LECTURE 1: PRELIMINARIES AND HYDRODYNAMICS OF INDEPENDENT RANDOM WALKS

Before discussing a basic example, illustrating possibilities in the study of 'hydrodynamics of stochastic particle systems', we recall some basic notions in Markov chains.

1. Markov chains

1.1. **Construction.** A family of random variables $\{X_n : n \geq 0\}$ taking values on a countable state space E is called a 'discrete time Markov chain' if the 'stationary Markov property' is satisfied:

$$\begin{aligned} P(X_n = x_n | X_0 = x_0, \ldots, X_m = x_m) &= P(X_n = x_n | X_m = x_m) \\ &= P(X_{n-m} = x_n | X_0 = x_m) \end{aligned}$$

for all $x_0, \ldots, x_m \in E$ and $n > m \geq 0$. When $n = 1$ and $m = 0$, the last quantity on the right-side represents a 'transition probability' of the Markov chain, denoted $p(x, y) = P(X_1 = y | X_0 = x)$.

We now construct a 'continuous time Markov chain' on E with 'skeleton' $\{X_n : n \geq 0\}$ and transition probability vanishing on the diagonal, that is $p(x, x) = 0$ for all $x \in E$. Let $\{\lambda_x : x \in E\}$ be a collection of positive numbers, and let $\{W_n : n \geq 0\}$ be a collection of independent identically distributed exponential random variables with rate 1, independent in particular the skeleton discrete time chain. Define now the process $\{Z_t : t \geq 0\}$ as follows: Initially, $Z_0 = X_0 \in E$. After time $\lambda_{X_0}^{-1} W_0$, the process jumps to value X_1, and after a subsequent time $\lambda_{X_1}^{-1} W_1$, the process jumps to value X_2, and so on. Let $T_k = \sum_{i=0}^{k} \lambda_{X_i}^{-1} W_i$ for $k \geq 0$. Then,

$$Z_t = \begin{cases} x & \text{for } 0 \leq t < T_0 \\ X_1 & \text{for } T_0 \leq t < T_1 \\ \vdots & \vdots \\ X_n & \text{for } T_{n-1} \leq t < T_n \end{cases}$$

for $0 \leq t < T_\infty = \lim_{n \uparrow \infty} T_n$. Sufficient conditions for $T_\infty = \infty$ include the cases if the state space E is finite, or if $\sup_{x \in E} \lambda_x < \infty$. We will assume from now on that the process is 'regular', that is $T_\infty = \infty$, so that the process Z_t is defined for all time $t \geq 0$.

One can show that $\{Z_t : t \geq 0\}$ satisfies the stationary Markov property, which in this context is equivalent to

$$\begin{aligned} P(Z_t = y | Z_{t_0} = x_0, \ldots, Z_{t_m} = x_m, Z_s = x) &= P(Z_t = y | Z_s = x) \\ &= P(Z_{t-s} = y | Z_0 = x) \quad (1.1) \end{aligned}$$

for all $x, y, x_0, \ldots, x_m \in E$, $0 \leq t_0 < \cdots t_m < s < t$ and $m \geq 0$.

Conversely, given a process $\{Z_t : t \geq 0\}$ satisfying (1.1) and also the 'jump property' that there exists a sequence of strictly increasing stopping times $\{T_n : n \geq 0\}$ such that $T_0 > 0 = T_{-1}$ and Z_t is constant on intervals $[T_n, T_{n+1})$ and $Z_{T_n^-} \neq Z_{T_n}$ for $n \geq 0$, one can determine a unique skeleton discrete time Markov chain $\{X_n : n \geq 0\}$ and positive jump parameters $\{\lambda_x : x \in E\}$ such that $X_n = Z_{T_n}$, $p(x,y) = P(Z_{T_{n+1}} = y | Z_{T_n} = x)$, and $T_n - T_{n-1}$ are independent exponentials with rates λ_{X_n} for $n \geq 0$. Chains with the same skeleton and jump parameters have the same joint distributions. The condition $Z_{T_n^-} \neq Z_{T_n}$ ensures p vanishes on the diagonal.

1.2. **Generators, Chapman-Komogorov equations.** In the discrete time setting, we can define the transition operator $P = (p(x,y) : x, y \in E)$, which is a matrix when E is finite. Then, the nth powers give the nth step probabilities, $P^n(x,y) = P(X_n = y | X_0 = x)$.

Computing the t-time probabilities for the continuous time Markov chain Z_t is more complicated. Define the transition probability

$$P_t(x,y) \;=\; P(Z_t = y | Z_0 = x).$$

Then, by the Markov property we have

$$P_{t+s}(x,y) \;=\; \sum_{z \in E} P_t(x,z) P_s(z,y)$$

or in terms of operators $P_{t+s} = P_t P_s$.

Given regularity of the process, the transition functions are differentiable in time, and satisfy the backward equation

$$\begin{aligned} \frac{d}{dt} P_t(x,y) &= \sum_{z \in E} \lambda_x p(x,z)[P_t(z,y) - P_t(x,y)] \\ P_0(x,y) &= \delta_{x,y}. \end{aligned}$$

Similarly, one has the forward equation

$$\frac{d}{dt} P_t(x,y) \;=\; \sum_{z \in E} P_t(x,z) \lambda_z p(z,y) - P_t(x,y)\lambda_y.$$

Integral versions of both the backward and forward equations can be rigorously derived from decomposing the transition probability on first and last jump times respectively, or analytically.

Exercise 1.1. Review this deriviation.

Define the operator

$$L(x,y) \;=\; \begin{cases} \lambda_x p(x,y) & \text{for } y \neq x \\ -\lambda_x & \text{for } y = x. \end{cases}$$

Then, compactly expressed, the backward and forward equations become

$$\frac{d}{dt} P_t \;=\; L P_t \quad \text{and} \quad \frac{d}{dt} P_t \;=\; P_t L.$$

Also, $\lim_{t \downarrow 0} t^{-1}[P_t - I] = L$ and $P_t(x,y) = \delta_{x,y} + tL(x,y) + o(t)$.

When the space E is finite, L is a 'generator' matrix, that is $L(x,y) \geq 0$ for $x \neq y$ and $L(x,x) = -\sum_{y \neq x} L(x,y)$, and by solving the ODE's, one obtains $P_t = e^{tL}$ which can be computed in some cases. More generally, the 'Trotter-Kato' formula holds $P_t = \lim_{n \uparrow \infty} (I + \frac{t}{n} L)^n$.

Let $f : E \to \mathbb{R}$ be a bounded function on the state space. In the discrete time case, define $Pf(x) = \sum_{y \in E} p(x,y) f(y)$, which is the conditional expectation of

$f(X_1)$ given $X_0 = x$. Then, $P^n f(x) = \sum_{y\in E} p^{(n)}(x,y)f(y)$ is the conditional expectation of $f(X_n)$ given $X_0 = x$, where $p^{(n)}(x,y)$ is the n-fold convolution, or nth step transition probability. In the continuous case, define $P_t f(x) = \sum_{y\in E} P_t(x,y)f(y)$ which is the conditional distribution of $f(Z_t)$ given $Z_0 = x$.

In this framework, the generator L is often expressed in terms of its action on f:

$$\begin{aligned}(Lf)(x) &= \sum_{y\in E} L(x,y)[f(y) - f(x)] \\ &= \sum_{y\in E} \lambda_x p(x,y)[f(y) - f(x)].\end{aligned}$$

1.3. **Invariant measures.** Let μ be a probability measure on E. In the discrete time situation, define $\mu P(x) = \sum_{y\in E} \mu(y)p(y,x)$. Hence, we see that μP^n is the distribution at time n when the initial state is distributed according to μ.

In the continuous model, define

$$\mu P_t(x) = \sum_{y\in E} \mu(y)P_t(y,x), \quad \text{and} \quad \mu L(x) = \sum_{y\in E} \mu(y)L(y,x).$$

We say that μ is an 'invariant measure' for discrete time chains if $\mu P = \mu$, and for continuous time chains if $\mu P_t = \mu$ for all $t \geq 0$. We also say that μ is a 'reversible' invariant measure in discrete time chains if $\mu(x)p(x,y) = \mu(y)p(y,x)$ for all $x, y \in E$. In continuous time chains, μ is 'reversible' when P_t is self-adjoint in $L^2(\mu)$, that is $\sum_{x\in E} \mu(x)f(x)P_t(x,y)g(y) = \sum_{x\in E} \mu(x)g(x)P_t(x,y)f(y)$, or in terms of the inner product on $L^2(\mu)$, $\langle f, P_t g\rangle_\mu = \langle P_t f, g\rangle_\mu$ for all $f, g \in L^2(\mu)$.

One can verify that in the continuous time setting that μ is invariant is equivalent to $\mu L = 0$, and μ is reversible is equivalent to $\mu(x)L(x,y) = \mu(y)L(y,x)$ for all $x, y \in E$, or in terms of inner products $\langle f, Lg\rangle_\mu = \langle Lf, g\rangle_\mu$ for all $f, g \in L^2(\mu)$.

Exercise 1.2. Review these equivalences. Hint: The Trotter-Kano formula is useful in this exercise.

In the finite state space case, invariant measures always exist, and if the skeleton chain can reach every state from any state in finite time, that is p is irreducible, the invariant measure is unique. However, in the countable state case there may be no invariant measures.

Reversibility has the following interesting implication. Fix a time $t > 0$, and consider $R_s = Z_{t-s}$ for $0 \leq s \leq t$. Suppose that initially Z_0 is distributed according to an invariant measure μ. Then, it can be seen that R_s is a continuous time Markov chain with transition probability $Q_t(x,y) = (\mu(y)/\mu(x))P_t(y,x)$. In particular, when μ is reversible, $Q_t(x,y) = P_t(x,y)$ and in this case the 'forward in time' and 'backward in time' chains have the same distribution!

We remark that it is sometimes easier to find directly a reversible measure and therefore an invariant measure by checking the reversibility conditions.

1.4. **Examples.** We will content ourselves for the moment with two basic continuous time examples, the two-state Markov chain, and random walk.

Example 1.3. Let $E = \{0, 1\}$ correspond to states 'on' and 'off', or sometimes 'empty' and 'occupied'. Here, state 0 can transition to state 1 and vice versa. Let

λ_0 and λ_1 be the corresponding jump rates. The skeleton chain probabilities are $p(0,1) = p(1,0) = 1$. The generator matrix is

$$L = \begin{bmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix}.$$

and correspondingly, it is an exercise in diagonalization to compute $P_t = e^{tL}$ and to find the unique invariant measure.

Exercise 1.4. Compute the invariant measure and P_t.

Example 1.5. Let $E = \mathbb{T}_N^d$ the d-dimensional torus where $\mathbb{T}_N = \mathbb{Z} \setminus N\mathbb{Z}$, or integers modulo N. Let also $\lambda_x \equiv 1$, and p be a finite-range, translation-invariant transition probability: For all $x, y \in E$, $p(x,y) = 0$ if $|x-y| \geq R$ some $R < \infty$, and $p(x,y) = p(0, y-x) =: p(y-x)$. We will also assume that p is irreducible. For instance, the nearest-neighbor, symmetric case is one possibility.

Then, $L(x,y) = p(x,y)$ for $x \neq y$ and $L(x,x) = -\sum_{y \neq x} L(x,y)$. In particular, the uniform distribution $\mu(x) \equiv N^{-1}$ is the unique invariant measure. Moreover, μ is reversible exactly when $p(x) = p(-x)$ for all $x \in E$.

Now let R_t be the number of jumps before time t. Since the jump rates are all 1, we see that R_t is a Poisson process with rate 1. In particular, the transition probability

$$P_t(y-x) := P_t(x,y) \;=\; E\big[p^{(R_t)}(x,y)\big] \;=\; \sum_{n \geq 0} \frac{e^{-1}}{n!} p^{(n)}(x,y).$$

We now consider a sequence of chains $\{Z_t^{(N)} : t \geq 0\}_{N \geq 1}$ on a sequence of torii.

Define $m = \sum_{x \in E} x p(x)$ be the mean displacement of the position. Then, it is not difficult to extablish the following law of large numbers,

$$\lim_{N \uparrow \infty} \frac{Z_{Nt}^{(N)}}{N} \;=\; m \quad \text{in probability.}$$

Here, $m \in \mathbb{T}^d$ whre $\mathbb{T}$ is the unit circle.

Exercise 1.6. Show this LLN by say variance computations. Noting $R_t^{(N)}$ is Poisson with variance t is useful. One can do it also by regeneration, and other methods.

Also, if $m = 0$, let σ be the matrix of covariances, $\sigma_{i,j} = \sum_x x_i x_j p(x)$ for $0 \leq i, j \leq d$. We have the central limit theorem,

$$\frac{Z_{N^2 t}^{(N)}}{N} \;\Rightarrow\; \mathrm{N}(0, \sigma t).$$

Exercise 1.7. There are a few ways to show this CLT. One way is to write

$$Z_{N^2 t}^{(N)} N = \frac{Z_{T_{R_{N^2 t}^{(N)}}}^{(N)}}{\sqrt{T_{R_{N^2 t}^{(N)}}}} \cdot \frac{\sqrt{T_{R_{N^2 t}^{(N)}}}}{N}$$

and to use a random index CLT. For instance, in this case, $R_t^{(N)}$ is independent of the displacements.

Finally, when $m = \sum_x xp(x) \neq 0$, we say the walk is asymmetric, and when $m = 0$ and $p(\cdot)$ is not symmetric, we say the walk is mean-zero asymmetric, and when $p(\cdot)$ is symmetric, we say the walk is symmetric.

2. Hydrodynamics of independent random walks

We would like to understand the space-time evolution of the mass in a system of particles with a conservation law. Perhaps the simplest model is that of non-interacting random walks on a d-dimensional torus with N locations. When N is large, and one looks at the system from afar, after long times, one can more discern the motion of the bulk of the mass rather than individual components. The goal is to make precise the evolution of the mass in this scale in terms of a continuum equation.

We will be working with a Markov chain on $E = \mathbb{N}^{\mathbb{T}_N^d}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, which govern the motion of K independent random walks on $\mathbb{T}_N^d$ as in Example 1.5. Since we are interested in the 'mass' of particles, we will consider the occupation numbers at each location on the lattice $\mathbb{T}_N^d$. That is, let Z_t^i be the position of the ith particle at time t. Define

$$\eta_t(x) \;=\; \sum_{i=1}^{K} 1(Z_t^i = x).$$

We now observe that the process $\eta_t = \{\eta_t(x) : x \in \mathbb{T}_N^d\}$ is a Markov chain. Indeed, given independence and the Markov property of the individual particle movement, by splitting over all possibilities, the Markov property of η_t can be deduced.

2.1. **Associated invariant measures.** What are the invariant measures for the process? Since the process η_t, corresponding to K particles, is irreducible, there is a unique invariant measure. It is not so easy to characterize it immediately. We will come back to this question later.

In fact, the following analysis will be easier if we relax the assumption there are K particles in the system. If we do not specify the initial number of particles, then η_t is no longer irreducible, since there is no birth or death present: For instance, a system with 10 particles cannot evolve into one with 20 random walks. However, we can more easily specify in nice form several invariant measures for this 'relaxed' system.

Recall the Poisson distribution with parameter α, $q_\alpha(k) = e^{-\alpha}\alpha^k/k!$ for $k \geq 0$. Its moment generating function is given by

$$\sum_{k\geq 0} e^{\lambda k} e^{-\alpha}\frac{\alpha^k}{k!} \;=\; e^{\alpha(e^\lambda - 1)}.$$

For a positive function $\rho : \mathbb{T}^d \to \mathbb{R}_+$, define the product measure $\nu^N_{\rho(\cdot)}$ on $\mathbb{N}^{\mathbb{T}_N^d}$ by

$$\nu^N_{\rho(\cdot)}(\eta(x) = k) \;=\; q_{\rho(x/N)}(k).$$

When $\rho(\cdot) \equiv \alpha$ is constant, we denote $\nu^N_{\rho(\cdot)} = \nu^N_\alpha$.

The process $\{\eta_t : t \geq 0\}$ belongs to the space of right-continuous paths with left limits in $E = \mathbb{N}^{\mathbb{T}_N^d}$, $D([0,\infty); \mathbb{N}^{\mathbb{T}_N^d})$. We will denote by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ the probability measure and expectation with respect to the evolution of the process when initially

η_0 is distributed according to μ. On the other hand, E_μ will refer to the expectation with respect to μ on E.

Proposition 2.1. *The measures ν_α^N are invariant for the Markov chain η_t.*

Proof. We need only compute the moment generating function of η_t. Write

$$\eta_t(x) \;=\; \sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)} 1(Z_t^{y,k}=x)$$

and

$$\sum_{x\in\mathbb{T}_N^d}\theta(x)\eta_t(x) \;=\; \sum_{x\in\mathbb{T}_N^d}\sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)}\theta(x)1(Z_t^{y,k}=x) \;=\; \sum_{y\in\mathbb{T}_N^d}\sum_{k=1}^{\eta_0(y)}\theta(Z_t^{y,k})$$

where $Z_t^{y,k}$ denotes the position at time t of the kth particle initially at location y.

Since particles move independently, and initially there are a Poisson number of particles on each site of the lattice,

$$\begin{aligned}
\mathbb{E}_{\nu_\alpha^N}\Big[\exp\sum_{x\in\mathbb{T}_N^d}\theta_x\eta_t(x)\Big] &= \prod_{y\in\mathbb{T}_N^d}\mathbb{E}_{\nu_\alpha^N}\Big[\exp\sum_{k=1}^{\eta_0(y)}\theta(Z_t^{y,k})\Big]\\
&= \prod_{y\in\mathbb{T}_N^d}E_{\nu_\alpha^N}\Big(E\big[\exp\theta(Z_t^{y,1})\big]\Big)^{\eta_0(y)}\\
&= \prod_{y\in\mathbb{T}_N^d}\exp\Big[\alpha\big(E\big[e^{\theta(y+Z_t)}\big]-1\big)\Big]
\end{aligned}$$

where Z_t is the position of a random walk on $\mathbb{T}_N^d$ starting at the origin.

Now,

$$E\big[e^{\theta(y+Z_t)}\big] \;=\; \sum_{x\in\mathbb{T}_N^d}P_t^N(x-y)e^{\theta(x)}$$

and

$$E\big[e^{\theta(y+Z_t)}\big]-1 \;= sum_{x\in\mathbb{T}_N^d}P_t^N(x-y)(e^{\theta(x)}-1).$$

Hence, after a calculation,

$$\begin{aligned}
E_{\nu_\alpha^N}\Big[\exp\sum_{x\in\mathbb{T}_N^d}\theta(x)\eta_t(x)\Big] &= \exp\sum_{y\in\mathbb{T}_N^d}\alpha\sum_{x\in\mathbb{T}_N^d}P_t^N(x-y)\big(e^{\theta(x)}-1\big)\\
&= \exp\sum_{x\in\mathbb{T}_N^d}\alpha\big(e^{\theta(x)}-1\big)
\end{aligned}$$

which finishes the proof. □

We now remark that the measure ν_α^N can be decomposed in terms of its restrictions to the sets $\{\eta\in E : \sum_{x\in\mathbb{T}_N^d}\eta(x)=K\}$ for $K\geq 0$ which are invariant for the motion. Then, each of these restrictions, $\nu_{\mathbb{T}_N^d,K} = \nu_\alpha^N(\cdot|\sum_{x\in\mathbb{T}_N^d}\eta(x)=K)$, is invariant, and does not depend on α. In physics terminology, ν_α^N is the 'grand canonical' measure and $\nu_{\mathbb{T}_N^d,K}$ is the 'canonical' one.

Since the mean of $\eta(x)$ under ν_α^N equals α, it makes sense to call α the 'mass density' of the process. In this way, $\{\nu_\alpha^N : \alpha\geq 0\}$ is a family of invariant measures indexed by density α. Moreover, we may interpret the measure $\nu_{\rho_0(\cdot)}^N$ as a 'local equilibrium' measure in the following sense: Let $u \in \mathbb{T}^d$ be a continuity point of $\rho_0(\cdot)$. Then, since $\rho_0(\cdot)$ is continuous at u, $\nu^N_{\rho_0(\cdot)}$ distributes nearby $\lfloor Nu \rfloor$ almost like the invariant measure $\nu^N_{\rho(u)}$. More precisely, for fixed l, we have

$$\begin{aligned}\lim_{N\uparrow\infty} E_{\nu^N_{\rho_0(\cdot)}}\Big[\exp\sum_{|x|\le l}\theta(x)\eta(x+\lfloor uN\rfloor)\Big] &= \lim_{N\uparrow\infty}\prod_{|x|\le l}\exp\rho_0(N^{-1}(x+\lfloor uN\rfloor))(e^{\theta(x)}-1)\\ &= E_{\nu_{\rho_0(u)}}\Big[\exp\sum_{|x|\le l}\theta(x)\eta(x)\Big].\end{aligned}$$

In fact, we will say that a sequence of probabiliy measures μ^N on $\mathbb{N}^{\mathbb{T}^d_N}$ is a 'local equilibrium' of profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ if

$$\lim_{N\uparrow\infty} E_{\mu^N}\Big[\exp\sum_{|x|\le l}\theta(x)\eta(x+\lfloor uN\rfloor)\Big] = E_{\nu_{\rho_0(u)}}\Big[\exp\sum_{|x|\le l}\theta(x)\eta(x)\Big]$$

for all $l \ge 1$ and $\theta(\cdot)$.

2.2. **Hydrodynamics.** The question now is if we start with a local equilibrium measure $\nu^N_{\rho_0(\cdot)}$, how to characterize the distribution at a later time? Initially, the 'density profile' is given by ρ_0. Is there a function which captures the density profile at future times?

The answers depend on the particular time and space scales chosen in the problem. We will think of $\mathbb{T}^d_N$ as embedded in $\mathbb{T}^d$ where grid points on $\mathbb{T}^d_N$ are separated by distance N^{-1}. In this way, a 'macroscopic' point u on $\mathbb{T}^d$ corresponds to the 'microscopic' point $\lfloor uN\rfloor$. As we will see, time should now be appropriately speeded up to see movement of the system. How fast will depend on the structure of the underlying jump probability $p(\cdot)$.

Following the computations above, the moment generating function, starting from $\nu^N_{\rho_0(\cdot)}$, satisfies

$$\begin{aligned}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big[\exp\sum_{x\in\mathbb{T}^d_N}\theta(x)\eta_t(x)\Big] &= \prod_{y\in\mathbb{T}^d_N}\exp\Big\{\rho_0(y/N)\big(E\big[e^{\theta(y+Z_t)}\big]-1\big)\Big\}\\ &= \exp\sum_{y\in\mathbb{T}^d_N}\rho_0(y/N)\sum_{x\in\mathbb{T}^d_N}P^N_t(x-y)\big(e^{\theta(x)}-1\big)\\ &= \exp\sum_{x\in\mathbb{T}^d_N}\big(e^{\theta(x)}-1\big)\psi_{N,t}(x)\end{aligned}$$

where

$$\begin{aligned}\psi_{N,t}(x) = \sum_{y\in\mathbb{T}^d_N}P^N_t(x-y)\rho_0(y/N) &= \sum_{z\in\mathbb{T}^d_N}P^N_t(z)\rho_0(N^{-1}(x-z))\\ &= E\big[\rho_0\big(N^{-1}(x-Z^N_t)\big)\big].\end{aligned}$$

Hence, the distribution at time t is still a product measure with varying intensity $\psi_{N,t}(\cdot)$.

When t is fixed, the t-time distributions $\{P^N_t(\cdot) : N \ge 1\}$ are tight, that is for all $\epsilon > 0$, there exists A such that $\sum_{|x|\le A} P^N_t(x) \ge 1-\epsilon$. Hence, at a continuity point u for $\rho_0(\cdot)$, we have $\lim_{N\uparrow\infty}\psi_{N,t}(\lfloor uN\rfloor) = \rho_0(u)$, the same limit we obtained earlier when $t = 0$. So, if we do not speed up time at all, in this time scale, the system does not move.

When $m = \sum xp(x) \neq 0$, the asymmetric case, since $N^{-1}Z^N_{tN} \to mt$ in probability, we have

$$\lim_{N\uparrow\infty} \sum_{|z/N - mt|\leq\epsilon} p^N_{tN}(z) \;=\; \lim_{N\uparrow\infty} P\Big[\Big|\frac{Z^N_{Nt}}{N} - mt\Big| \leq \epsilon\Big] \;=\; 1.$$

In this case, when the initial profile ρ_0 is continuous,

$$\lim_{N\uparrow\infty} \psi_{N,Nt}(\lfloor uN\rfloor) \;=\; \rho_0(u - mt) \;:=\; \rho(t,u).$$

Therefore, if we speed up time by a factor of N, we see that the density profile has translated by mt. In this sense Nt is referred to as the 'microscopic' time, and t as the 'macroscopic' time. The density $\rho(t,u)$ satisfies

$$\partial_t\rho + m\cdot\nabla\rho \;=\; 0. \tag{2.1}$$

This makes sense as individual particles displace an order N microscopic locations at microscopic time Nt.

However, when $m = 0$, particles do not displace as much, but follow the 'square root' law, in that displacements are on order $\sqrt{N}$ at time Nt, or alternatively on order N at times N^2t. The latter version fits in nicely with our space scaling. By the central limit theorem for random walks in this case, $N^{-1}Z^N_{N^2t} \Rightarrow \mathrm{N}(0,\sigma t)$, we have

$$\begin{aligned}
\lim_{N\uparrow\infty} \psi_{N,N^2t}(\lfloor Nu\rfloor) \;&=\; \lim_{N\uparrow\infty} \sum_{z\in\mathbb{T}^d_N} P^N_{N^2t}(z)\rho_0\big(N^{-1}(\lfloor Nu\rfloor - z)\big)\\
&=\; \lim_{N\uparrow\infty} E\Big[\rho_0(u - N^{-1}Z^N_{N^2t})\Big] \;=\; \int_{\mathbb{R}^d} \bar\rho_0(x)G_t(u-x)dx.
\end{aligned}$$

Here, $\bar\rho_0$ is the periodic extension of ρ_0 with period $\mathbb{T}^d$, and G_t is the Gaussian density with covariance $t\sigma$. It follows that $\rho(t,u) := \int_{\mathbb{R}^d} \bar\rho_0(x)G_t(u-x)dx$, as a convolution with the $\bar\rho_0$, satisfies the heat equation

$$\begin{aligned}
\partial_t\rho \;&=\; \sum_{1\leq i,j\leq d} \sigma_{i,j}\partial^2_{u_i,u_j}\rho\\
\rho(0,u) \;&=\; \rho_0(u).
\end{aligned} \tag{2.2}$$

As terminology, we call the equations (2.1) and (2.2) and their solutions $\rho(t,u)$ as 'hydrodynamic' equations and 'hydrodynamic' solutions for the space-time evolution of the macroscopic density. What we have proved is the following:

Theorem 2.2. *Suppose $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ is continuous. Let $v(N) = N$ if $m \neq 0$ and $v(N) = N^2$ if $m = 0$. Then, starting from the sequence $\nu^N_{\rho_0(\cdot)}$, the distribution at time $v(N)t$ is a local equilibrium sequence with respect to profile $\rho(t,u)$ corresponding to equation* (2.1) *if $m \neq 0$ and equation* (2.2) *if $m = 0$.*

3. Notes

The material on Markov chains was based on the development in [3], [8] and [4, Appendix 1]. The discussion about hydrodynamics of independent walks follows closely that in [4, Chapter 1] and [5].

Recent work on invariant measures and hydrodynamics of systems of independent particles includes [2], [6], [7]. The first rigorous work on hydrodynamics of independent particle systems, on which the above development rests, is [1].

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