{\nu_\rho}\Big[\Big(\int_0^{N^2t}\eta_s(y)-\rho ds\Big)^2\Big] \ \le\ C\rho(1-\rho)N^2t\cdot N\sqrt{t}.$$

Hence, the integral term in (1.1) is bounded by $Cn^{-2}N^3N^3t^{3/2} = Cn^{-2}$ which vanishes as $n\uparrow\infty$.

For the martingale term in (1.1), use the orthogonality of martingales to get

$$\begin{aligned}
&E_{\nu_\rho}\Big[\Big(\frac{1}{nN}\sum_{x=1}^{N+1}M_{x-1,x}(N^2t)\Big)^2\Big]\\
&\le\ \frac{1}{n^2N^2}\sum_{x=1}^{N+1}E_{\nu_\rho}M^2_{x-1,x}(N^2t)\\
&\le\ \frac{1}{2n^2N^2}\sum_{x=1}^{N+1}E_{\nu_\rho}\int_0^{N^2t}\eta(x)(1-\eta(x+1))+\eta(x+1)(1-\eta(x))ds\\
=\ & O(n^{-2}).
\end{aligned}$$

This finishes the proof. □

To complete the argument, we need to understand the limit fluctuation fields. This is the subject of the next sections.

2. Infinite dimensional Ornstein-Uhlenbeck process

Let us capture the evolution of $W^N_t(G)$ for a fixed G, smooth with compact support. Write

$$W^N_t(G)\ \ =\ \ W^N_0(G)+N^2\int_0^t LW^N_s(G)ds+\mathcal{M}^N_t(G)$$

where, after the usual summation-by-parts,

$$LW^N_t(G)\ =\ \frac{1}{N^{d/2}}\sum\triangle_N G(x/N)\eta_{N^2s}(x)$$

and $\mathcal{M}^N_t(G)$ is a martingale such that

$$\mathbb{N}^N_t(G)\ =\ (\mathcal{M}^N_t(G))^2-N^2\int_0^t L(Y^N_s(G))^2-2Y^N_s(G)LY^N_s(G)ds$$

is a martingale. The last integral in the display can be evaluated as

$$\begin{aligned}
&N^2\int_0^t L(Y^N_s(G))^2-2Y^N_s(G)LY^N_s(G)ds\\
&\quad=\ \frac{1}{2N^d}\int_0^t\sum[\nabla_N G(x/N)]^2\eta_{N^2s}(x)(1-\eta_{N^2s}(x+1))ds.
\end{aligned}$$

Now, as before with hydrodynamics, we have two steps:

Step 1: Show tightness of $\{Y^N_t : t\in[0,T]\}$ in an appropriate space, and continuity of limit trajectories under limit points.

Step 2: Identify the limit points in terms of a unique "infinite dimensional Ornstein-Uhlenbeck" process and a

The space will be $D([0,T],\mathbb{S}')$, containing distribution valued trajectories on the Schwarz space $\mathbb{S}$. In fact, we can restrict the domain functions to a specific subset of $\mathbb{S}$. Tightness will allow us to recover a limit "martingale" which is continuous, and hence will be a type of "infinite dimenensional Brownian motion".

Putting it together, we arrive at the following sketch: Y_t^N converges to Y_t where

$$dY_t \;=\; \frac{1}{2}\triangle Y_t dt + \sqrt{\rho(1-\rho)}\nabla d\mathcal{B}_t. \tag{2.1}$$

2.1. **Hermite polynomials.** To make sense of the above equation (2.1), we define a few spaces. Let $\{h_n\}$ be the Hermite polynomials on $L^2(\mathbb{R}^1)$, that is $h_0(u) = \pi^{-1/2}e^{-u^2/2}$ and for $n \geq 1$

$$h_n(u) \;=\; (2^n n!)^{-1/2}(-1)^n\pi^{-1/2}e^{u^2/2}\frac{d^n}{du^n}e^{-u^2}.$$

It is standard that $\{h_z = h_{z_1}\cdots h_{z_d}\}$, where $z = (z_1,\ldots,z_d)$ and $z_i \geq 1$, are orthonormal and complete on $L^2(\mathbb{R}^d)$. Each Hermite function is an eigenfunction with respect to operator $\mathcal{L} = |u|^2 - \triangle$, that is $|u|^2h_z - \triangle h_z = \lambda_z h_z$, with eigenvalues $\lambda_z = \sum_{i=1}^d(2z_i+1)$. See Reed-Simon [8][Chapter V] for more discussion.

Any function $f \in L^2(\mathbb{R}^d)$ can be expressed as

$$f \;=\; \sum_z \langle f, h_z\rangle h_z$$

where

$$\langle f, h_z\rangle = \int f(u)h_z(u)du.$$

Define for $k \geq 1$, the Hilbert spaces $\mathcal{H}_k$ which are the completions of smooth compactly supported functions with inner product

$$\langle f, g\rangle_k \;=\; \langle f, \mathcal{L}^k g\rangle.$$

In particular, $L^2 = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \cdots \supset \mathcal{H}_k$ are those functions such that

$$\sum_z \langle f, h_z\rangle^2\lambda_z^k \;<\; \infty.$$

The duals of $\mathcal{H}_k$ are $\mathcal{H}_{-k}$, relative to completion with respect to the innerproduct, can be identified as those functions such that

$$\sum_z \langle f, h_z\rangle^2\lambda_z^{-k} \;<\; \infty.$$

We have the ordering $\mathcal{H}_0 \subset \mathcal{H}_{-1} \subset \cdots \subset \mathcal{H}_{-k}$.

We shall endow $\mathcal{H}_{-k}$ with the uniform weak topology: $\{Y_i : i \geq 1\}$ in $D([0,T],\mathcal{H}_{-k})$ converges to Y if for all $f \in \mathcal{H}_k$ we have

$$\lim_{i\uparrow\infty}\sup_{0\leq t\leq T}\Big|\langle Y_i(t), f\rangle - \langle Y(t), f\rangle\Big| \;=\; 0.$$

We remark this follows our custom in hydrodynamics of proving tightness results in the uniform topology.

2.2. **A precise statement of** (2.1)**.** We will consider the fluctuation field Y_t^N acting on functions in $\mathcal{H}_k$ for a large enough k. Then, Y_t^N is thought of as an element of $\mathcal{H}_{-k}$. Let Q_N be the probability measure on $D([0,T],\mathcal{H}_{-k})$ governing $\{Y_t^N : t \in [0,T]\}$, when the underlying exclusion process starts from equilibrium ν_ρ.

Theorem 2.1. *For $k > 4+d$, we have that Q_N converges to Q concentrated on $C([0,T],\mathcal{H}_{-k})$ given by a generalized Ornstein Uhlenbeck process with mean* 0 *and covariance*

$$\begin{aligned} &E_Q[W_t(H)W_s(G)] \\ &= \frac{\rho(1-\rho)}{(2\pi(t-s))^d}\int\int H(u)G(v)\exp\Big\{-\frac{|u-v|^2}{2(t-s)}\Big\}dudv \end{aligned}$$

for all $0 \le s \le t$, $H, G \in \mathcal{H}_k$.

This generalized OU process is a Markov random field, and will be discussed in more detail later.

3. Application: Current fluctuations

We return to our motivating example with respect to current fluctuations. By Lemma 1.2, since $N^{-1/2}J_{-1,0}(N^2t)$ does not depend on n, we have uniformly in $N \ge 1$ that

$$\{W_t^N(G_n) + W_0^N(G_n) : n \ge 1\}$$

is a Cauchy sequence in $L^2(\nu_\rho)$.

By Theorem 2.1, by approximating G_n by smooth compactly supported functions, we have that for fixed n that

$$W_t^N(G_n) - W_0^N(G_n) \ \Rightarrow\ W_t(G_n) - W_0(G_n).$$

Since $\{W_t(G_n) - W_0(G_n) : n \ge 1\}$ is Cauchy in L^2, we denote its limit by $W_t(H_0) - W_0(H_0)$, whose distribution is a mean-zero Gaussian.

In particular,

$$\frac{1}{\sqrt{N}}J_{-1,0}(N^2t) \ \Rightarrow\ W_t(H_0) - W_0(H_0).$$

By Lemma 1.2, one may identify the variance by computing

$$\mathrm{Var}(N^{-1/2}J_{-1,0}(N^2t) \ = \ \lim_{n\uparrow\infty}\lim_{N\uparrow\infty} E_{\nu_\rho}\Big[\big(W_t^N(G_n) - W_0^N(G_n)\big)^2\Big].$$

Exercise 3.1. Use duality, that is $E_{\nu_\rho}[(\eta_t(x)-\rho)(\eta_0(y)-\rho)] = \rho(1-\rho)P_t(0,y-x)$, to verify the variance equals $\sqrt{2/\pi}\rho(1-\rho)$.

4. Holley-Stroock martingale problem

The main vehicle behind Theorem 2.1 is the following "martingale problem" characterization of Q. Let $U = (1/2)\triangle$ be the nonnegative self-adjoint operator defined on domain $L^2(\mathbb{R}^2)$ and let T_t be the associated semigroup. Let $B = \rho(1-\rho)\nabla$ be the linear gradient operator. Let also $\mathcal{F}_t$ be the sigma-field in $D([0,T],\mathcal{H}_k)$ generated by a process $W_s(H)$ for $s \le t$ and H smooth with compact support.

Theorem 4.1. *For $k \geq 2$, suppose Q is a probability measure governing W_t which concentrates on $C([0,T],\mathcal{H}_k)$, and for each smooth, compactly supported H,*

$$M_t^{U,H} \;=\; W_t(H) - W_0(H) - \int_0^t W_s(UH)ds$$

and

$$N_t^{U,H} \;=\; (M_t^{U,H})^2 - \|BH\|_{L^2}^2 t$$

are $L^1(Q)$, $\mathcal{F}_t$ martingales. Then, for all $0 \leq s < t$, and subsets $A \subset \mathbb{R}^d$, Q a.s.,

$$\begin{aligned} &Q\Big[W_t(H) \in A|\mathcal{F}_s\Big] \\ &= \int_A \frac{1}{\sqrt{2\pi \int_0^{t-s} \|BT_r H\|_{L^2}^2 dr}} \exp\Big\{\frac{-|y - W_s(T_{t-s}H)|^2}{2\int_0^{t-s}\|BT_rH\|_{L^2}^2 dr}\Big\}dy. \end{aligned}$$

Also, Q is determined by its restriction to $\mathcal{F}_0$.

We can now sketch the proof of Theorem 2.1. In our context, the restriction to $\mathcal{F}_0$ is already known: In equililbrium ν_ρ, we have that W_0^N converges to a Gaussian field with mean zero and covariance

$$E_Q[W_0(H)W_0(G)] \;=\; \rho(1-\rho)\langle H, G\rangle_{L^2}. \tag{4.1}$$

Exercise 4.2. Show that the joint distribution of $W_0^N(H_1),\ldots,W_0^N(H_\ell)$ is Gaussian with covariance (4.1). One can do this by computing the moment generating function.

Also, from the martingale property in Theorem 4.1 and that $d/dtT_t = T_tU$, we can argue

$$\begin{aligned} E_Q[W_t(H)W_s(G)] &= E_Q[(M_t^{U,H} - M_s^{U,H})W_s(G)] \\ &\quad + \int_s^t E_Q[W_u(UH)W_s(G)]du + E_Q[W_s(H)W_s(G)] \\ &= E_Q[W_s(G)T_{t-s}W_s(H)] \;=\; E_Q[W_0(G)T_{t-s}W_0(H)]. \end{aligned}$$

In our context, the last expression can be directly found using "duality." This covariance is exactly what is given in Theorem 2.1.

Now, since $\mathcal{B}_t^H = \|BH\|_{L^2}^{-1}M_t^{U,H}$ is a continuous martingale with quadratic variation t, by Levy's characterization, we have that $\mathcal{B}_t^H$ is a Brownian motion. Hence, we have

$$W_t(H) \;=\; W_0(H) + \int_0^t W_s(UH)ds + \|BH\|_{L^2}\mathcal{B}_t^H$$

where $\mathcal{B}_t$ as the infinite dimensional Brownian motion with covariance

$$E[\mathcal{B}_s^G\mathcal{B}_t^H] \;=\; (\min\{s,t\})\int_{\mathbb{R}^d} \frac{\nabla G(u)}{\|\nabla G\|_{L^2}}\frac{\nabla H(u)}{\|\nabla H\|_{L^2}}du.$$

In this way, we give a meaning to (2.1).

Exercise 4.3. Use polarization with the martingales $M_s^{U,G}$ and $M_t^{U,H}$ to show the formula in the last display.

What remains in the proof of Theorem 2.1 is to show that Q_N converges to a Q, supported on continuous trajectories, which satisfies the "martingale problem" conditions in Theorem 4.1.

5. Proof of Steps 1,2

The proof of Step 2 is easy in the symmetric simple exclusion context. Suppose that Q is a limit point of $\{Q_N\}$ supported on continuous trajectories. We show that $M_t^{U,H}$ and $N_t^{U,H}$ are $L^1(Q)$ martingales. Consider the martingales $\mathcal{M}_t^N(G)$ and $\mathbb{N}_t^N(G)$ defined earlier. By the L^1 ergodic theorem and as limits of martingales are martingales, their limits are $M_t^{U,H}$ and $N_t^{U,H}$ respectively, which are identified as $L^1(Q)$ martingales.

We now address Step 1. Define the uniform modulus of continuity for a path in $D([0,T],\mathcal{H}_{-k})$ and $\delta > 0$:

$$w_\delta(W) \;=\; \sup_{\substack{|t-s|\leq\delta\\ s,t\in[0,T]}} \|W_t - W_s\|_{\mathcal{H}_{-k}}.$$

The uniform modulus of course bounds the Skorohod modulus as before in the study of hydrodynamics.

Also, note that $A\subset D([0,T],\mathcal{H}_{-k})$ is relatively compact if uniformly over paths $Y\in A$ and times $t\in[0,T]$ we have $\|Y_t\|_{\mathcal{H}_{-k}}$ is bounded, and the modulus $w_\delta(Y)$ vanishes as $\delta\downarrow 0$.

We have therefore the following tightness criterion. Let $k > 4+d$.

Lemma 5.1. *A family of probability measures $\{Q_N\}$ on $D([0,T],\mathcal{H}_{-k})$ is tight if*

$$\lim_{A\uparrow\infty}\lim_{N\uparrow\infty} Q_N\Big[\sup_{t\in[0,T]}\|W_t\|_{-k} > A\Big] \;=\; 0$$

and, for all $\epsilon > 0$,

$$\lim_{\delta\downarrow 0}\lim_{N\uparrow\infty} Q_N\big[w_\delta(W)\geq\epsilon\big] \;=\; 0.$$

The proof now is divided into a few lemmas.

Lemma 5.2. *There is a constant $C(\rho,T)$ such that for all z,*

$$\begin{aligned}\limsup_{N\uparrow\infty} E_{Q_N}\Big[\sup_{t\in[0,T]}|\langle W_t,h_z\rangle|^2\Big] \\ \leq\; C(\rho,T)\big\{\langle h_z,h_z\rangle + \langle\triangle h_z,\triangle h_z\rangle\big\}.\end{aligned}$$

Proof. Prelimit, we have

$$\langle W_t^N, h_z\rangle \;=\; M_t^z + W_0^N(h_z) + \frac{1}{2N^{d/2}}\int_0^t \sum_x (\triangle_N h_z)(x/N)(\eta_{N^2 s}(x)-\rho)ds \tag{5.1}$$

where $M_t^z = \mathcal{M}_t^N(h_z)$.

We need to show each of these terms is bounded appropriately. The mean square of the time 0 term is clearly bounded in the limit by $\rho(1-\rho)\|h_z\|_{L^2}^2$. The martingale term is bounded by Doob's inequality:

$$E_Q\big[\sup_t |M_t^z|^2\big] \;\leq\; CE_Q\big[|M_T^z|^2\big]$$

which is bounded by the mean of the quadratic variation of M_T^z, namely

$$\int_0^T \frac{1}{2N^d}\sum_x |\nabla_N h_z|^2 E_{Q_N}[\eta_{N^2s}(x)(1-\eta_{N^2s}(x))]ds \;\leq\; C\rho(1-\rho)T\|\nabla h_z\|_{L^2}^2.$$

The last L^2 norm by integration by parts can be bounded in terms of $\|h_z\|_{L^2}$ and $\|\triangle h_z\|_{L^2}$.

The integral term can also be bounded

$$E_{Q_N}\big[\sup_t \big(\int_0^t \Gamma_1^z(s)ds\big)^2\big] \ \le \ TE_{Q_N}\big[\int_0^T (\Gamma_1^z(s))^2 ds\big]$$

where Γ_1^z is the integrand. Inspection of the integrand shows that its mean-square is bounded by a constant times $\|\triangle h_z\|_{L^2}^2$. □

Lemma 5.3. *For $k > 4+d$, we have*

$$\limsup_{N\uparrow\infty} E_{Q_N}\big[\sup_t \|Y_t\|_{-k}^2\big] \ < \ \infty$$

and

$$\lim_{\ell\uparrow\infty}\limsup_{N\uparrow\infty} E_{Q_N}\big[\sup_t \sum_{|z|\ge\ell}\langle W_t, h_z\rangle^2\gamma_z^{-k}\big] \ = \ 0.$$

Proof. The first display is bounded by

$$\sum_z \gamma_z^{-k} E_{Q_N}\big[\sup_t \langle W_t, h_z\rangle^2\big] \ \le \ C\sum_z \gamma_z^{-k}\big\{\|h_z\|_{L^2}^2 + \langle\triangle h_z, \triangle h_z\rangle\big\}. \qquad (5.2)$$

By properties of h_z, we have

$$\langle \triangle h_z, \triangle h_z\rangle \ \le \ \|C(1+|z|+|z|^2)h_z\|_{L^2}^2.$$

Then, noting the asymptotics of γ_z, we have the bound on (5.2) of

$$C\sum_z \frac{(1+|z|)^4}{(1+|z|)^k}$$

which is finite if $k > 4+d$.

A similar argument holds for the second term. □

Therefore, the first statement of Lemma 5.3 shows that the first condition in Lemma 5.1 is satisfied. Moreover, tightness of $\{Q_N\}$ will follow, given the second statement of Lemma 5.3, if the following result holds.

Lemma 5.4. *For all ℓ, we have*

$$\lim_{\delta\downarrow 0}\limsup_{N\uparrow\infty} Q_N\Big[\sup_{\substack{|t-s|\le\delta \\ t,s\in[0,T]}} \sum_{|z|\le\ell}\langle W_t - W_s, h_z\rangle^2\gamma_z^{-k} > \epsilon\Big] \ = \ 0.$$

Proof. Since $|z|\le\ell$, with ℓ fixed, we need only prove

$$Q_N\Big[\sup_{\substack{|t-s|\le\delta \\ t,s\in[0,T]}} \langle W_t - W_s, h_z\rangle^2\gamma_z^{-k} > \epsilon\Big]$$

vanishes as $N\uparrow\infty$ and $\delta\downarrow 0$ for each z.

Recall (5.1). We need to show the estimate with $W_\cdot$ replaced by $M_\cdot^z$, and also when replaced with the additive functional.

For the additive functional, by Chebychev and Schwarz inequalities, we may bound

$$Q_N\Big[\sup_{\substack{|t-s|\le\delta \\ s,t\in[0,T]}} \Big|\int_s^t \frac{N^{d/2}}{\sum}_x \big(\triangle_N G\big)(x/N)[\eta_{N^2u}(x)-\rho]du\Big| > \epsilon\Big]$$

by

$$\frac{\delta}{\epsilon^2}E_{Q_N}\int_0^T \Big[\frac{1}{N^{d/2}}\sum_x \big(\triangle_N G\big)(x/N)[\eta_{N^2u}(x)-\rho]du\Big]^2.$$

Since, we are starting from ν_ρ, the last quantity is easily bounded by

$$\delta\epsilon^{-2}\|\triangle G\|^2_{L^2}\rho(1-\rho)T$$

which vanishes as $\delta \downarrow 0$.

The martingale estimate is handled by the following lemmas. $\square$

We will take the following approach which will allow to bound the fourth moment of M^z_t. Other arguments using Aldous's stopping time argument, as in [6], are available also.

Lemma 5.5. *For all local functions F, we have that*

$$Z^\lambda_{s,t} \;=\; \exp\Big\{\lambda F(\eta_t) - \lambda F(\eta_s) - \int_s^t e^{-\lambda F(\eta_u)} L e^{\lambda F(\eta_u)} du\Big\}$$

is a martingale.

Proof. The lemma is a type of "Girsanov" formula. See [2][around p. 175] which shows $Z_{s,t}$ is a local martingale. Since we are dealing with the exclusion process, where occupation numbers are bounded, one can show it is a martingale. $\square$

Exercise 5.6. Investigate more the proof of the above lemma.

Lemma 5.7. *We have*

$$E_{Q_N}\big[\big(M^z_t - M^z_s\big)^4\big] \;\le\; C\big(|t-s|^2 + N^{-d/2}|t-s|\big).$$

Proof. First, we can approximate h_z by a smooth compactly supported G in L^4 say. Then, $Z^\lambda_{s,t}$ with $F(\eta) = N^{-d/2}\sum_x G(x/N)\big(\eta(x)-\rho\big)$, is a martingale. By explicit calculation,

$$e^{-\lambda F(\eta_u)} L e^{\lambda F(\eta_u)} \;=\; \frac{1}{2}\sum_{x,y} p(y-x)\big[e^{\lambda N^{-d/2}[G(y/N)-G(x/N)]} - 1\big]\eta(x).$$

Morevoer,

$$E_{Q_N} Z^\lambda_{s,t} \;=\; 1. \tag{5.3}$$

Now, we may expand the left hand side and equate in powers of λ. We have (5.3) is evaluated as

$$\begin{aligned} &E_{Q_N}\exp\Big\{\lambda\big(M^G_N(t) - M^G_N(s)\big)\\ &-\frac{\lambda^2}{N^d}\int_s^t \sum_{x,y} p(y-x)\big(G(y/N)-G(x/N)\big)^2 \eta_{N^2u}(x)du + O(N^{-3d/2})\Big\} \;=\; 1.\end{aligned}$$

It is not difficult now to obtain the desired statement, noting that $E_{Q_N}[(M^G_N(t) - M^G_N(s))^2] \le C|t-s|$ from computing the quadratic variation. $\square$

Exercise 5.8. Make the computation in the proof of the above lemma to match fourth powers of λ to obtain the lemma statement.

Lemma 5.9. *We have*

$$\lim_{\delta\downarrow 0}\lim_{N\uparrow\infty} Q_N\Big[\sup_{\substack{|t-s|\le\delta\\ t,s\in[0,T]}} \langle M^z_t - M^z_s, h_z\rangle^2\gamma_z^{-k} > \epsilon\Big] \;=\; 0$$

Proof. By the usual 3 ϵ argument, Doob's inequality, and stationarity, we can bound the display by

$$\frac{C}{\delta\epsilon^4} E_{Q_N}\left[\left(M_\delta^z\right)^4\right].$$

However, by the previous lemma, we have the right hand side is bounded above by

$$\frac{C}{\delta\epsilon^4} \cdot \left(\delta^2 + N^{-d/2}\delta\right)$$

which as $N \uparrow \infty$ and $\delta \downarrow 0$ vanishes. □

6. Notes

The proof given for current fluctuations follows [5]. The idea of truncation goes back to [9]. Equilibrium fluctuations of particle systems is relatively well understood [6], [10]. In symmetric simple exclusion, "nonequilibrium fluctuations" are also known [7].

However, a main open problem is to understand "nonequilibrium" fluctuations of the density field in more general particle systems, say without the duality property. Until now, only results in dimension $d = 1$ are known [1], [3].

The theory of generalized Ornstein Uhlenbeck processes originates in [4].

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