Hydrodynamic Limit of Interacting Particle Systems

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Abstract

We present in these notes two methods to derive the hydrodynamic equation of conservative interacting particle systems. The intention is to present the main ideas in the simplest possible context and refer to [5] for details and references.

Contents

1 Independent Random Walks

1.1 Equilibrium States

In this chapter we investigate in detail the case of indistinguishable particles moving as discrete time independent random walks. Denote by $\mathbb{Z}$ the set of integers and by $\mathbb{Z}^d$ the d-dimensional lattice. For a positive integer N, denote by $\mathbb{T}_N$ the torus with N points: $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ and let $\mathbb{T}_N^d = (\mathbb{T}_N)^d$. The points of $\mathbb{T}_N^d$, called sites, are represented by the last characters of the alphabet (x, y and z).

To describe the evolution of the system, we begin by distinguishing all particles. Let K denote the total number of particles at time 0 and let $x_1, \dots, x_K$ denote their initial positions. Particles evolve as independent translation invariant discrete time random walks on the torus. Fix a translation invariant transition probability $p(x,y)$ on $\mathbb{Z}^d$: $p(x,y) = p(0, y-x) =: p(y-x)$ for some probability $p(\cdot)$ on $\mathbb{Z}^d$, called the elementary transition probability of the system.

Let $p_t(x,y)$ represent the probability of being at time t on site y for a discrete time random walk with elementary transition probability $p(\cdot)$ starting from x. $p_t(\,\cdot\,,\,\cdot\,)$ inherits the translation invariance property from $p(\,\cdot\,,\,\cdot\,)$: $p_t(x,y) = p_t(0, y-x) =: p_t(y-x)$.

We are now in a position to describe the motion of each particle. Denote by $\{Z_t^i,\ 1 \le i \le K\}$ K independent copies of a discrete time random walk with elementary transition probability $p(\cdot)$ and initially at the origin. For $1 \le i \le K$, let X_t^i represent the position at time t of the i-th particle. We set

$$X_t^i \;=\; x_i \;+\; Z_t^i \mod N\ .$$

Since particles are considered indistinguishable, we are not interested in the individual position of each particle but only in the total number of particles at each site. In particular, the state space of the system, also called configuration space, is $\mathbb{N}^{\mathbb{T}_N^d}$. Configurations are denoted by Greek letters η, ξ and ζ. In this way, for a site x of $\mathbb{T}_N^d$, $\eta(x)$ represents the number of particles at site x for the configuration η. Therefore, if the initial positions are $x_1, \dots, x_K$, for every $x \in \mathbb{T}_N^d$:

$$\eta(x) \;=\; \sum_{i=1}^{K} \mathbf{1}\{x_i = x\}\ .$$

Inversely, given $\{\eta(x);\ x \in \mathbb{T}_N^d\}$, to define the evolution of the system, we can first label all particles and then let them evolve according to the stochastic dynamics described above.

Clearly, if we denote by η_t the configuration at time t, we have

$$\eta_t(x) \;=\; \sum_{i=1}^{K} \mathbf{1}\{X_t^i = x\}\;.$$

The process $(\eta_t)_{t\ge 0}$ inherits the Markov property from the random walks $\{X_t^i,\ 1\le i\le K\}$ because all particles have the same elementary transition probability and they do not interact with each other.

The first question raised in the study of Markov processes is the characterization of all invariant measures. Since the state space is finite and since the total number of particles is the unique quantity conserved by the dynamics, for every positive integer K representing the total number of particles, there is only one invariant measure, as long as the support of the elementary transition probability $p(\,\cdot\,)$ generates $\mathbb{Z}^d$. The Poisson measures will, however, play a central role.

Recall that a Poisson distribution of parameter $\alpha\ge 0$ is the probability measure $\{p_{\alpha,k}=p_k,\ k\ge 1\}$ on $\mathbb{N}$ given by

$$p_k \;=\; e^{-\alpha}\,\frac{\alpha^k}{k!}\,,\quad k\in\mathbb{N},$$

and its Laplace transform is equal to

$$e^{-\alpha}\sum_{k=0}^{\infty} e^{-\lambda k}\,\frac{\alpha^k}{k!} \;=\; e^{-\alpha}\,e^{\alpha e^{-\lambda}} \;=\; \exp\,\alpha(e^{-\lambda}-1)$$

for all positive λ.

For a fixed positive function $\rho\colon \mathbb{T}_N^d\to\mathbb{R}_+$, we call Poisson measure on $\mathbb{T}_N^d$ associated to the function ρ a probability on the configuration space $\mathbb{N}^{\mathbb{T}_N^d}$, denoted by $\nu^N_{\rho(\cdot)}$, having the following two properties. Under $\nu^N_{\rho(\cdot)}$, the random variables $\{\eta(x),\ x\in\mathbb{T}_N^d\}$ representing the number of particles at each site are independent and, for every fixed site $x\in\mathbb{T}_N^d$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho(x)$. In the case where the function ρ is constant equal to α, we denote $\nu^N_{\rho(\cdot)}$ just by ν^N_α. Throughout these note, expectation with respect to a measure ν will be denoted by E_ν.

The measure $\nu^N_{\rho(\cdot)}$ is characterized by its multidimensional Laplace transform:

$$
\begin{aligned}
E_{\nu^N_{\rho(\cdot)}}\Big[\exp\Big\{-\sum_{x\in\mathbb{T}^d_N}\lambda(x)\,\eta(x)\Big\}\Big] \quad &= \prod\nolimits_{x\in\mathbb{T}^d_N}\exp\,\rho(x)\big(e^{-\lambda(x)}-1\big)\\
&= \exp\,\sum\nolimits_{x\in\mathbb{T}^d_N}\rho(x)\,\big(e^{-\lambda(x)}-1\big)
\end{aligned}
$$

for all positive sequences $\{\lambda(x);\ x\in\mathbb{T}^d_N\}$ (cf. [3], Chap. VII).

The first result consists in proving that the Poisson measures associated to constant functions are invariant for a system of independent random walks.

Proposition 1.1 *If particles are initially distributed according to a Poisson measure associated to a constant function equal to α then the distribution at time t is exactly the same Poisson measure.*

Proof: Denote by $\mathbb{P}_{\nu^N_\alpha}$ the probability measure on the path space $\Omega_N = \mathbb{N}^{\mathbb{T}^d_N}\times\mathbb{N}^{\mathbb{T}^d_N}\times\cdots$ induced by the independent random walks dynamics and the initial measure ν^N_α. Expectation with respect to $\mathbb{P}_{\nu^N_\alpha}$ is denoted by $\mathbb{E}_{\nu^N_\alpha}$. Notice the difference between $E_{\nu^N_\alpha}$ and $\mathbb{E}_{\nu^N_\alpha}$. The first expectation is an expectation with respect to the probability measure ν^N_α defined on $\mathbb{N}^{\mathbb{T}^d_N}$, while the second is an expectation with respect to the probability measure $\mathbb{P}_{\nu^N_\alpha}$ defined on the path space Ω_N. In particular, $\mathbb{E}_{\nu^N_\alpha}[F(\eta_0)] = E_{\nu^N_\alpha}[F(\eta)]$ for all bounded continuous function F on $\mathbb{N}^{\mathbb{T}^d_N}$.

Since a probability measure on $\mathbb{N}^{\mathbb{T}^d_N}$ is characterized by its multidimensional Laplace transform, we are naturally led to compute the expectation

$$
\mathbb{E}_{\nu^N_\alpha}\Big[\exp-\sum_{x\in\mathbb{T}^d_N}\lambda(x)\,\eta_t(x)\Big]
$$

for all positive sequences $\{\lambda(x);\ x\in\mathbb{T}^d_N\}$. For a site y in $\mathbb{T}^d_N$, we will denote by $X^{y,k}_t$ the position at time t of the k-th particle initially at y. In this way, the number of particles at site x at time t is equal to

$$
\eta_t(x) \;=\; \sum_{y\in\mathbb{T}^d_N}\sum_{k=1}^{\eta_0(y)}\mathbf{1}\{X^{y,k}_t = x\}\ .
$$

From this formula and inverting the order of summations we obtain the identity

$$\sum_{x\in\mathbb{T}_N^d} \lambda(x)\,\eta_t(x) = \sum_{y\in\mathbb{T}_N^d} \sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k})\ .$$

Since each particle evolves independently and the total number of particles at each site at time 0 is distributed according to a Poisson distribution of parameter α,

$$\begin{aligned}
\mathbb{E}_{\nu_\alpha^N}\Big[\exp - \sum_{x\in\mathbb{T}_N^d} \lambda(x)\,\eta_t(x)\Big] &= \prod_{y\in\mathbb{T}_N^d} \mathbb{E}_{\nu_\alpha^N}\Big[\exp\Big\{ - \sum_{k=1}^{\eta_0(y)} \lambda(X_t^{y,k})\Big\}\Big] \\
&= \prod_{y\in\mathbb{T}_N^d} \int \nu_\alpha^N(d\eta)\Big(E\Big[\exp\big\{ - \lambda(X_t^{y,1})\big\}\Big]\Big)^{\eta_0(y)} \\
&= \prod_{y\in\mathbb{T}_N^d} \exp\Big\{\alpha\Big(E\Big[e^{-\lambda(y+X_t)}\Big] - 1\Big)\Big\}\,,
\end{aligned}$$

where X_t is the position at time t of a random walk on the torus $\mathbb{T}_N^d$ starting from the origin and with transition probability $p_t^N(\,\cdot\,)$ defined by

$$p_t^N(x,y) \ := \ \sum_{z\in\mathbb{Z}^d} p_t(x, y+Nz)$$

for x and y in $\mathbb{T}_N^d$. Since

$$E\Big[e^{-\lambda(y+X_t)}\Big] \ = \ \sum_{x\in\mathbb{T}_N^d} p_t^N(x-y) e^{-\lambda(x)},$$

inverting the order of summation, we obtain that

$$\mathbb{E}_{\nu_\alpha^N}\Big[\exp - \sum_{x\in\mathbb{T}_N^d} \lambda(x)\,\eta_t(x)\Big] \ = \ \exp\Big\{\sum_{x\in\mathbb{T}_N^d} \alpha\left(e^{-\lambda(x)} - 1\right)\Big\}\ .$$

□

Remark 1.2 *Since the total number of particles $\sum_{x\in\mathbb{T}_N^d}\eta(x)$ is conserved by the stochastic dynamics it might seem more natural to consider as reference probability measures the extremal invariant measures that are concentrated* *on the "hyper–planes" of all configurations with a fixed total number of particles. These measures are given by*

$$\nu_{\mathbb{T}_N^d, K}(\,\cdot\,) \;:=\; \nu_\alpha^N\Big(\,\cdot\,\Big|\sum_{x\in\mathbb{T}_N^d}\eta(x)=K\Big)\,.$$

Besides the fact that they enable easier computations, the Poisson distributions present other intrinsic advantages that will be seen in the forthcoming sections.

Notice that only one quantity is conserved by the dynamics: the total number of particles. On the other hand, Poisson distributions are such that their expectation is equal to

$$\sum_{k\geq 0} e^{-\alpha}\,\frac{\alpha^k}{k!}\,k \;=\; \alpha\,.$$

The Poisson measures are in this way naturally parametrized by the density of particles. Furthermore, by the weak law of large numbers, if the number of sites of the set $\mathbb{T}_N^d$ is denoted by $|\mathbb{T}_N^d|$,

$$\lim_{N\to\infty}\frac{1}{|\mathbb{T}_N^d|}\sum_{x\in\mathbb{T}_N^d}\eta(x) \;=\; \alpha$$

in probability with respect to ν_α^N. The parameter α describes therefore the "mean" density of particles in a "large" box.

In conclusion, we obtained above in Proposition (1.1) a one–parameter family of invariant and translation invariant measures indexed by the density of particles, which is the unique quantity conserved by the time evolution.

1.2 Local Equilibrium

We said that the passage from microscopic to macroscopic would be done performing a limit in which the distance between particles converges to zero. This point does not present any difficulty in formalization. We just have to consider the torus $\mathbb{T}_N^d$ as embedded in the d-dimensional torus $\mathbb{T}^d=[0,1)^d$, that is, taking the lattice $\mathbb{T}^d$ with "vertices" x/N, $x\in\mathbb{T}_N^d$. In this way the distances between molecules is $1/N$ and tends to zero as $N\uparrow\infty$.

We shall refer to $\mathbb{T}^d$ as the macroscopic space and to $\mathbb{T}_N^d$ as the microscopic space. In this way each macroscopic point u in $\mathbb{T}^d$ is associated to a microscopic site $x = [uN]$ in $\mathbb{T}_N^d$ and, reciprocally, each site x is associated to a macroscopic point x/N in $\mathbb{T}^d$. Here and below, for a d-dimensional real $r = (r_1, \dots, r_d)$, $[r]$ denotes the integer part of r: $[r] = ([r_1], \dots, [r_d])$.

On the other hand, since we have a one–parameter family of invariant measures, one way to describe a local equilibrium with density profile $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}_+$ is the following. We distribute particles according to a Poisson measure with slowly varying parameter on $\mathbb{T}_N^d$, that is, for each positive N we fix the parameter of the Poisson distribution at site x to be equal to $\rho_0(x/N)$. Since this type of measure will appear frequently in these notes, we introduce the following terminology.

Definition 1.3 *For each smooth function $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}_+$, we represent by $\nu^N_{\rho_0(\cdot)}$ the measure on the state space $\Sigma_{\mathbb{T}_N^d} = \mathbb{N}^{\mathbb{T}_N^d}$ having the following two properties. Under $\nu^N_{\rho_0(\cdot)}$ the variables $\{\eta(x);\ x \in \mathbb{T}_N^d\}$ are independent and, for a site $x \in \mathbb{T}_N^d$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho_0(x/N)$:*

$$\nu^N_{\rho_0(\cdot)}\{\eta;\ \eta(x) = k\} \;=\; \nu^N_{\rho_0(x/N)}\{\eta;\ \eta(0) = k\}$$

for all x in $\mathbb{T}_N^d$ and k in $\mathbb{N}$.

We have thus associated to each profile $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}_+$ and each positive integer N a Poisson measure on the torus $\mathbb{T}_N^d$.

As the parameter N increases to infinity, the discrete torus $\mathbb{T}_N^d$ tends to the full lattice $\mathbb{Z}^d$. We can also define a Poisson measure on the space of configurations over $\mathbb{Z}^d$. For each $\alpha \ge 0$ we will denote by ν_α the probability on $\mathbb{N}^{\mathbb{Z}^d}$ that makes the variables $\{\eta(x);\ x \in \mathbb{Z}^d\}$ independent and under which, for every x in $\mathbb{Z}^d$, $\eta(x)$ is distributed according to a Poisson law of parameter α.

With the definition we have given of $\nu^N_{\rho_0(\cdot)}$, and since $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ is assumed to be smooth, as $N \uparrow \infty$ and we look "close" to a point $u \in \mathbb{T}^d$ – that is "around" $x = [Nu]$ – we observe a Poisson measure of parameter (almost) constant equal to $\rho_0(u)$. In fact, since the function $\rho_0(\cdot)$ is smooth, for every positive integer ℓ and for every positive family of parameters $\{\lambda(x);\ |x| \le \ell\}$,

$$\lim_{N\to\infty} E_{\nu^N_{\rho_0(\cdot)}}\Big[e^{-\sum_{|x|\le \ell}\lambda(x)\eta([uN]+x)}\Big] \;=\; E_{\nu_{\rho_0(u)}}\Big[e^{-\sum_{|x|\le\ell}\lambda(x)\eta(x)}\Big]. \qquad (1.1)$$

In this formula and throughout these notes, for $u = (u_1, \dots, u_d)$ in $\mathbb{R}^d$, $\|u\|$

denotes the Euclidean norm of u and $|u|$ the max norm:

$$\|u\|^2 \;=\; \sum_{1\le i\le d} u_i^2\;, \qquad |u| \;=\; \max_{1\le i\le d} |u_i|\;.$$

In this sense the sequence $\nu^N_{\rho_0(\cdot)}$ describes an example of local equilibrium. This definition of product measure with slowly varying parameter is of course too restrictive. We refer to [5] for some generalization.

In the configuration space $\mathbb{N}^{\mathbb{T}^d_N}$, endowed with its natural discrete topology, we denote by $\{\tau_x,\ x \in \mathbb{T}^d_N\}$ the group of translations. Thus, for a site x, $\tau_x\eta$ is the configuration that, at site y, has $\eta(x+y)$ particles:

$$(\tau_x\eta)(y) \;=\; \eta(y+x), \quad y \in \mathbb{T}^d_N.$$

The action of the translation group extends in a natural way to the space of functions and to the space of probability measures on $\mathbb{N}^{\mathbb{T}^d_N}$. In fact, for a site x and a probability measure μ, $(\tau_x\mu)$ is the measure such that

$$\int f(\eta)(\tau_x\mu)(d\eta) \;=\; \int f(\tau_x\eta)\mu(d\eta),$$

for every bounded continuous function f.

To perform the limit $N \uparrow \infty$ we embed the space $\mathbb{N}^{\mathbb{T}^d_N}$ in $\mathbb{N}^{\mathbb{Z}^d}$ identifying a configuration on the torus to a periodic configuration on the full lattice. We will endow the configuration space $\mathbb{N}^{\mathbb{Z}^d}$ with its natural topology, the product topology. By $\mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^d})$ or simply by $\mathcal{M}_1$, we represent the space of probability measures on $\mathbb{N}^{\mathbb{Z}^d}$ endowed with the weak topology.

In this topological setting, formula (1.1) establishes that for all points u of $\mathbb{T}^d$ the sequence $\tau_{[uN]}\nu^N_{\rho_0(\cdot)}$ converges weakly to the measure $\nu_{\rho_0(u)}$.

1.3 Hydrodynamic Equation

We turn now to the study of the distribution of particles at a later time t starting from a product measure with slowly varying parameter. Repeating the computations we did to prove Proposition (1.1) we see that if we start from a Poisson measure with slowly varying parameter then

$$
\begin{aligned}
&\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\Big(\exp - \sum_{x\in\mathbb{T}^d_N} \lambda(x)\,\eta_t(x) \Big) \\
&\quad = \exp \sum_{x\in\mathbb{T}^d_N} \rho_0(x/N) \sum_{y\in\mathbb{T}^d_N} p^N_t(y-x)\,\big(e^{-\lambda(y)} - 1\big) \\
&\quad = \exp \sum_{y\in\mathbb{T}^d_N} \big(e^{-\lambda(y)} - 1\big) \sum_{x\in\mathbb{T}^d_N} p^N_t(y-x)\,\rho_0(x/N) \\
&\quad =: \exp \sum_{y\in\mathbb{T}^d_N} \big(e^{-\lambda(y)} - 1\big)\,\psi_{N,t}(y)\ .
\end{aligned}
$$

Therefore, at time t, we still have a Poisson measure with slowly varying parameter, which is now $\psi_{N,t}(\cdot)$ instead of $\rho_0(\cdot/N)$. Up to this point we have not used the particular form of $p_t(\cdot)$ besides the fact that it makes $p_t(\cdot,\cdot)$ translation invariant and thus bistochastic: $\sum_x p_t(x,y) = 1$ for every y. We shall now see what happens when t is fixed and N increases to infinity. In this case $p_t(\cdot)$ is a function with essentially finite support, that is, for all $\varepsilon > 0$, there exists $A = A(t,\varepsilon) > 0$ so that

$$
\sum_{|x|\le A} p_t(x) \ \ge\ 1 - \varepsilon\ .
$$

From the explicit form of $\psi_{N,t}$, we have that for every continuity point u of ρ_0,

$$
\lim_{N\to\infty} \psi_{N,t}([uN]) \ =\ \rho_0(u)\ .
$$

The profile remained unchanged. The system did not have time to evolve and this reflects the fact that at the macroscopic scale particles did not move. Indeed, consider a test particle initially at the origin. Since it evolves as a discrete time random walk, if X_t denotes its position at time t, for every $\varepsilon > 0$, there exists $A = A(t,\varepsilon) > 0$ such that $P[\,|X_t| > A] \le \varepsilon$. Therefore, with probability close to 1, in the macroscopic scale, the test particle at time t is at distance of order N^{-1} from the origin. In a fluid, however, a "test" particle traverses a macroscopic distance in a macroscopic time.

We solve this problem distinguishing between two time scales (as we have two space scales: $\mathbb{T}^d$ and $N^{-1}\,\mathbb{T}^d_N$): a microscopic time t and a macroscopic time which is infinitely large with respect to t.

To introduce the macroscopic time scale, notice that the transition probabilities $p_t(\,\cdot\,)$ are equal to

$$p_t(x) \;=\; p^t(x)\;,$$

where p^t stands for the k-th convolution power of the elementary transition probability of each particle.

Assume that the elementary transition probability $p(\,\cdot\,)$ has finite expectation: $m := \sum x\,p(x) \in \mathbb{R}^d$. We say that the random walk is *asymmetric* if $m \neq 0$, that it is *mean–zero asymmetric* if $p(\cdot)$ is not symmetric but $m = 0$ and that it is *symmetric* if $p(\cdot)$ is symmetric. Recall that X_t stands for the position at time t of a discrete time random walk with transition probability $p(\cdot)$ and initially at the origin. By the law of large numbers for random walks, for all $\varepsilon > 0$,

$$\lim_{N\to\infty} \sum_{x;\;|x/N-mt|\le\varepsilon} p_{tN}(x) \;=\; \lim_{N\to\infty} P\Big[\Big|\frac{X_{tN}}{N} - mt\Big| \,\le\, \varepsilon\Big] \;=\; 1\,.$$

In particular, from the explicit expression for $\psi_{N,tN}$ and since we assumed the initial profile to be smooth, we have that

$$\lim_{N\to\infty} \psi_{N,tN}([uN]) \;=\; \rho_0(u-mt) \;=:\; \rho(t,u)$$

for every u in $\mathbb{T}^d$.

We obtained in this way a new time scale, tN, in which we observe a new macroscopic profile: the original one translated by mt. More precisely, in this macroscopic scale tN we observe a local equilibrium profile that has been translated by mt since $\psi_{N,tN}$ is itself slowly varying in the macroscopic scale.

Of course, the profile $\rho(t,u)$ satisfies the partial differential equation

$$\partial_t\rho \;+\; m\cdot\nabla\rho \;=\; 0 \tag{1.2}$$

if $\nabla\rho$ denotes the gradient of ρ: $\nabla\rho = (\partial_{u_1}\rho, \dots, \partial_{u_d}\rho)$.

In conclusion, if we restrict ourselves to a particular class of initial measures, we have established the existence of a time and space scales in which the particles density evolves according to the linear partial differential equation (1.2). We have thus derived from the microscopic stochastic dynamics a macroscopic deterministic evolution for the unique conserved quantity.

An interacting particle system for which there exists a time and space macroscopic scales in which the conserved quantities evolve according to some partial differential equation is said to have a hydrodynamic description. Moreover, the P.D.E. is called the hydrodynamic equation associated to the system.

We summarize this result in the following proposition.

Proposition 1.4 *A system of particles evolving as independent asymmetric random walks with finite first moment on a d-dimensional torus has a hydrodynamic description. The evolution of the density profile is described by the solution of the differential equation*

$$\partial_t \rho \;+\; m \cdot \nabla \rho \;=\; 0 \,.$$

When the expectation m vanishes, the solution of this differential equation is constant, which means that the profile didn't change in the time scale tN. Nothing imposed, however, the choice of tN as macroscopic time scale. In fact, when the mean displacement m vanishes, to observe an interesting time evolution, we need to consider a larger time scale, times of order N^2.

Assume that the elementary transition probability that describes the displacement of each particle has a second moment. Let $\sigma = (\sigma_{i,j})_{1\le i,j\le d}$ be the covariance matrix of this distribution:

$$\sigma_{i,j} = \sum_{x\in\mathbb{T}_N^d} x_i\, x_j\, p(x), \quad 1\le i,j\le d\,.$$

By the central limit theorem for random walks, we see that

$$\begin{aligned}\lim_{N\to\infty} \psi_{N,N^2t}([Nu]) \;&=\; \lim_{N\to\infty} \sum_{x\in\mathbb{T}_N^d} p_t^N([Nu]-x)\rho_0(x/N)\\ &=\; \lim_{N\to\infty} E\Big[\rho_0(u - N^{-1}X_{tN^2})\Big] \;=\; \int_{\mathbb{R}^d} \overline{\rho_0}(\theta) G_t(u-\theta)\, d\theta\,,\end{aligned}$$

where $\overline{\rho_0}\colon \mathbb{R}^d \to \mathbb{R}_+$ is the periodic function, with period $\mathbb{T}^d$ and equal to ρ_0 on the torus $\mathbb{T}^d$ and G_t is the density of the Gaussian distribution with covariance matrix $t\,\sigma$.

Since the Gaussian distribution is the fundamental solution of the heat equation (which can be checked by a simple computation) we obtain the following result.

Proposition 1.5 *A system of particles evolving as independent mean–zero asymmetric random walks with finite second moment on a d-dimensional torus has a hydrodynamic description. The evolution of the density profile is described by the solution of the differential equation*

$$\begin{cases} \partial_t \rho \;=\; \sum_{1\le i,j\le d} \sigma_{i,j}\, \partial^2_{u_i,u_j} \rho \;, \\ \rho(0,u) \;=\; \rho_0(u) \;. \end{cases}$$

Let $\{S^N(t),\, t \ge 0\}$ be the semigroup on $\mathcal{M}_1$ associated to the Markov process $(\eta_t)_{t\ge 0}$. In Propositions (1.4) and (1.5), we have proved that there is a time renormalization θ_N such that

$$\lim_{N\to\infty} S^N(t\theta_N)\, \tau_{[uN]}\, \nu^N_{\rho_0(\cdot)} \;=\; \nu_{\rho(t,u)} \;,$$

for all $t \ge 0$ and all continuity points u of $\rho(t,\cdot)$.

Usually $\rho(t,\cdot)$ is the solution of a Cauchy problem with initial condition $\rho_0(\cdot)$. As we said earlier, this differential equation is called the hydrodynamic equation of the interacting particle system.

In this section we took advantage of several special features of the evolution of independent random walks to obtain an explicit formula for the profile $\rho(t,\cdot)$. The type of result, however, is characteristic of the subject. We have proved:

1. Description of the equilibrium states of the system.

2. Conservation of the local equilibrium in time evolution.

3. Characterization at a later time of the new parameters describing the local equilibrium and derivation of a partial differential equation that determines how the parameters evolve in time.

The aim of these notes is to present some general methods which lead to the derivation of a weak version of the conservation of local equilibrium for a class of interacting particle systems. We would like in fact to prove a more general result, that is, one for initial states that are not product measures with slowly varying parameter – thus without assuming a strong form of local equilibrium at time 0 – but for initial states having a density profile and imposing that it is not too far, in a sense to be defined later, from a local equilibrium; the process establishing by itself a local equilibrium at later times.

2 Exclusion Processes

2.1 Exclusion Processes

We present in these notes the theory of the hydrodynamic behavior of interacting particle system in the context of exclusion processes. In contrast with superpositions of random walks presented in Chapter 1, the exclusion process allows at most one particle per site.

The state space is therefore $\{0,1\}^{\mathbb{T}_N^d}$.

Fix a positive integer A and a finite collection of local *strictly positive* functions $\{a_z : |z| \leq A\}$. Consider the Markov process on $\{0,1\}^{\mathbb{T}_N^d}$ with generator L_N given by

$$(L_N f)(\eta) \;=\; \sum_{x,z\in\mathbb{T}_N^d} \eta(x)\,[1-\eta(x+z)]\, a_{x,x+z}(\eta)\,[f(\sigma^{x,x+z}\eta) - f(\eta)]\;, \tag{2.1}$$

where $a_{x,x+z}(\eta) = a_z(\tau_x\eta)$ and $\sigma^{x,y}\eta$ is the configuration obtained from η exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) \;=\; \begin{cases} \eta(z) & \text{if } z \neq x,y\;, \\ \eta(y) & \text{if } z = x\;, \\ \eta(x) & \text{if } z = y\;. \end{cases} \tag{2.2}$$

The exclusion processes can be divided in two types. The first ones, in which the rates $a_z(\eta)$ do not depend on the configurations, are called simple exclusion processes and may be themselves subdivided in three categories:

1. **Symmetric simple exclusion process**: $a_z(\eta) = p(z)$, where $p : \mathbb{Z}^d \to \mathbb{R}_+$ is a positive finite range symmetric function.

2. **Zero-mean asymmetric simple exclusion process**: $a_z(\eta) = p(z)$ and $p : \mathbb{Z}^d \to \mathbb{R}_+$ is a positive finite range asymmetric function such that $\sum_{z\in\mathbb{Z}^d} zp(z) = 0$.

3. **Asymmetric simple exclusion process**: $a_z(\eta) = p(z)$ and $p : \mathbb{Z}^d \to \mathbb{R}_+$ is a positive finite range asymmetric function such that $\sum_{z\in\mathbb{Z}^d} zp(z) \neq 0$.

We will only consider a sub-class of the second category of exclusion processes, interacting systems in which the rates

$$\begin{gathered} a_z(\eta) = a_{-z}(\tau_z\eta) \text{ for all } z, \\ a_z(\eta) \text{ does not depend on } \eta(0),\, \eta(z). \end{gathered} \tag{2.3}$$

Notice that $a_{-z}(\tau_z \eta) = a_{z,0}(\eta)$ is the rate at which a particle jumps from z to the origin when the configuration is η. This formula is thus saying that $a_z(\eta)$, the rate at which a particle is jumping from the origin to z, is equal to the rate at which a particle jumps from z to the origin.

We always assume the process to be irreducible, which means that the set $\{z \, , a_z(\eta) > 0\}$ generates $\mathbb{Z}^d$, i.e., for any pair of sites x, y in $\mathbb{Z}^d$, there exists $M \geq 1$ and a sequence $x = x_0, \ldots, x_M = y$ such that $a_{x_{i+1} - x_i} > 0$ for $0 \leq i \leq M-1$.

We denote by $\{S^N(t), \ t \geq 0\}$ the semigroup of the Markov process with generator L_N. We use the same notation for semigroups acting on continuous functions or on the space $\mathcal{M}_1(\{0,1\}^{\mathbb{T}_N^d})$ of probability measures on $\{0,1\}^{\mathbb{T}_N^d}$.

For $0 \leq \alpha \leq 1$, we denote by $\nu_\alpha = \nu_\alpha^N$ the Bernoulli product measure of parameter α, that is, the product measure on $\{0,1\}^{\mathbb{T}_N^d}$ with density α. Under ν_α, the variables $\{\eta(x), \ x \in \mathbb{T}_N^d\}$ are independent with marginals given by

$$\nu_\alpha\{\eta(x) = 1\} \ = \ \alpha \ = \ 1 - \nu_\alpha\{\eta(x) = 0\} \, .$$

Proposition 2.1 *The Bernoulli measures $\{\nu_\alpha \, , 0 \leq \alpha \leq 1\}$ are invariant for simple exclusion processes. With respect to each ν_α, simple exclusion processes with elementary jump probability $\check{p}(z) := p(-z)$ are adjoint to processes with elementary jump probability $p(z)$. Symmetric simple exclusion processes and exclusion processes satisfying (2.3) are self-adjoint with respect to each ν_α.*

The proof is elementary and can be found in [5]. Notice that the family of invariant measures ν_α is parametrized by the density, for

$$E_{\nu_\alpha}[\eta(0)] \ = \ \nu_\alpha\{\eta(0) = 1\} \ = \ \alpha \, .$$

Since the total number of particles is conserved by the dynamics the measures

$$\nu_{N,K}(\,\cdot\,) \ := \ \nu_\alpha \left(\cdot \,\Big|\, \sum_{x \in \mathbb{T}_N^d} \eta(x) = K \right)$$

are invariant and it could have seemed more natural to consider them instead of the Bernoulli product measures ν_α. Nevertheless, a simple computation on binomials shows that for all finite subsets E of $\mathbb{Z}^d$, for all sequences

$\{\varepsilon_x;\ x \in E\}$ with values in $\{0,1\}$ and for all $0 \le \alpha \le 1$,

$$\lim_{N\to\infty} \nu_\alpha\Big\{\eta(x) = \varepsilon_x\ ,\ x \in E \Big|\ \sum_{y\in\mathbb{T}_N^d} \eta(y) = [\alpha_0 N^d]\Big\}$$
$$=\ \nu_{\alpha_0}\big\{\eta(x) = \varepsilon_x\ ,\ x \in E\big\}$$

uniformly in α_0. Therefore, the Bernoulli product measures are obtained as limits of the invariant measures $\nu_{N,K}$, as the total number of sites increases to infinity.

For each $0 \le K \le N^d$, denote by $\Sigma^1_{N,K}$ the "hyperplanes" of all configurations with K particles:

$$\Sigma^1_{N,K}\ =\ \Big\{\eta \in \{0,1\}^{\mathbb{T}_N^d};\ \sum_{x\in\mathbb{T}_N^d} \eta(x) = K\Big\}\ .$$

Invariant measures of density preserving particle systems that are concentrated on hyperplanes with a fixed total number of particles are called canonical measures (the family $\{\nu_{N,K},\ 0 \le K \le N^d\}$ in the context of simple exclusion processes, for instance). In contrast, the measures obtained as weak limits of the canonical measures, as the number of sites increases to infinity, are called the grand canonical measures (here the Bernoulli measures as we have just seen above).

2.2 Hydrodynamic Limit

In Chapter 1 we presented a quite restrictive definition of local equilibrium associated to a density profile. We required the marginal probabilities of the state of the system at a macroscopic time t, that is, of $S^N(t\theta_N)\mu^N$, to converge to one of the extremal invariant and translation invariant measures of the infinite system at each continuity point of the profile. Even if this definition is "physically" natural, it is difficult to prove it. With this in mind, we introduce a weaker notion of local equilibrium. We first recall the notion of product measures with slowly varying parameter associated to a profile.

Definition 2.2 *Given a continuous profile $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}_+$, we denote by $\nu^N_{\rho_0(\cdot)}$ the product measure on $\{0,1\}^{\mathbb{T}_N^d}$ with marginals given by*

$$\nu^N_{\rho_0(\cdot)}\{\eta,\, \eta(x) = k\}\ =\ \nu^N_{\rho_0(x/N)}\{\eta,\, \eta(0) = k\}$$

for all x in $\mathbb{T}_N^d$, $k = 0, 1$. This measure is called the product measure with slowly varying parameter associated to $\rho_0(\cdot)$.

Measures on $\{0,1\}^{\mathbb{Z}^d}$ are characterized by the way they integrate cylinder functions. In order to keep notation simple, to each cylinder function Ψ in $\{0,1\}^{\mathbb{Z}^d}$ we associate the real bounded function $\tilde{\Psi} : \mathbb{R}_+ \to \mathbb{R}$ that at α is equal to the expected value of Ψ under ν_α:

$$\tilde{\Psi}(\alpha) \; : \; = \; E_{\nu_\alpha}[\Psi] \; = \; \int \Psi(\eta)\, \nu_\alpha(d\eta)\; . \tag{2.4}$$

Since Ψ is a local function, $\tilde{\Psi}$ is a polynomial.

In the same way that we have defined in Chapter 1 translations of configurations, for all continuous functions Ψ, we denote by $\tau_x \Psi$ the translation of Ψ by x units:

$$(\tau_x \Psi)(\eta) \;=\; \Psi(\tau_x \eta)$$

for every configuration η.

By Chebychev inequality and the dominated convergence theorem, for every smooth profile $\rho_0 \colon \mathbb{T}^d \to \mathbb{R}_+$ and every sequence $\nu^N_{\rho_0(\cdot)}$ of Poisson measures with slowly varying parameter associated to this profile,

$$\lim_{N\to\infty} \nu^N_{\rho_0(\cdot)} \Big[\Big| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} G(x/N)\, (\tau_x \Psi)(\eta) \;-\; \int_{\mathbb{T}^d} G(u)\;\; \tilde{\Psi}(\rho_0(u))\, du \Big| \;>\; \delta \Big] \;=\; 0$$

for all continuous functions $G \colon \mathbb{T}^d \to \mathbb{R}$, all bounded cylinder functions Ψ and all strictly positive δ.

This last statement asserts that the sequence of measures $\nu^N_{\rho_0(\cdot)}$ integrates the cylinder function Ψ around the macroscopic point u in $\mathbb{T}^d$ in the same way as an equilibrium measure of density $\rho_0(u)$ would do. We do not require anymore the marginals of the sequence of measures to converge to an extremal equilibrium measure at every continuity point of the profile ρ_0. We only impose that its spatial mean converges to the corresponding spatial mean. This notion is therefore weaker than the one of local equilibrium introduced in Chapter 1. The main difference is the spatial average over small macroscopic boxes of volume of order N^d that is implicit in this new concept and absent in the definition of local equilibrium.

The function $\Psi(\eta) = \eta(0)$ plays a special role in the whole study of hydrodynamics being closely related to the conserved quantity. The definition that follows is a particularly weak notion since it demands only convergence for this cylinder function.

Definition 2.3 *A sequence $(\mu^N)_{N\geq 1}$ of probability measures on $\{0,1\}^{\mathbb{T}_N^d}$ is associated to a profile $\rho_0\colon \mathbb{T}^d \to \mathbb{R}_+$ if for every continuous function $G\colon \mathbb{T}^d \to \mathbb{R}$, and for every $\delta > 0$ we have*

$$\lim_{N\to\infty} \mu^N \Big[\Big| N^{-d} \sum_{x\in\mathbb{T}_N^d} G(x/N)\eta(x) \; - \; \int_{\mathbb{T}^d} G(u)\,\rho_0(u)\,du \Big| > \delta \Big] = 0\,.$$

The quantity just introduced in the definition above can be reformulated in terms of empirical measures. Let π^N be the positive measure on the torus $\mathbb{T}^d$ obtained by assigning to each particle a mass N^{-d}:

$$\pi^N(\eta, du) \; : \; = \; N^{-d} \sum_{x\in\mathbb{T}_N^d} \eta(x)\,\delta_{x/N}(du). \tag{2.5}$$

In this formula, for a d-dimensional vector u, δ_u represents the Dirac measure concentrated on u. The measure $\pi^N(\eta, du)$ is called the empirical measure associated to the configuration η. The dependence in η will frequently be omitted to keep notation as simple as possible. With this notation $N^{-d}\sum_{x\in\mathbb{T}_N^d} G(x/N)\eta(x)$ is the integral of G with respect to the empirical measure π^N, denoted by $<\pi^N, G>$.

Let $\mathcal{M}_+(\mathbb{T}^d)$ be the space of finite positive measures on the torus $\mathbb{T}^d$ endowed with the weak topology. Notice that for each N, π^N is a continuous function from $\mathbb{N}^{\mathbb{T}_N^d}$ to $\mathcal{M}_+$. For a probability measure μ on $\mathbb{N}^{\mathbb{T}_N^d}$, throughout these notes, we also denote by μ the measure $\mu(\pi^N)^{-1}$ on $\mathcal{M}_+$ induced by μ and π^N $(\mu(\pi^N)^{-1}[A] = \mu[\pi^N \in A])$. With this convention, a sequence of probability measures $(\mu^N)_{N\geq 1}$ in $\mathbb{N}^{\mathbb{T}_N^d}$ is associated to a density profile ρ_0 if the sequence of random measures $\pi^N(du)$ converges in probability to the deterministic measure $\rho_0(u)du$.

In the study of the hydrodynamic behavior of interacting particle systems we will sometimes be forced to reduce our goals and to content ourselves in proving that starting from a sequence of measures associated to a density profile ρ_0 then, at a later suitably renormalized time, we obtain a state ($S^N(t\theta_N)\mu^N$ in the notation of Chapter 1) associated to a new density profile $\rho(t,\cdot)$ which is the solution of some partial differential equation. More precisely, we want to prove that for all sequences of probability measures

μ^N associated to a profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ and not too far, in a sense to be specified later, from a global equilibrium ν_α^N,

$$\lim_{N\to\infty} \mu^N \Big[\Big| N^{-d} \sum_{x\in\mathbb{T}_N^d} G(x/N)\eta_{t\theta_N}(x) \; - \; \int_{\mathbb{T}^d} G(u)\,\rho(t,u)\,du \Big| > \delta \Big] = 0$$

for a suitable renormalization θ_N, for every continuous function $G\colon \mathbb{T}^d \to \mathbb{R}$ and every δ strictly positive. As in the case of local equilibrium, in this formula $\rho(\cdot,\cdot)$ will be the solution of a Cauchy problem with initial condition $\rho_0(\cdot)$. This constitutes the program of the next chapters.

2.3 The Approach

In this section we prove the hydrodynamic behavior of nearest neighbor symmetric simple exclusion processes and show that the hydrodynamic equation is the heat equation:

$$\partial_t \rho \;=\; (1/2)\,\Delta\rho\;.$$

In this formula, $\Delta\rho$ stands for the Laplacian of ρ: $\Delta\rho = \sum_{1\le i\le d} \partial_{u_i}^2 \rho$.

We briefly present the strategy of the proof. We first show that the empirical measure solves the heat equation in a weak sense in an integral form. More precisely, for a positive measure π on $\mathbb{T}^d$ of finite total mass and for a continuous function $G\colon \mathbb{T}^d \to \mathbb{R}$, denote by $<\pi, G>$ the integral of G with respect to π:

$$<\pi, G> = \int_{\mathbb{T}^d} G(u)\,\pi(du)\;.$$

We shall prove that the empirical measure π_t^N associated to the symmetric simple exclusion process converges, in a way to be specified later, to a measure π_t absolutely continuous with respect to the Lebesgue measure and satisfying:

$$<\pi_t, G> = <\pi_0, G> \;+\; (1/2)\int_0^t <\pi_s, \Delta G>\,ds \tag{2.6}$$

for a sufficiently large class of functions $G : \mathbb{T}^d \to \mathbb{R}$ and for every t in an interval $[0,T]$ fixed in advance.

Recall that we denoted by $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{T}^d)$ the space of finite positive measures on $\mathbb{T}^d$ endowed with the weak topology. In order to work in a fixed space as N increases, we consider the time evolution of the empirical measure π_t^N associated to the particle system defined by:

$$\pi_t^N(du) \;=\; \pi^N(\eta_t, du) \;:=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N}(du)\;. \tag{2.7}$$

Notice that there is a one to one correspondence between configurations η and empirical measures $\pi^N(\eta, du)$. In particular, π_t^N inherits the Markov property from η_t.

We consider the distribution of the empirical measure as a sequence of probability measures on a fixed space. Since there are jumps this space must be $D([0,T], \mathcal{M}_+)$, the space of right continuous functions with left limits taking values in $\mathcal{M}_+$.

Fix a profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and denote by μ^N a sequence of probability measures associated to ρ_0. For each $N \geq 1$, let Q^N be the probability measure on $D([0,T], \mathcal{M}_+)$ corresponding to the Markov process π_t^N speeded up by N^2 and starting from μ^N. We speeded up the process by N^2 because we have seen in Chapter 1 that to obtain a non trivial hydrodynamic evolution for mean-zero processes we need to consider time scales of order N^2.

Our goal is to prove that, for each fixed time t, the empirical measure π_t^N converges in probability to $\rho(t,u)du$ where $\rho(t,u)$ is the solution of the heat equation with initial condition ρ_0. We shall proceed in two steps. We first prove that the process π_t^N converges in distribution to the probability measure concentrated on the deterministic path $\{\rho(t,u)du,\ 0 \leq t \leq T\}$ and then argue that convergence in distribution to a deterministic weakly continuous trajectory implies convergence in probability at any fixed time $0 \leq t \leq T$.

A deterministic trajectory can be interpreted as the support of a Dirac probability measure on $D([0,T], \mathcal{M}_+)$ concentrated on this trajectory. The proof of the hydrodynamic behavior of symmetric simple exclusion processes is therefore reduced to show the convergence of the sequence of probability measures Q^N to the Dirac measure concentrated on the solution of the heat equation.

An indirect standard method to prove the convergence of a sequence is to show that this sequence is relatively compact and then to show that all converging subsequences converge to the same limit. To show the relative compactness we will use Prohorov's criterion. At this point it will remain the identification of all limit points of subsequences.

To characterize all limit points of the sequence Q^N, we have to investigate how we may use the random evolution to make an equation of type (2.6) appear. Notice first that, under Q^N, for every function $G\colon \mathbb{T}^d \to \mathbb{R}$, the quantities

$$< \pi_t^N, G > = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G(x/N)\, \eta_t(x) \tag{2.8}$$

verify the identity

$$< \pi_t^N, G > = < \pi_0^N, G > + \int_0^t N^2 L_N < \pi_s^N, G > ds + M_t^{G,N}$$

where $M_t^{G,N}$ are martingales with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta_s,\, s \le t)$. The factor N^2 in front of the generator L_N appears because we speeded up the process by N^2. In the particular case of nearest neighbor symmetric simple exclusion processes, the second term on the right hand side may be rewritten as a function of the empirical measure. Indeed, applying the generator to the function $\eta \to \eta(x)$ we have:

$$L_N\, \eta(x) = (1/2) \sum_{j=1}^{d} \Big[\eta(x + e_j) + \eta(x - e_j) - 2\eta(x) \Big] .$$

After two summations by parts we obtain that under Q^N

$$< \pi_t^N, G > = < \pi_0^N, G > + (1/2) \int_0^t < \pi_s^N, \Delta_N G > ds + M_t^{G,N}$$

where Δ_N is the discrete Laplacian:

$$(\Delta_N G)(x/N) = N^2 \sum_{j=1}^{d} \Big[G\big((x + e_j)/N\big) + G\big((x - e_j)/N\big) - 2G(x/N) \Big] .$$

To conclude the proof of the hydrodynamic behavior of symmetric simple exclusion processes, it remains to show, on the one hand, an uniqueness theorem for solutions of equations (2.6); uniqueness theorem that will require to prove identity (2.6) for a certain class of functions G; and, on the other hand, to prove that the martingales $M_t^{G,N}$ vanish in the limit as $N \uparrow \infty$ for this family of functions G. From these two results it follows that the sequence Q^N has a unique limit point Q^* which is the probability measure concentrated on the unique solution of (2.6).

2.4 A Rigorous Proof

We may now turn to a rigorous proof.

Theorem 2.4 *Fix a profile $\rho_0 : \mathbb{T}^d \to [0,1]$ and let μ^N be the sequence of Bernoulli product measures of slowly varying parameter associated to the profile ρ:*

$$\mu^N\{\eta;\, \eta(x) = 1\} \;=\; \rho_0(x/N)\,, \quad x \in \mathbb{T}_N^d\,.$$

Then, for every $t > 0$, the sequence of random measures

$$\pi_t^N(du) \;=\; \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N}(du)$$

converges in probability to the absolutely continuous measure $\pi_t(du) = \rho(t,u)$ du whose density is the solution of the heat equation:

$$\begin{cases} \partial_t \rho \;=\; (1/2)\,\Delta\rho \\ \rho(0,\cdot) \;=\; \rho_0(\cdot)\,. \end{cases} \tag{2.9}$$

We start fixing a time $T > 0$ and considering the sequence of probability measures Q^N on $D([0,T],\mathcal{M}_+)$ corresponding to the Markov process π_t^N, defined by (2.7), speeded up by N^2 and starting from μ^N.

First step: Relative compactness. The first step in the proof of the hydrodynamic behavior consists in showing that the sequence Q^N is relatively compact. We skip this step here and refer the reader to [5] for details.

Second step: Uniqueness of limit points. Now that we know that the sequence Q^N is weakly relatively compact, it remains to characterize all limit points of Q^N. Let Q^* be a limit point and let Q^{N_k} be a sub-sequence converging to Q^*.

We first prove that Q^* is concentrated on absolutely continuous measures with respect to the Lebesgue measure. Since there is at most one particle per site,

$$\sup_{0\le t\le T} \Big| < \pi_t^N, G > \Big| \;\le\; N^{-d} \sum_{x\in\mathbb{T}_N^d} |G(x/N)|\,.$$

Since, on the other hand, for fixed continuous functions G, the application which associates to a trajectory π the value $\sup_{0\le t\le T} | < \pi_t, G > |$ is continuous, by weak convergence, all limit points are concentrated on trajectories π_t such that

$$\Big| < \pi_t, G > \Big| \;\le\; \int |G(u)|\, du$$

for all continuous function G and for all $0 \le t \le T$. All limit points are thus concentrated on absolutely continuous trajectories with respect to the Lebesgue measure:

$$Q^* \left[\pi;\ \pi_t(du) = \pi_t(u)du\right] \ = \ 1 \ .$$

All limit points of the sequence Q^N are concentrated on trajectories that at time 0 are equal to $\rho_0(u)du$. Indeed, by weak convergence, for every $\varepsilon > 0$,

$$\begin{aligned} & Q^* \Big[\Big| N^{-d} \sum_{x \in \mathbb{T}_N^d} G(x/N) \eta_0(x) \ - \ \int G(u) \rho_0(u)\, du \Big| > \varepsilon \Big] \\ & \quad \le \ \liminf_{k \to \infty} Q^{N_k} \Big[\Big| N^{-d} \sum_{x \in \mathbb{T}_N^d} G(x/N) \eta_0(x) \ - \ \int G(u) \rho_0(u)\, du \Big| > \varepsilon \Big] \\ & \quad = \ \lim_{k \to \infty} \mu^{N_k} \Big[\Big| N^{-d} \sum_{x \in \mathbb{T}_N^d} G(x/N) \eta(x) \ - \ \int G(u) \rho_0(u)\, du \Big| > \varepsilon \Big] \ = \ 0 \ . \end{aligned}$$

We now prove that Q^* is concentrated on paths $\pi(t, du) = \rho(t, u)du$ whose density is a weak solution of the heat equation. For positive integers m and n, denote by $C^{m,n}([0,T] \times \mathbb{T}^d)$ the space of continuous functions with m continuous derivatives in time and n continuous derivatives in space. For $G : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ of class $C^{1,2}$, consider the martingale $M_t^G = M_t^{G,N}$ given by

$$M_t^G \ = < \pi_t^N, G_t > \ - \ < \pi_0^N, G_0 > \ - \ \int_0^t (\partial_s + N^2 L_N) < \pi_s^N, G_s > \, ds \ .$$

It is well known (cf. Appendix 1 in [5]) that the process $N_t^G = N_t^{G,N}$ defined by

$$\begin{aligned} N_t^G \ &= \ \left(M_t^G\right)^2 - N^2 \int_0^t A_G(s)\, ds \\ \text{with} \quad A_G(s) \ &= \ L_N < \pi_s^N, G_s >^2 \ - 2 < \pi_s^N, G_s > L_N < \pi_s^N, G_s > \end{aligned}$$

is a martingale with respect to the natural filtration. Straightforward computations show that

$$A_G(s) \ = \ \frac{1}{N^{2(d-1)}} \sum_{|x-y|=1} \left[G_s(y/N) - G_s(x/N)\right]^2 \eta_s(x)(1 - \eta_s(y)) \ .$$

In particular, by Chebychev and Doob inequality,

$$\begin{aligned} Q^N\Big[\sup_{0\le t\le T}\Big|M_t^G\Big|\ge\varepsilon\Big] \;&\le\; 4\varepsilon^{-2}E_{Q^N}\left[(M_T^G)^2\right] \\ =\; 4\varepsilon^{-2}E_{Q^N}\Big[\int_0^T A_G(s)\,ds\Big] \;&\le\; \frac{C(G)T}{\varepsilon^2N^d}\,. \end{aligned}$$

If we now observe that the difference between $2N^2L_N < \pi_s^N, G_s >= < \pi_t^N, \Delta_N G >$ and $< \pi_t^N, \Delta G >$ is of order $o_N(1)$ because the total mass of π_t^N is bounded by 1 and G is of class C^2, we may conclude that every limit point Q^* is concentrated on trajectories such that

$$< \pi_t, G_t > = < \pi_0, G_0 > \;+\; \int_0^t < \pi_s, \partial_s G_s - (1/2)\Delta G_s > \,ds\;, \qquad (2.10)$$

for all $0\le t\le T$ because the application from $D([0,T],\mathcal{M}_+)$ to $\mathbb{R}$ which associates to a trajectory $\{\pi_t, 0\le t\le T\}$ the number

$$\sup_{t\le T}\Big| < \pi_t, G > - < \pi_0, G > - (1/2)\int_0^t < \pi_s, \Delta G > \,ds\Big|$$

is continuous as long as G is of class C^2.

In conclusion, all limit points are concentrated on absolutely continuous trajectories $\pi_t(du) = \pi_t(u)du$ that are weak solutions of the heat equation in the sense of (2.10) and whose density at time 0 is $\rho_0(\cdot)$.

Third step: Uniqueness of weak solutions of the heat equation. We just proved that every limit point of the sequence Q^N is concentrated on weak solutions of the heat equation with initial profile ρ_0. To conclude the proof of the uniqueness of limit points, it remains to show that there exists only one weak solution of this equation.

There exists many methods. Brezis and Crandall (1979) proved such a result for a class of quasi-linear second order equations. Their theorem gives us immediately the uniqueness result we need.

We present in Appendix A2.4 [5] a uniqueness result based on the investigation of the time evolution of the H_{-1} norm. This method requires, however, supplementary properties of weak solutions that are not difficult to check in the case of symmetric simple exclusion processes.

Finally, since the hydrodynamic equation of symmetric simple exclusion is linear, the methods developed by Oelschläger (1985) give a third possible approach.

Note that for any bounded profile ρ_0 the heat equation admits a strong solution given by

$$\rho(t,u) \;=\; \int \bar\rho_0(v)\, G_t(u-v)\, dv$$

if $\bar\rho_0 : \mathbb{R}^d \to \mathbb{R}$ represents the $\mathbb{T}^d$-periodic function identically equal to ρ_0 on $\mathbb{T}^d$ and if $G_t(w)$ is the usual d-dimensional Gaussian kernel: $G_t(w) = (2\pi t)^{-d/2}\exp\{-(1/2t)|w|^2\}$. In particular the weak solution is in fact a strong solution.

In conclusion, with any of these uniqueness results, we proved that the sequence Q^N converges to the Dirac measure concentrated on this strong solution.

Fourth step (Convergence in probability at fixed time). Even if in general it is false that the application from $D([0,T],\mathcal{M}_+)$ to $\mathcal{M}_+$ obtained by taking the value at time $0<t<T$ of the process is continuous, this statement is true if the process is almost surely continuous at time t for the limiting probability measure. In the present context, the limiting probability measure is concentrated on weakly continuous trajectories. Thus π_t^N converges in distribution to the deterministic measure $\pi_t(u)du$. Since convergence in distribution to a deterministic variable implies convergence in probability, the theorem is proved.

In the previous proof, the initial state μ^N appeared only in the second step. It was necessary to show that the limit points Q^* were concentrated on trajectories $\pi_t(du)$ that at time 0 were given by

$$\pi_0(du) \;=\; \rho_0(u)\, du\ .$$

Therefore, the special structure of the measure μ^N did not play any particular role in the proof and the hypothesis of Theorem 2.4 concerning the initial state can be considerably relaxed:

Theorem 2.5 *For $\rho_0\colon \mathbb{T}^d \to [0,1]$, consider a sequence of measures $\{\mu^N : N\ge 1\}$ on $\{0,1\}^{\mathbb{T}_N^d}$ associated to the profile ρ_0:*

$$\limsup_{N\to\infty} \mu^N\Big[\Big|N^{-d}\sum_x G(x/N)\eta(x) \;-\; \int G(u)\rho_0(u)\,du\Big| > \delta\Big] \;=\; 0\ .$$

for every $\delta>0$ and every continuous function $G\colon \mathbb{T}^d\to\mathbb{R}$. The conclusions of Theorem 2.4 remain in force.

In conclusion, under the hypothesis of a weak law of large numbers at time 0 for the empirical measure π^N, we have proved a law of large numbers for any later time t.

3 The Entropy Method for Gradient Systems

There are essentially two general methods to prove the hydrodynamic behavior of an interacting particle system. We present here the first one, called the entropy method, due to Guo, Papanicolaou and Varadhan [4].

To illustrate this method, we prove in this chapter the hydrodynamic behavior of an exclusion process satisfying assumptions (2.3). We have seen in Chapter 1 that to address such a question, we need first to describe the equilibrium states of the process. The next result shows that the Bernoulli product measures are reversible for exclusion processes satisfying assumptions (2.3), explaining also the interest of this set of hypotheses.

Lemma 3.1 *Consider an exclusion process with generator given by (2.1) and rates $\{a_z(\eta)\}$ satisfying (2.3). Then, the Bernoulli product measures ν_α^N are reversible.*

Proof: By assumption $a_{x,x+z}(\eta) = a_{x+z,x}(\eta)$ and both functions do not depend on $\eta(x)$, $\eta(x+z)$. Fix two local functions f, g. A change of variables $\xi = \sigma^{x,x+z}\eta$ gives that

$$\begin{aligned}
&\int a_{x,x+z}(\eta)\, \eta(x)\, [1-\eta(x+z)]\, f(\sigma^{x,x+z}\eta)\, g(\eta)\, d\nu_\alpha^N \\
&\quad = \int a_{x,x+z}(\sigma^{x,x+z}\eta)\, \eta(x+z)\, [1-\eta(x)]\, f(\eta)\, g(\sigma^{x,x+z}\eta)\, d\nu_\alpha^N \\
&\quad = \int a_{x,x+z}(\eta)\, \eta(x+z)\, [1-\eta(x)]\, f(\eta)\, g(\sigma^{x,x+z}\eta)\, d\nu_\alpha^N \\
&\quad = \int a_{x+z,x}(\eta)\, \eta(x+z)\, [1-\eta(x)]\, f(\eta)\, g(\sigma^{x,x+z}\eta)\, d\nu_\alpha^N\ .
\end{aligned}$$

This identity proves that the measures ν_α^N are reversible. □

For z in $\mathbb{T}_N^d$, denote by $W_{x,x+z}$ the current over the bond $\{x, x+z\}$. This is the rate at which a particle jumps from x to $x+z$ minus the rate at which a particle jumps from $x+z$ to x. With the notation introduced in the definition of the exclusion process,

$$W_{x,x+z} \;=\; a_{x,x+z}(\eta)\, \eta(x)\, [1-\eta(x+z)] \;-\; a_{x+z,x}(\eta)\, \eta(x+z)\, [1-\eta(x)]\ .$$

We will say that an interacting particle system is gradient if the current can be written as a local function minus its translation:

Definition 3.2 *An exclusion process with jump rates $\{a_z(\eta)\}$ is said to be gradient if for each z there exist a positive integer $n_z \geq 1$, local functions $h_{z,i}$, $1 \leq i \leq n_z$, and sites $x_{z,i}$, $1 \leq i \leq n_z$, such that*

$$W_{0,z} \;=\; \sum_{i=1}^{n_z} \{\tau_{x_{z,i}} h_{z,i} - h_{z,i}\} \;.$$

We will see below the reason and the importance of this terminology.

3.1 A Gradient Exclusion Process

Everything presented in this chapter relies on the assumptions that the process is gradient and that the one-parameter family of invariant states is explicitly known. However, to fix ideas and keep notation simple, we focus hereafter on a one-dimensional gradient exclusion process satisfying assumptions (2.3).

Consider the one-dimensional exclusion process with rates $\{a_z(\eta)\}$ given by

$$a_z(\eta) \;=\; \begin{cases} 1 + \mathfrak{a}[\eta(-1) + \eta(2)] & \text{for } z = 1, \\ 1 + \mathfrak{a}[\eta(-2) + \eta(1)] & \text{for } z = -1, \\ 0 & \text{if } |z| > 1 \end{cases}$$

for some $\mathfrak{a} > -1/2$. These rates satisfy assumption (2.3) so that the product Bernoulli measures are reversible. An elementary computation shows that the current $W_{0,1}$ is given by

$$W_{0,1} \;=\; \Big\{h_1(\eta) - \tau_1 h_1(\eta)\Big\} \;+\; \Big\{h_2(\eta) - \tau_2 h_2(\eta)\Big\} \;, \tag{3.1}$$

where,

$$h_1(\eta) \;=\; \eta(0) - \mathfrak{a}\eta(-1)\eta(1) \quad \text{and} \quad h_2(\eta) \;=\; \mathfrak{a}\eta(-1)\eta(0) \;.$$

This process is thus gradient and reversible with respect to the Bernoulli product measures. Of course, since the process is translation invariant, the current over the bond $\{x, x+1\}$, denoted by $W_{x,x+1}$, is just the current over the bond $\{0,1\}$ translated by x:

$$W_{x,x+1} \;=\; \tau_x W_{0,1} \;.$$

Our goal is to prove that under some assumptions on the initial state, the empirical measure converges to some absolutely continuous measure $\rho(t,u)du$, whose density is the solution of some partial differential equation. For gradient systems, it is easy to anticipate the PDE which describes the evolution of the density.

Here is the heuristic derivation of the PDE. Assume that the conservation of local equilibrium holds and that the empirical measure π_t^N converges to some $\rho(t,u)du$. Fix a smooth function $G : \mathbb{T} \to \mathbb{R}$ and consider the martingale M_t^G associated to the empirical measure:

$$M_t^G \;=< \pi_t^N, G > \;-\; < \pi_0^N, G > \;-\; \int_0^t N^2 L_N < \pi_s^N, G > \, ds \; .$$

By definition of the empirical measure,

$$N^2 L_N < \pi^N, G > = \; N^{-1} \sum_{x \in \mathbb{T}_N} G(x/N) N^2 L_N \eta(x) \; .$$

$L_N \eta(x)$ is the rate at which $\eta(x)$ changes. In our example this happens only if a particle jumps from a neighbor site to x or from x to one of the neighbors. By definition of the current, this may be written as

$$L_N \eta(x) \;=\; W_{x-1,x} \;-\; W_{x,x+1} \; .$$

Replacing in the previous sum $L_N \eta(x)$ by this expression and performing a summation by parts, we obtain that the integral term in the martingale M_t^G is equal to

$$N^{-1} \sum_{x \in \mathbb{T}_N} (\nabla_N G)(x/N) N W_{x,x+1} \; ,$$

where ∇_N is the discrete derivative given by

$$(\nabla_N G)(x/N) \;=\; N\{G(x+1/N) - G(x/N)\} \; .$$

This first summation by parts, always possible, canceled one of the factors N. A second summation by parts is also possible because we assumed the current to be itself the difference of two local functions. Here enters the gradient assumptions which, as you can see, simplifies considerably the problem. Without a second integration by parts, we will have to estimate an expression of order N.

By (3.1) and a summation by parts, the previous sum is equal to

$$
\begin{aligned}
&N^{-1} \sum_{x\in\mathbb{T}_N} N\{(\nabla_N G)(x/N) - (\nabla_N G)(x-1/N)\}\tau_x h_1 \\
&\quad + N^{-1} \sum_{x\in\mathbb{T}_N} N\{(\nabla_N G)(x/N) - (\nabla_N G)(x-2/N)\}\tau_x h_2 \\
&\quad = N^{-1} \sum_{x\in\mathbb{T}_N} G''(x/N)\tau_x h_1 \;+\; 2N^{-1} \sum_{x\in\mathbb{T}_N} G''(x/N)\tau_x h_2 \;+\; O(N^{-1})\;.
\end{aligned}
$$

We now return to the martingale. Since the martingale vanishes at time 0, its expectation is constant equal to 0. In particular,

$$
\begin{aligned}
&\mathbb{E}_{\mu^N}\Big[< \pi_t^N, G > \;-\; < \pi_0^N, G > \Big] \qquad (3.2)\\
&\quad = \mathbb{E}_{\mu^N}\Big[\int_0^t ds\, N^{-1} \sum_{x\in\mathbb{T}_N} G''(x/N)\tau_x h(\eta_s)\Big] \;+\; O(N^{-1})\;,
\end{aligned}
$$

where

$$
h \;=\; h_1 + 2h_2\;.
$$

Recall that we are assuming that the empirical measure converges to $\rho(t,u)du$. In particular, the left hand side of this expression converges to

$$
\int_{\mathbb{T}} G(u)\rho(t,u)\, du \;-\; \int_{\mathbb{T}} G(u)\rho_0(u)\, du\;.
$$

On the other hand, the right hand side is equal to

$$
\int_0^t ds\, N^{-1} \sum_{x\in\mathbb{T}_N} G''(x/N)\mathbb{E}_{\mu^N}[\tau_x h(\eta_s)]\;.
$$

Under the assumption of conservation of local equilibrium, for any fixed macroscopic time $0 \le s \le t$ and any local function f,

$$
\mathbb{E}_{\mu^N}[\tau_x f(\eta_s)] \;\sim\; E_{\nu^N_{\rho(s,x/N)}}[f]\;.
$$

By (2.4), this last expression can be written as $\tilde{f}(\rho(s,x/N))$. Therefore, under the assumption of conservation of local equilibrium, the right hand side of (3.2) is close, as $N\uparrow\infty$, to

$$
\int_0^t ds\, N^{-1} \sum_{x\in\mathbb{T}_N} G''(x/N)\tilde{h}(\rho(s,x/N))\;.
$$

In conclusion, we proved that if there is conservation of local equilibrium and if the empirical measure converges, it must converges to a measure $\rho(t,u)du$ whose density satisfies the identity

$$\int_{\mathbb{T}} G(u)\rho(t,u)\,du \;-\; \int_{\mathbb{T}} G(u)\rho_0(u)\,du \;=\; \int_0^t ds \int_{\mathbb{T}} du\, G''(u)\tilde{h}(\rho(s,u))$$

for all smooth functions G and all time t. This means that ρ is a weak solution of the PDE

$$\begin{cases} \partial_t \rho = \Delta \tilde{h}(\rho)\ , \\ \rho(0,\cdot) = \rho_0(\cdot)\ . \end{cases} \tag{3.3}$$

A simple computation shows that in our example, $\tilde{h}(\alpha) = \alpha + \mathfrak{a}\alpha^2$ so that the equation is

$$\partial_t \rho = \partial_u (D(\rho)\partial_u \rho)$$

with $D(\alpha) = 1 + 2\mathfrak{a}\alpha$.

3.2 The Hydrodynamic Behavior

We may now turn to a sketch of a rigorous proof. Technical details can be find in Chapter 5 of [5]. We presented in Chapter 2 a general method to prove the hydrodynamic behavior of an interacting particle system, which consisted in three steps.

Denote by Q^N the probability measure on the path space $D([0,T],\mathcal{M}_+)$ corresponding to the Markov process π_t^N, speeded up by N^2, and starting from some measure μ_N. We first prove that Q^N is tight. This step is technical and omitted.

We then need to characterize all limit points of the sequence Q^N. This step is divided in three statements. We show that all limit points Q^* are concentrated on

1. absolutely continuous trajectories $\pi(t,du) = \rho(t,u)du$;
2. weak solutions of the PDE (3.3);
3. trajectories $\pi(t,du) = \rho(t,u)du$ whose density satisfy an energy estimate.

Just as in Chapter 2, since there is at most one particle per site, all limit points are concentrated on paths $\pi(t,du)$ which are absolutely continuous with respect to the Lebesgue measure and whose density $\rho(t,u)$ is positive and bounded by 1. The first statement is thus trivial.

To prove that all limit points are concentrated on weak solutions, we follow again the strategy proposed in Chapter 2. For a smooth function $G : \mathbb{T} \to \mathbb{R}$, consider the martingale M_t^G introduced in the previous section. An elementary estimate shows that this martingale vanishes in L^2 as $N \uparrow \infty$. Therefore, by Doob's inequality, for every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}\Big[\sup_{0\le t\le T} |M_T^G| \; > \; \delta \Big] \; = \; 0 \, . \tag{3.4}$$

Here appears the first main difference between this gradient exclusion process and the symmetric simple exclusion process. While in the simple symmetric case the martingale M_t^G could be written as a function of the empirical measure, here M_t^G, given by

$$< \pi_t^N, G > \; - \; < \pi_0^N, G > \; - \; \int_0^t ds\, N^{-1} \sum_{x\in\mathbb{T}_N} G''(x/N)\tau_x h(\eta_s) \; ,$$

is not a function of the empirical measure because product terms as $\eta(0)\eta(1)$ appear in the definition of $h(\eta)$. If we want to proceed as in Chapter 2, we need to "close" this equation in terms of the empirical measure. This is done by the replacement lemma below due to Guo, Papanicolaou and Varadhan. To state this result, we need to introduce some notation.

For a probability measure μ^N on $\{0,1\}^{\mathbb{T}_N}$, denote by $H(\mu^N|\nu_\alpha^N)$ the entropy of μ_N with respect to ν_α^N:

$$H(\mu^N|\nu_\alpha^N) \; = \; \int \log \frac{d\mu^N}{d\nu_\alpha^N}\, d\mu^N \; .$$

The reader can find in Appendix 1 of [5] the main properties of the entropy in the context of Markov processes. Just observe that for any local equilibrium $\nu^N_{\rho_0(\cdot)}$

$$\begin{aligned} &\lim_{N\to\infty} N^{-1} H(\nu^N_{\rho_0(\cdot)}|\nu_\alpha^N) \\ &\quad = \int du \left\{ \rho_0(u)\log\frac{\rho_0(u)}{\alpha} + [1-\rho_0(u)]\log\frac{1-\rho_0(u)}{1-\alpha} \right\} . \end{aligned}$$

Thus, the entropy of a local equilibrium state with respect to any non-degenerate invariant state is of order N.

For each positive integer ℓ and integer x, denote by $\eta^\ell(x)$ the empirical density of particles in a box of length $2\ell+1$ centered at x:

$$\eta^\ell(x) \;=\; \frac{1}{(2\ell+1)} \sum_{|y-x|\le \ell} \eta(y)\;. \tag{3.5}$$

Lemma 3.3 *Fix a sequence μ^N of probability measures μ^N on $\{0,1\}^{\mathbb{T}_N}$. Assume that $H(\mu^N|\nu^N_\alpha) \le C_0 N$ for some finite constant C_0. Then, for every $\delta>0$ and every local function g,*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} \mathbb{P}_{\mu^N}\Big[\int_0^T N^{-1}\sum_{x\in\mathbb{T}_N} \tau_x V_{\varepsilon N}(\eta_s)\,ds \ge \delta\Big] \;=\; 0\;,$$

where

$$V_\ell(\eta) \;=\; \Big|\frac{1}{(2\ell+1)}\sum_{|y|\le\ell}\tau_y g(\eta) \;-\; \tilde g\left(\eta^\ell(0)\right)\Big|\;.$$

We claim that this lemma permits to express the martingale M^G_t in terms of the the empirical measure π^N_t. For $\varepsilon>0$, denote by ι_ε the approximation of the identity

$$\iota_\varepsilon(u) \;=\; \frac{1}{2\varepsilon}\mathbf{1}\{|u|\le\varepsilon\}\;.$$

In particular,

$$\eta^{\varepsilon N}(0) \;=\; \frac{1}{2\varepsilon N+1}\sum_{|y|\le \varepsilon N}\eta(y) \;\sim\; \frac{1}{2\varepsilon N}\sum_{|y|\le\varepsilon N}\eta(y) \;=<\pi^N,\iota_\varepsilon>\;. \tag{3.6}$$

Recall the expression for the integral part of the martingale M^G_t and fix a smooth function $H:\mathbb{T}\to\mathbb{R}$. Since H is a smooth function, a second order Taylor expansion shows that

$$\begin{aligned} & N^{-1}\sum_{x\in\mathbb{T}_N} H(x/N)\tau_x h(\eta_s) \\ &\quad =\; N^{-1}\sum_{x\in\mathbb{T}_N}\frac{1}{2\varepsilon N+1}\sum_{|y-x|\le\varepsilon N} H(y/N)\tau_x h(\eta_s) \;+\; O(\varepsilon^2)\;. \end{aligned}$$

A summation by parts shows that the previous term can be written as

$$N^{-1}\sum_{x\in\mathbb{T}_N} H(x/N)\frac{1}{2\varepsilon N+1}\sum_{|y-x|\le\varepsilon N}\tau_y h(\eta_s) \;+\; O(\varepsilon^2)\;.$$

By Lemma 3.3 and identity (3.6), this expression is equal to

$$N^{-1} \sum_{x \in \mathbb{T}_N} H(x/N) \tau_x \tilde{h}(< \pi_s^N, \iota_\varepsilon >) \ + \ R_{N,\varepsilon} \ ,$$

where $R_{N,\varepsilon}$ is an expression which vanishes in probability as $N \uparrow \infty$ and then $\varepsilon \downarrow 0$.

In conclusion, it follows from Lemma 3.3 that the martingale M_t^G can be written as

$$\begin{aligned} &< \pi_t^N, G > \ - \ < \pi_0^N, G > \\ &- \int_0^t ds \, N^{-1} \sum_{x \in \mathbb{T}_N} G''(x/N) \tau_x \tilde{h}(< \pi_s^N, \iota_\varepsilon >) \ + \ R_{N,\varepsilon} \end{aligned}$$

where $R_{N,\varepsilon}$ is an expression which vanishes in probability as $N \uparrow \infty$ and then $\varepsilon \downarrow 0$. This fact, together with (3.4) and the strategy presented in Chapter 2 shows that all limit points Q^* of the sequence Q^N are concentrated on paths $\pi(t, du) = \rho(t, u) du$ such that

$$\int_{\mathbb{T}} G(u) \rho(t,u) \, du \ - \ \int_{\mathbb{T}} G(u) \rho_0(u) \, du \ = \ \int_0^t ds \int_{\mathbb{T}} du \, G''(u) \tilde{h}(\rho(s,u))$$

for all smooth functions G and all times t. Thus all limit points are concentrated on weak solutions of (3.3).

In contrast with the symmetric simple exclusion process, whose macroscopic evolution is described by the heat equation, we have here a non-linear parabolic equation and some further properties are needed in order to ensure uniqueness of weak solutions.

One classical proof of uniqueness relies on the time evolution of the H_{-1} norm and requires properties on the space derivative of the solution. Such result is summarized in the next lemma.

Definition 3.4 *A measurable function $\rho \colon [0,T] \times \mathbb{T} \to \mathbb{R}_+$ is said to be a weak solution in $\mathcal{H}_1$ of (3.3) provided*

1. *$\rho(t, \cdot)$ belongs to $L^1(\mathbb{T}^d)$ for every $0 \le t \le T$ and $\sup_{0 \le t \le T} \|\rho_t\|_{L^1} < \infty$.*
2. *There exists a function in $L^2([0,T] \times \mathbb{T})$, denoted by $\partial_u \tilde{h}(\rho(s,u))$, such that for every smooth function $G \colon [0,T] \times \mathbb{T} \to \mathbb{R}$*

$$\begin{aligned} &\int_0^T ds \int_{\mathbb{T}} du \, (\partial_u G)(s,u) \tilde{h}(\rho(s,u)) \\ &\qquad = \ - \int_0^T ds \int_{\mathbb{T}} du \, G(s,u) \partial_u \tilde{h}(\rho(s,u)) \ . \end{aligned}$$

3. For every smooth function $G\colon \mathbb{R}\to\mathbb{R}$ and for every $0<t\le T$,

$$\int_{\mathbb{T}} du\, \rho(t,u)G(u) \;-\; \int_{\mathbb{T}} du\, \rho_0(u)G(u) \\ = \;-\; \int_0^t ds\, \int_{\mathbb{T}} du\, G'(u)\partial_u \tilde{h}(\rho(s,u))\;.$$

Lemma 3.5 *There exists a unique weak solution in $\mathcal{H}_1$ of (3.3).*

In view of this result, to conclude the proof of the uniqueness of limit points of the sequence Q^N, we need to prove the following lemma:

Lemma 3.6 *All limit points Q^* of the sequence Q^N are concentrated on paths $\rho(t,u)du$ with the property that there exists an $L^2([0,T]\times\mathbb{T}^d)$ function denoted by $\partial_u\tilde{h}(\rho(s,u))$ such that*

$$\int_0^T ds\, \int_{\mathbb{T}^d} du\, (\partial_u G)(s,u)\tilde{h}(\rho(s,u)) \;=\; -\int_0^T ds\, \int_{\mathbb{T}^d} du\, G(s,u)\partial_u\tilde{h}(\rho(s,u))$$

for all smooth functions G. Moreover,

$$\int_0^T ds\, \int_{\mathbb{T}^d} du\, (\partial_u\tilde{h}(\rho(s,u)))^2 \;<\; \infty\;.$$

These results together show that the sequence Q^N converges to the measure Q concentrated on the unique weak solution in $\mathcal{H}_1$ of (3.3) provided the entropy of μ^N with respect to a reference invariant state ν_α^N is bounded by C_0N, for some finite constant C_0, and provided the empirical measure converges in probability, under μ^N, to $\rho_0(u)du$.

To conclude the proof of the convergence of π_t^N to $\rho(t,u)du$, we need to argue, as in Chapter 2, that convergence in the Skorohod space to a continuous path implies convergence in the uniform norm and that convergence in distribution to a Dirac measure implies convergence in probability. In summary, we have proved in this chapter the following result.

Theorem 3.7 *Let μ^N be a sequence of probability measures on the path space $\{0,1\}^{\mathbb{T}_N}$. Assume there exists a finite constant C_0 and a density $0<\alpha<1$ such that $H(\mu^N|\nu_\alpha^N)\le C_0N$. Suppose furthermore that for every continuous functions $H:\mathbb{T}\to\mathbb{R}$ and every $\delta>0$,*

$$\lim_{N\to\infty}\mu^N\Big\{\,\Big|<\pi^N,H> - \int_{\mathbb{T}} du\, H(u)\rho_0(u)\Big|>\delta\Big\}\;=\;0\;.$$

Then, for every $t > 0$, every continuous functions $H : \mathbb{T} \to \mathbb{R}$ and every $\delta > 0$

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \Big\{ \Big| < \pi_t^N , H > - \int_{\mathbb{T}} du\, H(u) \rho(t,u) \Big| > \delta \Big\} \;=\; 0 \;,$$

where $\rho(t,u)$ is the unique weak solution in $\mathcal{H}_1$ of (3.3).

4 The Relative Entropy Method

We present in this lecture the second general method to prove the hydrodynamic behavior of an interacting particle system. Known as the relative entropy method, it is due to Yau [9]. While the entropy method presented in the previous chapter relies on the uniqueness of weak solutions of the hydrodynamic equation, the relative entropy method relies on the existence of a smooth solution.

As in the previous chapters, all proofs omitted here can be found in Chapter 6 of [5].

To fix ideas, consider the one-dimensional exclusion process with rates $\{a_z(\eta)\}$ given by

$$a_z(\eta) \;=\; \begin{cases} 1 + \mathfrak{a}[\eta(-1) + \eta(2)] & \text{for } z = 1, \\ 1 + \mathfrak{a}[\eta(-2) + \eta(1)] & \text{for } z = -1, \\ \qquad 0 & \text{if } |z| > 1 \end{cases}$$

for some $\mathfrak{a} > -1/2$. We have seen that the hydrodynamic equation of this model is the non-linear parabolic equation

$$\partial_t \rho = \partial_u (D(\rho) \partial_u \rho) \;,$$

with $D(\alpha) = 1 + 2\mathfrak{a}\alpha$. This equation admits a smooth solution, denoted by $\rho(t,u)$, twice continuously differentiable in space and once continuously differentiable in time (cf. [8]).

To avoid uninteresting technical difficulties, we assume that the initial profile $\rho_0(\cdot)$ is bounded below and above by strictly positive and finite constants:

$$0 < a \le \rho_0(u) \le a^{-1} \;.$$

Hereafter, for $t \ge 0$, we denote by $\nu^N_{\rho(t,\cdot)}$ the product measure with slowly varying parameter associated to the profile $\rho(t,\cdot)$:

$$\nu^N_{\rho(t,\cdot)}\{\eta; \eta(x) = 1\} \;=\; \nu_{\rho(t,x/N)}\{\eta, \eta(0) = 1\} \;, \quad \text{for} \quad x \in \mathbb{T}_N \;.$$

We may now state the main result of this chapter.

Theorem 4.1 *Let $(\mu^N)_{N\ge 1}$ be a sequence of measures on $\{0,1\}^{\mathbb{T}_N}$ whose entropy with respect to $\nu^N_{\rho(0,\cdot)}$ is of order $o(N)$:*

$$H(\mu^N | \nu^N_{\rho(0,\cdot)}) \; = \; o(N) \; .$$

Then, the relative entropy of the state of the process at the macroscopic time t with respect to $\nu^N_{\rho(t,\cdot)}$ is also of order $o(N)$:

$$H\big(\mu^N S^N_t | \nu^N_{\rho(t,\cdot)}\big) \; = \; o(N) \qquad \text{for every} \quad t \ge 0 \; .$$

In this formula, S^N_t stands for the semi-group associated to the generator L_N speeded up by N^2.

It is not difficult to deduce from this result a strong version of the hydrodynamic limit behavior of the interacting particle system:

Corollary 4.2 *Under the assumptions of the theorem, for every cylinder function Ψ and every continuous function $H \colon \mathbb{T} \to \mathbb{R}$,*

$$\lim_{N\to\infty} E_{\mu^N S^N_t} \Big[\Big| N^{-1} \sum_{x\in\mathbb{T}_N} H(x/N) \tau_x \Psi(\eta) \; - \; \int_{\mathbb{T}} H(u) E_{\nu_{\rho(t,u)}}[\Psi] \, du \Big| \Big] \; = \; 0 \; .$$

The proof of Theorem 4.1 is divided in several lemmas. We start introducing some notation. For $0 < \alpha < 1$, ν^N_α stands for a reference invariant measure and $\psi_N(t)$ for the Radon–Nikodym derivative of $\nu^N_{\rho(t,\cdot)}$ with respect to ν^N_α:

$$\psi_N(t) \; := \; \frac{d\nu^N_{\rho(t,\cdot)}}{d\nu^N_\alpha} \; .$$

A simple computation allows to obtain an explicit formula for $\psi_N(t)$ because the measures $\nu^N_{\rho(t,\cdot)}$, ν^N_α are product and the profile $\rho(t,\cdot)$ is bounded below by a strictly positive constant uniformly in time:

$$\psi_N(t) \; = \; \exp\Big\{ \sum_{x\in\mathbb{T}_N} \eta(x) \, \log \frac{\rho(t,x/N)}{\alpha} + [1-\eta(x)] \, \log \frac{1-\rho(t,x/N)}{1-\alpha} \Big\} \; .$$

Denote by μ^N_t the measure μ^N at macroscopic time t:

$$\mu^N_t \; := \; \mu^N S^N_t \; ,$$

by $f_N(t) = f_t^N$ the Radon–Nikodym derivative of μ_t^N with respect to the reference measure ν_α^N:

$$f_N(t) \;:=\; \frac{d\mu_t^N}{d\nu_\alpha^N} \;=\; \frac{d\mu^N S_t^N}{d\nu_\alpha^N}$$

and by $H_N(t)$ the relative entropy of μ_t^N with respect to $\nu_{\rho(t,\cdot)}^N$:

$$H_N(t) \;=\; H\big(\mu_t^N | \nu_{\rho(t,\cdot)}^N\big)\;.$$

Since μ_t^N is absolutely continuous with respect to $\nu_{\rho(t,\cdot)}^N$, by the explicit formula for the relative entropy

$$\begin{aligned} H_N(t) \;&=\; \int \log \frac{d\mu_t^N}{d\nu_{\rho(t,\cdot)}^N}\, d\mu_t^N \\ &=\; \int f_t^N(\eta)\;\log\left[\frac{f_t^N(\eta)}{\psi_t^N(\eta)}\right]\;\nu_\alpha^N(d\eta)\;. \end{aligned}$$

We turn now to the proof of Theorem 4.1. The strategy consists in estimating the relative entropy $H_N(t)$ by a term of order $o(N)$ and the time integral of the entropy multiplied by a constant:

$$H_N(t) \;\le\; o(N) \;+\; \gamma^{-1}\int_0^t H_N(s)\,ds$$

and apply Gronwall lemma to conclude. The first step stated in Lemma 4.3 below gives an upper bound for the entropy production.

Lemma 4.3 *For every $t \ge 0$,*

$$\partial_t H_N(t) \;\le\; \int \frac{1}{\psi_t^N}\,\big\{N^2 L_N^* \psi_t^N - \partial_t \psi_t^N\big\}\, f_t^N\, d\nu_\alpha^N\;,$$

where L_N^* is the adjoint of L_N in $L^2(\nu_\alpha^N)$.

Proof: It is easy to check that $f_N(t)$ is the solution of the Kolmogorov forward equation

$$\partial_t f_N(t) \;=\; N^2 L_N^* f_N(t)\;.$$

Since the profile $\rho(t,\cdot)$ is smooth, a simple computation shows that

$$\begin{aligned} \partial_t H_N(t) \;&=\; \int N^2 L_N^* f_t^N \cdot \log\left[\frac{f_t^N}{\psi_t^N}\right]\, d\nu_\alpha^N \\ &+\; \int \left\{N^2 L_N^* f_t^N - f_t^N\,\frac{\partial_t \psi_t^N}{\psi_t^N}\right\}\, d\nu_\alpha^N\;. \end{aligned}$$

Since L_N^* is the adjoint of L_N in $L^2(\nu_\alpha)$,

$$\int L_N^* f_t^N \, d\nu_\alpha^N \;=\; 0\,.$$

By the same reason, the first expression on the right hand side may be rewritten as

$$N^2 \int \psi_t^N \, \frac{f_t^N}{\psi_t^N} \, L_N \left(\log \frac{f_t^N}{\psi_t^N} \right) \, d\nu_\alpha\,.$$

The elementary inequality

$$a\,[\log b - \log a] \;\le\; (b-a)$$

that holds for positive reals a, b, shows that for every positive function h and for every generator L of a jump process,

$$h\, L(\log h) \;\le\; L\,h\,.$$

In particular, the last integral is bounded above by

$$N^2 \int \psi_t^N \, L_N \left(\frac{f_t^N}{\psi_t^N} \right) \, d\nu_\alpha^N \;=\; N^2 \int \frac{f_t^N}{\psi_t^N} \, L_N^* \psi_t^N \, d\nu_\alpha^N\,.$$

□

We claim that

$$\frac{N^2 L_N^* \psi_t^N - \partial_t \psi_t^N}{\psi_t^N} \;=\; O(1) \tag{4.1}$$

$$+ \sum_{x=0}^{N-1} H_t''(x/N) \Big\{ \tau_x h_0(\eta) - \tilde{h}_0(\rho(t,x/N)) - \tilde{h}_0'(\rho(t,x/N))[\eta(x) - \rho(t,x/N)] \Big\}$$

$$+ \sum_{x=0}^{N-1} [H_t'(x/N)]^2 \Big\{ \tau_x g_0(\eta) - \tilde{g}_0(\rho(t,x/N)) - \tilde{g}_0'(\rho(t,x/N))[\eta(x) - \rho(t,x/N)] \Big\},$$

where $O(1)$ is an expression bounded by a constant uniformly in N and

$$\begin{aligned} H_t(x/N) \;&=\; \log \frac{\rho(t,x/N)}{1-\rho(t,x/N)}\,, \\ h_0(\eta) \;&=\; 2\mathfrak{a}\eta(0)\eta(1) \;-\; \mathfrak{a}\eta(-1)\eta(1)\,, \\ g_0(\eta) \;&=\; -\eta(0)\eta(1) \;+\; \mathfrak{a}\eta(0)\eta(1) \;+\; \mathfrak{a}\eta(-1)\eta(1) \;-\; 2\mathfrak{a}\eta(-1)\eta(0)\eta(1)\,. \end{aligned}$$

To obtain the above simple formulas, we need to perform some integration by parts. For instance, instead of obtaining $2\mathfrak{a}\eta(0)\eta(1)$ in h_0, one gets

$$\mathfrak{a}\sum_{x\in\mathbb{Z}} r(x)\eta(x)\eta(x+1)$$

for some positive, finite-supported function r such that $\sum_x r(x) = 2$. We can rewrite this sum as

$$2\mathfrak{a}\eta(0)\eta(1) \;+\; \mathfrak{a}\sum_{x\in\mathbb{Z}} r(x)\{\eta(x)\eta(x+1) - \eta(0)\eta(1)\}$$

and integrate by parts the second expression to incorporate it in the order 1 term. This procedure has been done repeatedly to almost all terms.

To prove formula (4.1), first observe that $L_N^* = L_N$ because the process is reversible. Moreover, since $a_{x,x+1}(\eta) = a_{x+1,x}(\eta)$, we can write the generator L_N as

$$(L_N f)(\eta) \;=\; \sum_{x\in\mathbb{T}_N^d} a_{x,x+1}(\eta)\,[f(\sigma^{x,x+z}\eta) - f(\eta)]\;.$$

A straightforward computation shows that

$$\begin{aligned}\frac{N^2 L_N \psi_t^N}{\psi_t^N} \;&=\; \sum_{x=0}^{N-1} a_{x,x+1}(\eta)\Big\{\frac{\psi_t^N(\sigma^{x,x+1}\eta)}{\psi_t^N(\eta)} - 1\Big\}\\ &=\; \sum_{x=0}^{N-1} H_t''(x/N)\tau_x h(\eta) \;+\; \sum_{x=0}^{N-1}[H_t'(x/N)]^2\tau_x g(\eta) \;+\; O(1)\;,\end{aligned}$$

where $O(1)$ is an expression bounded by a constant uniformly in N, $h = h_1 + 2h_2$ is the local function introduced in the previous chapter and

$$g(\eta) \;=\; \eta(0)[1-\eta(1)] \;+\; \mathfrak{a}\eta(0)\eta(1) \;+\; \mathfrak{a}\eta(-1)\eta(1) \;-\; 2\mathfrak{a}\eta(-1)\eta(0)\eta(1)\;.$$

Since

$$\begin{aligned}\int_{\mathbb{T}} H_t''(u)\tilde{h}(\rho(t,u))\,du \;&=\; -\int_{\mathbb{T}} du\, H_t'(u)\,\tilde{h}'(\rho(t,u))\,(\partial_u\rho)(t,u)\\ &=\; -\int_{\mathbb{T}} du\,[H_t'(u)]^2\,\tilde{g}(\rho(t,u))\;,\end{aligned}$$

we obtain that

$$\begin{aligned}\frac{N^2 L_N \psi_t^N}{\psi_t^N} \;&=\; \sum_{x=0}^{N-1} H_t''(x/N)\Big\{\tau_x h(\eta) - \tilde{h}(\rho(t,x/N))\Big\}\\ &\quad+\; \sum_{x=0}^{N-1}[H_t'(x/N)]^2\Big\{\tau_x g(\eta) - \tilde{g}(\rho(t,x/N))\Big\} \;+\; O(1)\;,\end{aligned}$$

On the other hand an elementary computation shows that

$$\frac{\partial_t \psi_t^N}{\psi_t^N} \;=\; \partial_t \log \psi_t^N \;=\; \sum_{x\in\mathbb{T}_N} \frac{\Delta \tilde h(\rho(t,x/N))}{\rho(t,x/N)[1-\rho(t,x/N)]}[\eta(x) - \rho(t,x/N)]$$

because $\rho(t,u)$ is the solution of the hydrodynamic equation. Adding the previous two expressions we get that

$$\begin{aligned}
&\frac{N^2 L_N^* \psi_t^N - \partial_t \psi_t^N}{\psi_t^N} \;=\; O(1)\\
&+ \sum_{x=0}^{N-1} H_t''(x/N)\Big\{\tau_x h(\eta) - \tilde h(\rho(t,x/N)) - \tilde h'(\rho(t,x/N))[\eta(x)-\rho(t,x/N)]\Big\}\\
&+ \sum_{x=0}^{N-1} [H_t'(x/N)]^2\Big\{\tau_x g(\eta) - \tilde g(\rho(t,x/N)) - \tilde g'(\rho(t,x/N))[\eta(x)-\rho(t,x/N)]\Big\}\;.
\end{aligned}$$

It remains to observe that all linear terms cancel to deduce (4.1).

A microscopic Taylor expansion up to the second order appeared in the formula (4.1). To take advantage of this expansion, we apply the one block estimate to replace the cylinder functions h_0, g_0 by $\tilde h_0(\eta_t^\ell(x))$, $\tilde g_0(\eta_t^\ell(x))$.

In view of Lemma 4.3 and after the one block estimate, since $H_N(0) = o(N)$, we obtain that

$$\begin{aligned}
N^{-1} H_N(t) \;&\le\; C_0 \int_0^t ds E_{\mu^N S_s^N}\Big[N^{-1}\sum_{x=0}^{N-1}\Big\{\eta^\ell(x) - \rho(s,x/N)\Big\}^2\Big]\\
&+\; o(N,\ell)\;,
\end{aligned}$$

for some finite constant C_0 which depends only on the solution $\rho(t,u)$ of the hydrodynamic equation. In this formula, $o(N,\ell)$ is an expression which vanishes as $N\uparrow\infty$ and then $\ell\uparrow\infty$ and comes from the one block estimate.

By the entropy inequality, the main term on the right hand side of the previous expression is bounded above by

$$C_0\int_0^t ds \frac{H_N(s)}{\gamma N} \;+\; \frac{C_0}{\gamma N}\int_0^t ds \log E_{\nu^N_{\rho(s,\cdot)}}\Big[\exp\Big\{\gamma\sum_{x=0}^{N-1}[\eta^\ell(x)-\rho(s,x/N)]^2\Big\}\Big] \tag{4.2}$$

for every $\gamma>0$. Since $\nu^N_{\rho(s,\cdot)}$ is a product measure, by Hölder inequality, the second term is less than or equal to

$$\frac{C_0}{\gamma(2\ell+1)N}\int_0^t ds\sum_{x=0}^{N-1}\log E_{\nu^N_{\rho(s,\cdot)}}\Big[\exp\Big\{\gamma(2\ell+1)[\eta^\ell(x)-\rho(s,x/N)]^2\Big\}\Big]\;.$$

As $N \uparrow \infty$ this expression converges to

$$\frac{C_0}{\gamma(2\ell+1)} \int_0^t ds \int_{\mathbb{T}} du \, \log E_{\nu^N_{\rho(s,u)}} \Big[\exp \Big\{ \gamma(2\ell+1)[\eta^\ell(0) - \rho(s,u)]^2 \Big\} \Big] \, .$$

By the large deviations for i.i.d. Bernoulli random variables and Laplace-Varadhan lemma, as $\ell \uparrow \infty$, this expression converges to

$$\frac{C_0}{\gamma} \int_0^t ds \int_{\mathbb{T}} du \sup_{\theta \in [0,1]} \Big\{ \gamma[\theta - \rho(t,u)]^2 - I_{\rho(t,u)}(\theta) \Big\} \, ,$$

where $I_\alpha(\cdot)$ is the large deviations rate functional given by

$$I_\alpha(\theta) \;=\; \theta \log \frac{\theta}{\alpha} \;+\; [1-\theta] \log \frac{1-\theta}{1-\alpha} \, \cdot$$

Since $[\theta - \rho(t,u)]^2$ is quadratic around $\rho(t,u)$, it is not difficult to show that there exists $\gamma_0 > 0$, depending only on the solution of the hydrodynamic equation, such that

$$\sup_{\theta \in [0,1]} \Big\{ \gamma[\theta - \rho]^2 - I_\rho(\theta) \Big\} \;\le\; 0$$

for all $\gamma < \gamma_0$ and all ρ in the interval $[\inf_u \rho_0(u), \sup_u \rho_0(u)]$. This proves that the second term in (4.2) vanishes as $N \uparrow \infty$ and then $\ell \uparrow \infty$, provide we choose γ small enough.

Putting all previous estimates together, we obtain that

$$N^{-1} H_N(t) \;\le\; C_0 \int_0^t ds \frac{H_N(s)}{\gamma N} \;+\; o(N, \ell)$$

for γ small enough. Theorem 4.1 follows now from Gronwall inequality.

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